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Dissipative solitons of the discrete complex cubic–quintic Ginzburg–Landau equation

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Abstract

We study, analytically, the discrete complex cubic–quintic Ginzburg–Landau (dCCQGL) equation with a non-local quintic term. We find a set of exact solutions which includes, as particular cases, bright and dark soliton solutions, constant magnitude solutions with phase shifts, periodic solutions in terms of elliptic Jacobi functions in general forms, and various particular periodic solutions.

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1. Introduction

Light interaction with nonlinear periodic media has attracted much attention in recent years. Various optical phenomena in photonic crystals and arrays of planar waveguides or fibers have been predicted and observed. In particular, discrete solitons in nonlinear lattices have been the focus of considerable attention in diverse branches of science [1–4]. Discrete solitons exist in several physical settings, such as biological systems [5], atomic chains [6,7], solid state physics [8], electrical lattices [9] and Bose–Einstein condensates [10]. Discrete solitons also exist in photonic structures (in arrays of coupled nonlinear optical waveguides [11–17] as well as in a nonlinear photonic crystal structures [18]). Photonic crystals, which are artificial microstructures having photonic bandgaps, can be used to precisely control propagation of optical pulses and beams. When using discrete waveguides and photonic crystals, "discrete solitons"

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appear naturally and have a number of interesting properties.

Various models of discrete nonlinear systems admit soliton solutions. The discrete nonlinear Schrödinger (dNLS) equation is

$$i\frac{d\psi_n}{dt}+\frac{D}{2}(\psi_{n+1}-2\psi_n+\psi_{n-1}) + \beta_1|\psi_n|^2\psi_n=0, \tag{1}$$

where ψ_n are complex variables defined for all integer values of the site index n. The term $\psi_{n+1} - 2\psi_n + \psi_{n-1}$ plainly approximates a second derivative term for a continuous system and so physically represents diffraction. A simple transformation [11] eliminates the term $-2\psi_n$, thus indicating that what is occurring is nearest-neighbour coupling. Hence, a realistic discrete system features diffraction-type effects. The dNLS equation was used by Christodoulides and Joseph [12] to model the propagation of discrete self-trapped beams in an array of weakly-coupled nonlinear optical waveguides. It is well known that the standard DNLS equation (1) is not completely integrable and that it does not have any non-trivial exact solutions. The integrable discrete nonlinear Schrödinger equation (Ablowitz–Ladik (AL) system)

$$i\frac{d\psi_n}{dt}+\frac{D}{2}(\psi_{n+1}-2\psi_n+\psi_{n-1}) + |\psi_n|^2(\psi_{n+1}+\psi_{n-1})=0, \tag{2}$$

was found in [19] and it can be solved using the inverse scattering method and other methods.

Periodic structures have an even richer variety of properties when they are dissipative, i.e., have gain and loss in the system. In particular, discrete analogues of the complex Ginzburg–Landau equation have attracted attention in the field of the pattern formation in nonlinear coupled oscillators [20,21]. Specific laser and amplification systems can be designed using active waveguide arrays. In [22], we studied the discrete complex cubic Ginzburg–Landau (dCCGL) equation

$$i\frac{d\psi_n}{dt}+\left(\frac{D}{2}-i\beta\right)(\psi_{n+1}-2\psi_n+\psi_{n-1}) + (1-i\epsilon)|\psi_n|^2(\psi_{n+1}+\psi_{n-1})=i\delta\psi_n. \tag{3}$$

Several exact solutions can be derived in this case. However, in the model where the highest nonlinearity is cubic, the solitons are unstable, as a numerical study [23] clearly shows. Thus, just as in the continuous case, the dCCGL equation does not provide conditions for stability. Quintic terms have to be introduced into the continuous complex Ginzburg–Landau equation to allow for stable soliton solutions. The same requirement should be fulfilled in the case of discrete system. Hence analysis of the cubic–quintic discrete complex Ginzburg–Landau equation is a necessary step for further progress.

There are several ways of introducing the quintic terms. They can be local, as in the simple discretization of the NLSE, or non-local, as in the Ablowitz–Ladik equation. Higher-order nonlinearity can also include a higher degree of non-locality. Each of these cases has to be studied separately in order to understand the influence of the quintic nonlinearity on soliton solutions and the dynamics of the system in general.

Abdullaev et al. studied the discrete complex cubic–quintic Ginzburg–Landau equation [24] with non-local cubic and local quintic nonlinear terms:

$$i\frac{d\psi_n}{dt}+(\psi_{n+1}+\psi_{n-1})\left(1-\lambda|\psi_n|^2\right) = -\delta\psi_n+i\gamma|\psi_n|^2\psi_n+i\beta(\psi_{n+1}-2\psi_n+\psi_{n-1}) - i\kappa|\psi_n|^4\psi_n. \tag{4}$$

Note that all nonlinear terms responsible for dissipation are local. Using a perturbation technique, when δ, γ, β and κ are small, they found a soliton solution which is approximate and valid at small dissipation. Efremidis and Christodoulides studied a different discrete complex cubic–quintic Ginzburg–Landau equation [25]

$$i\frac{d\psi_n}{dt}+\left(\frac{D}{2}-i\beta\right)(\psi_{n+1}-2\psi_n+\psi_{n-1}) + (1-i\epsilon)|\psi_n|^2\psi_n+(\nu-i\mu)|\psi_n|^4\psi_n - i\delta\psi_n=0, \tag{5}$$

where all nonlinear terms are local. They found that discrete solitons of the discrete complex cubic–quintic Ginzburg–Landau equation have several features that have no counterparts in either the continuous limit or in other conservative discrete models.

In this Letter, we study a discrete equation set, similar to that in [24,25], but include a quintic nonlinearity with non-locality of the lowest order. At the same time, the cubic terms remain local. For this particular model, we derive exact soliton and periodic solutions. These are exact solutions for arbitrary values of the dissipative terms, so they are different from the approximate solutions obtained under the restriction that dissipative terms be small. In particular, we consider a model of a dissipative system, viz., the following discrete complex cubic–quintic Ginzburg–Landau (dCCQGL) equation

$$i\frac{d\psi_n}{dt} + \left(\frac{D}{2} - i\beta\right)(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + (1 - i\epsilon)|\psi_n|^2\psi_n + (\nu - i\mu)|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) - i\delta\psi_n = 0. \quad (6)$$

As we can see, the quintic nonlinear term differs from that in (4) and (5). Other generalizations of the dCCQGL equation are also possible and may have exact solutions. Each of these cases requires a separate study. The continuous limit of (6), as well as (4) and (5), is the complex Ginzburg–Landau equation (CGLE) [26]

$$i\frac{d\psi}{dt} + \left(\frac{D}{2} - i\beta\right)\psi_{xx} + (1 - i\epsilon)|\psi|^2\psi + (\nu - i\mu)|\psi|^4\psi = i\delta\psi, \quad (7)$$

which has many applications in describing superconductivity, superfluidity, non-equilibrium systems, phase transitions, and wave propagation phenomena. In the limit of $\beta = \epsilon = \nu = \mu = \delta = 0$, Eq. (6) is reduced to the (non-integrable) discrete nonlinear Schrödinger equation (1).

If we take the continuous limit of Eq. (6) with $\psi_n = a\Psi_n$, $\tau = a^2 t$, $x = na$ and $\delta = \delta_a a^2$ (a is a small lattice parameter), we have a complex quintic Swift–Hohenberg type equation [27,28]

$$i\Psi_\tau + \left(\frac{D}{2} - i\beta\right)\Psi_{xx} + (1 - i\epsilon)|\Psi_n|^2\Psi_n + 2a^2(\nu_a - i\mu_a)|\Psi_n|^4\Psi_n + \frac{a^2}{12}\left(\frac{D}{2} - i\beta\right)\Psi_{xxxx} = i\delta_a\Psi, \quad (8)$$

which also has many applications in describing non-equilibrium systems.

2. Exact soliton solutions

Stationary solutions of Eq. (6) are defined by

$$\psi_n = \phi_n e^{-i\omega t}. \quad (9)$$

The Hirota method can be applied to obtain selected exact solutions of Eq. (6). In order to do this we substitute

$$\psi_n(t) = \phi_n(t)e^{-i\omega t} = \frac{g_n(t)}{f_n(t)}e^{-i\omega t},$$

$$\psi_n^*(t) = \phi_n^*(t)e^{i\omega t} = \frac{g_n^*(t)}{f_n(t)}e^{i\omega t}$$

with real f_n, into Eq. (6). As a result, we obtain the multi-linear form:

$$i f_n^2 f_{n+1} f_{n-1}(\dot{g}_n f_n - g_n \dot{f}_n) + \omega g_n g_n^3 f_{n+1} f_{n-1} - (1 - i\epsilon) g_n g_n^* g_n f_n f_{n+1} f_{n-1} + \left(\frac{D}{2} - i\beta\right)(g_{n+1} f_n f_{n-1} - 2 g_n f_{n+1} f_{n-1} + g_{n-1} f_{n+1} f_n) f_n^3 + (\nu - i\mu)\left(g_n g_n^*\right)^2 (g_{n+1} f_{n-1} + f_{n+1} g_{n-1}) - i\delta g_n f_n^3 f_{n+1} f_{n-1}. \quad (10)$$

Then, the standard procedure of the Hirota method can be used to obtain the exact solutions listed in the following sections.

Solutions can be obtained only for certain relations between the coefficients of the equations. Namely, we set

$$\delta = \epsilon\omega, \qquad \beta = \frac{\epsilon D}{2}, \qquad \mu = \epsilon\nu, \quad (11)$$

and note that in this case the system simplifies to

$$\omega\phi_n + \frac{D}{2}(\phi_{n+1} - 2\phi_n + \phi_{n-1}) + \phi_n^3 + (\phi_{n+1} + \phi_{n-1})\nu\phi_n^4 = 0. \quad (12)$$

Simple (constant) solution.

We take D, ν and ϵ arbitrary. Direct substitution shows that any constant a is a solution, so long as $\omega = -a^2(1 + 2a^2\nu)$.

Alternating constant solution.

Furthermore, $(-1)^n a$ is a solution for any constant a, so long as $\omega = 2D + 2a^4\nu - a^2$.

2.1. Bright soliton

By using, for example, the Hirota method, we can find the explicit solution for the fundamental soliton with a constant phase across its profile.

We can write the two relevant solutions separately using the function sech. For the bright pulse solution, we need $D > 0$ but arbitrary, and $\nu < 0$ but arbitrary.

For convenience, we define k using

$$\operatorname{sech}(k) = \sqrt{-2\nu D}\ (< 1).$$

The solution is then:

$$\phi_n = \sqrt{D\cosh(k)}\sinh(k)\operatorname{sech}(kn+\alpha), \tag{13}$$

with α arbitrary and

$$\omega = -2D\sinh^2\left(\frac{k}{2}\right) = D - \sqrt{-\frac{D}{2\nu}}.$$

This can also be expressed as

$$\phi_n = \frac{(p-p^{-1})\sqrt{D(p+p^{-1})/2}}{p^{n+n_a}+p^{-n-n_a}}, \tag{14}$$

where $k = \log p$ and $n_a = \alpha/k$.

The soliton profile is shown in Fig. 1(a). The numerical simulations based on the original equation (6) show that this solution is stable. The results of the simulation are shown in Fig. 1(b). Small perturbations do not destroy the solution and tend to disappear as the soliton evolves in time. We recall that solitons of the discrete complex cubic Ginzburg–Landau equation are unstable [23]. This shows that quintic terms are important in making the soliton stable.

Let us consider particular examples. For $D = 1$ and $\nu = -1/4$, we have the all-positive solution:

$$\phi_n = 2^{1/4}\operatorname{sech}\left(n\,\operatorname{arcsech}\left(\frac{1}{\sqrt{2}}\right)\right) \approx 1.189207\operatorname{sech}(0.8813736n).$$

This can also be expressed as

$$\phi_n = \frac{2.37841}{q^n + q^{-n}}, \quad q = 0.414214, \tag{15}$$

and is clearly positive everywhere.

2.2. Bright soliton with alternating sign

In the case of the bright alternating sign (spiked) soliton solution, we need $D < 0$ but arbitrary, and $\nu > 0$ but arbitrary. As before, we define k through $\operatorname{sech}(k) = \sqrt{-2\nu D}\ (< 1)$. The solution is:

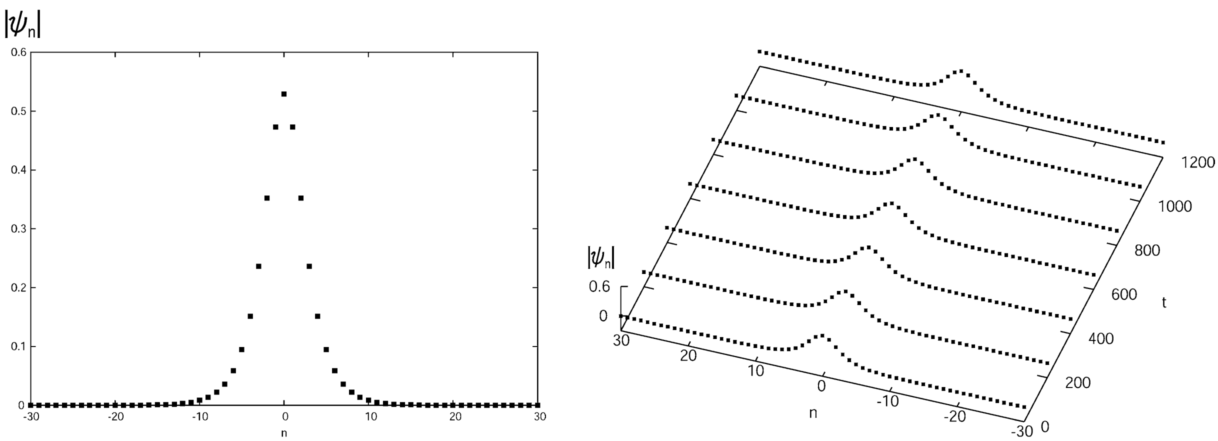
$$\phi_n = (-1)^n\sqrt{-D\cosh(k)}\sinh(k)\operatorname{sech}(kn+\beta), \tag{16}$$

with β arbitrary and

$$\omega = +2D\cosh^2\left(\frac{k}{2}\right) = D + \sqrt{-\frac{D}{2\nu}}.$$

For $D = -1$ and $\nu = 1/4$, we have the alternating sign solution:

$$\phi_n = 1.189207(-1)^n\operatorname{sech}(0.8813736n),$$



(a) (b)

Fig. 1. (a) Bright soliton profile for the equation parameters $D = 1$, $\nu = -0.4$. This implies that $k = 0.481212$ and the soliton is given by $\phi_n = 0.528686\operatorname{sech}(0.481212n)$. (b) Numerical simulation showing stationary soliton evolution in time for $\epsilon = 0.05$.

which can also be expressed as

$$\phi_n = \frac{2.37841}{q^n + q^{-n}}, \quad q = -0.414214. \tag{17}$$

It is clearly positive for n even and negative for n odd.

2.3. Dark soliton

In analogy with the previous case we can find the solution for the dark solitons. We can also introduce the parameter k through $\cosh^4 k = -2D\nu$ (> 1). For $D < 0$ but arbitrary and $\nu > 0$ but arbitrary, such that $2D\nu < -1$, the plain dark soliton solution can be written in terms of hyperbolic functions:

$$\phi_n = \sqrt{-D}\,\mathrm{sech}(k)\tanh(k)\tanh(kn + \alpha_2), \tag{18}$$

where α_2 is arbitrary and the frequency is

$$\omega = D\tanh^2 k = D - \sqrt{-\frac{D}{2\nu}}.$$

This can also be expressed as

$$\phi_n = \frac{2\sqrt{-D}(p - p^{-1})}{(p + p^{-1})^2}\,\frac{p^{n+n_a} - p^{-n-n_a}}{p^{n+n_a} + p^{-n-n_a}}, \tag{19}$$

where $n_a = \alpha/k$.

For example, for $D = -1$, $\nu = 2$, we have $\phi_n = \frac{1}{2}\tanh(0.88137n)$.

The dark soliton profile is shown in Fig. 2(a). Numerical simulations showing the evolution of this solution in time are presented in Fig. 2(b). The solution is stable and evolves in time without changing. Moreover, small perturbations do not grow but rather disappear exponentially with time.

2.4. Alternating sign dark soliton

We now describe the solution for the dark soliton with alternating sign values of ϕ_n. We again define the parameter k through $\cosh^4 k = -2D\nu$ (> 1). For $D > 0$ but arbitrary and with $\nu < 0$ but arbitrary, such that $2D\nu < -1$, as before, we can express this solution in terms of hyperbolic functions. The alternating sign dark solution can be written as

$$\phi_n = (-1)^n\sqrt{D}\,\mathrm{sech}(k)\tanh(k)\tanh(kn + \beta_2), \tag{20}$$

where β_2 is arbitrary and the frequency ω is

$$\omega = D\bigl(1 + \mathrm{sech}^2 k\bigr) = D + \sqrt{-\frac{D}{2\nu}}.$$

This can be expressed as

$$\phi_n = (-1)^n\frac{2\sqrt{D}(p - p^{-1})}{(p + p^{-1})^2}\,\frac{p^{n+n_a} - p^{-n-n_a}}{p^{n+n_a} + p^{-n-n_b}}, \tag{21}$$

where n_b is arbitrary.

For example, for $D = +1$, $\nu = -2$, we have $\phi_n = (-1)^n\frac{1}{2}\tanh(0.88137n)$.

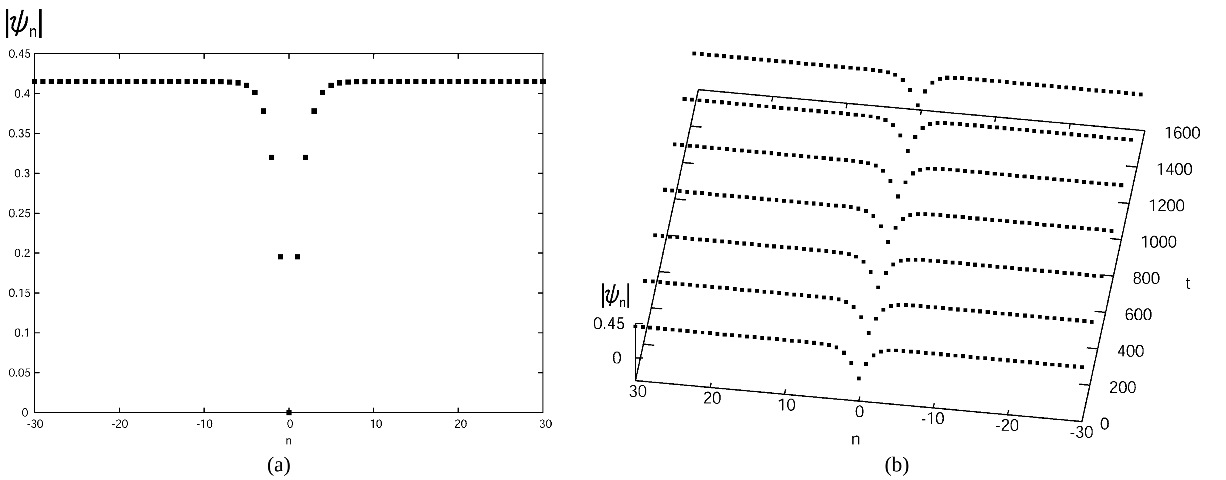


Fig. 2. (a) Dark soliton profile for the equation parameters $D = -1$, $\nu = 0.824898765$. (b) Numerical simulation showing stable stationary dark soliton evolution in time for $\epsilon = 0.001$.

3. Periodic trigonometric solutions

3.1. *Periodic solutions in terms of* cos *and* sin *functions*

There is a variety of periodic solutions with ν arbitrary, apart from sign, and D arbitrary. Here we give five examples. These five solutions differ in the value of the period along the lattice and in the frequency ω:

$$\phi_n = \sqrt{\frac{-2}{5\nu}}\cos(n\pi/3), \tag{22}$$

where $\omega = \frac{D}{2} + \frac{4}{25\nu}$;

$$\phi_n = \sqrt{\frac{-\sqrt{2}}{3\nu}}\cos(n\pi/4), \tag{23}$$

where $\omega = D - \frac{D}{\sqrt{2}} + \frac{\sqrt{2}}{9\nu}$;

$$\phi_n = \sqrt{\frac{-1}{\nu\sqrt{2}}}\cos\left[\frac{\pi}{8}(1+2n)\right], \tag{24}$$

where $\omega = D - \frac{D}{\sqrt{2}} + \frac{1}{8\sqrt{2}\nu}$;

$$\phi_n = -\sqrt{\frac{2}{5\nu}(1+\sqrt{5})} \times \sin\left[(n-2)\arccos\left(\frac{1-\sqrt{5}}{4}\right)\right], \tag{25}$$

where $\omega = \frac{5(1+\sqrt{5})D\nu-4}{10(\sqrt{5}-1)\nu}$.

Eqs. (22)–(25) plainly require $\nu < 0$, while the following one needs $\nu > 0$:

$$\phi_n = \sqrt{\frac{2}{3\nu}}\cos\left[\frac{\pi}{6}(5-4n)\right], \tag{26}$$

where $\omega = \frac{6D\nu-1}{4\nu}$.

4. Periodic solutions in terms of elliptic Jacobi functions

Periodic solutions can also be obtained for the relation given (11) between the coefficients of the equations. Then, the system simplifies to (12). The latter can be written in the form of a discrete map:

$$\phi_{n+1} + \phi_{n-1} = -\frac{\phi_n(\phi_n^2 - (D-\omega))}{\nu\left(\phi_n^4 + \frac{D}{2\nu}\right)}. \tag{27}$$

Once again, we take ϕ_n real. In general, this map is not integrable, but it includes integrable cases [29]. Only those cases in which the map (27) reduces to a quadratic difference equation are integrable. When $\frac{D}{2\nu} < 0$, we can rewrite the map (27) as

$$\phi_{n+1} + \phi_{n-1} = -\frac{\phi_n(\phi_n^2 - (D-\omega))}{\nu\left(\phi_n^2 + \sqrt{-\frac{D}{2\nu}}\right)\left(\phi_n^2 - \sqrt{-\frac{D}{2\nu}}\right)}. \tag{28}$$

The quadratic common factor of the denominator and numerator on the right-hand side of (28) cancels out for some sets of coefficients. There are two cases which include exact periodic solutions.

Case 1. In the case of $\omega = D - \sqrt{-\frac{D}{2\nu}}$, the map (28) becomes

$$\phi_{n+1} + \phi_{n-1} = -\frac{\phi_n}{\nu(\phi_n^2 + y^2)}, \tag{29}$$

where $y^2 \equiv \sqrt{-D/2\nu}$. This turns out to be the form of ω used in the soliton pulse solutions given by Eqs. (13) and (18).

Case 2. When $\omega = D + \sqrt{-\frac{D}{2\nu}}$, the map (28) becomes

$$\phi_{n+1} + \phi_{n-1} = -\frac{\phi_n}{\nu(\phi_n^2 - y^2)}, \tag{30}$$

where $y^2 \equiv \sqrt{-D/2\nu}$. This is the form of ω used in the soliton pulse solutions given by Eqs. (16) and (20).

Both maps are well-known to be integrable ones. Now we can construct exact periodic solutions, following Pott's paper [30].

4.1. *Periodic solutions in Case 1*

4.1.1. cn *solution*

To solve Eq. (29), we consider the periodic solution

$$\phi_n = A\,\mathrm{cn}[2nK/p; m], \tag{31}$$

where A is an amplitude, K is the complete elliptic integral of the first kind:

$$K(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - m\sin^2\varphi}},$$

m is modulus ($0 \leqslant m < 1$), and p is an integer (> 2).

The boundary condition $\phi_0 = A$, $\phi_{-1} = \phi_1$ and $\phi_{p-1} = \phi_{p+1}$. For $n = 0$, Eq. (29) gives

$$\phi_1 = -\frac{A}{2\nu(A^2 + y^2)}. \tag{32}$$

Substituting (31) into (32),

$$\mathrm{cn}(2K/p; m) = -\frac{1}{2\nu(A^2 + y^2)}, \tag{33}$$

and then (31) into (29), we get

$$\frac{2A\,\mathrm{cn}(2nK/p; m)\,\mathrm{cn}(2K/p; m)}{1 - m\,\mathrm{sn}^2(2nK/p; m)\,\mathrm{sn}^2(2K/p; m)} = -\frac{A\,\mathrm{cn}(2nK/p; m)}{\nu(A^2\,\mathrm{cn}^2(2nK/p; m) + y^2)}, \tag{34}$$

where we used the formula

$$\mathrm{cn}(u+v) + \mathrm{cn}(u-v) = \frac{2\,\mathrm{cn}\,u\,\mathrm{cn}\,v}{1 - m\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}. \tag{35}$$

With the assistance of (33), Eq. (34) simplifies to

$$\bigl((A^2 + y^2)m\,\mathrm{sn}^2(2K/p; m) - A^2\bigr)\,\mathrm{sn}^2(2nK/p; m) = 0, \tag{36}$$

which is satisfied for all n provided

$$m\,\mathrm{sn}^2(2K/p; m) = \frac{A^2}{A^2 + y^2}. \tag{37}$$

From (37) and (33), we obtain

$$m = \frac{4\nu^2 A^2 (A^2 + y^2)}{4\nu^2 (A^2 + y^2)^2 - 1}. \tag{38}$$

For $p = 2$ we get the solution

$$\phi_n = A\cos\left(n\frac{\pi}{2}\right),$$

which has period 4, and is valid for any A, so long as $\omega = D - A^2$.

If $p = 4$, all expressions here simplify to hyperbolic functions. Then the full solution can be written explicitly:

$$\phi_n = A\,\mathrm{cn}\left[\frac{n}{2}K(m), m\right]. \tag{39}$$

For convenience, we write m as $m = 1 - \sinh^4 b$, where b is a new real variable which is given by

$$b = \frac{1}{2}\,\mathrm{arcsinh}\bigl(-4\nu y^2\bigr).$$

Then the amplitude squared, A^2, is given by

$$A^2 = \frac{-2 \pm \sqrt{r}}{2\nu(3 + \cosh(2b))}\coth(b)\ (> 0), \tag{40}$$

where

$$r = 4 + 17D\nu - D\nu\cosh(4b)\ (> 0),$$

and the frequency ω is given by

$$\omega = D - A^2 - \bigl(D + 2A^4\nu\bigr)\tanh(b).$$

We clearly require that r be positive.

Eq. (39) can be written in different form:

$$\phi_n = A\bigl(\mathrm{mod}(n+1, 2) + \sqrt{2}\,\mathrm{mod}(n, 2)\tanh(b)\bigr) \times \cos\left(n\frac{\pi}{4}\right),$$

where $b = \frac{1}{2}\mathrm{arcsinh}(-4\nu y^2)$. The amplitude squared is

$$A^2 = \frac{-1}{2\nu\tanh(b)} - y^2.$$

Thus,

$$\phi_n = A\bigl(1, \tanh(b), 0, -\tanh(b), -1, -\tanh(b), 0, \tanh(b), 1, \tanh(b), \ldots\bigr).$$

This produces a sequence of period 8, with the ϕ_n ($n = 0, 1, \ldots$) being given by

$$\bigl(A, A\tanh(b), 0, -A\tanh(b), -A, -A\tanh(b), 0, A\tanh(b), A, \ldots\bigr),$$

where b is arbitrary.

Another special case of the solution (40) appears when $m = 0$. The plus sign case in the expression for the amplitude (40) gives the zero solution. On the other hand, the minus sign gives the following result:

$$\phi = A\cos(n\pi/4),$$

where the amplitude A and the frequency ω are given by

$$A^2 = -\frac{\sqrt{2}}{3\nu}, \qquad \omega = D - \frac{D}{\sqrt{2}} + \frac{\sqrt{2}}{9\nu}.$$

This solution is clearly the same as in Eq. (23).

4.1.2. dn *solution*

To solve Eq. (29), we consider the periodic solution

$$\phi_n = A\,\mathrm{dn}(nK/p;m), \tag{41}$$

where the boundary condition $\phi_0 = A$, $\phi_{-1} = \phi_1$ and $\phi_{p-1} = \phi_{p+1}$. For $n = 0$, Eq. (29) gives (32). Substituting (41) into (32),

$$\mathrm{dn}(K/p;m) = -\frac{1}{2\nu(A^2+\gamma^2)}, \tag{42}$$

and then (41) into (29), we get

$$\frac{2A\,\mathrm{dn}(nK/p;m)\,\mathrm{dn}(K/p;m)}{1-m\,\mathrm{sn}^2(nK/p;m)\,\mathrm{sn}^2(K/p;m)} = -\frac{A\,\mathrm{dn}(nK/p;m)}{\nu(A^2\,\mathrm{dn}^2(nK/p;m)+\gamma^2)}, \tag{43}$$

where we used the formula

$$\mathrm{dn}(u+v) + \mathrm{dn}(u-v) = \frac{2\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{1-m\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}. \tag{44}$$

With the assistance of (42), Eq. (43) simplifies to

$$\big((A^2+\gamma^2)\,\mathrm{sn}^2(K/p;m) - A^2\big)\,\mathrm{sn}^2(nK/p;m) = 0, \tag{45}$$

which is satisfied for all n provided

$$\mathrm{sn}^2(K/p;m) = \frac{A^2}{A^2+\gamma^2}. \tag{46}$$

From (46) and (42), we obtain

$$m = \frac{4\nu^2(A^2+\gamma^2)^2 - 1}{4\nu^2A^2(A^2+\gamma^2)}. \tag{47}$$

4.1.3. sn *solution*

To solve Eq. (29), we consider the periodic solution

$$\phi_n = A\,\mathrm{sn}(nK/p;m), \tag{48}$$

with the boundary condition $\phi_0 = 0$, $\phi_p = A$ and $\phi_{p-1} = \phi_{p+1}$ for any positive integer p. For $n = p$, Eq. (29) gives

$$\phi_{p-1} = -\frac{A}{2\nu(A^2+\gamma^2)}, \tag{49}$$

and (48) with $n = p-1$ gives

$$\phi_{p-1} = A\,\mathrm{sn}\big(K-(K/p);m\big) = A\frac{\mathrm{cn}(K/p;m)}{\mathrm{dn}(K/p;m)}. \tag{50}$$

Using the formula

$$\mathrm{sn}(u+v) + \mathrm{sn}(u-v) = 2\frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v}{1-m\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}, \tag{51}$$

and substituting (48) into (49), we obtain

$$\frac{2A\,\mathrm{sn}(nK/p;m)\,\mathrm{cn}(K/p;m)\,\mathrm{dn}(K/p;m)}{1-m\,\mathrm{sn}^2(nK/p;m)\,\mathrm{sn}^2(K/p;m)} = -\frac{A\,\mathrm{sn}(nK/p;m)}{\nu(A^2\,\mathrm{sn}^2(nK/p;m)+\gamma^2)}, \tag{52}$$

which simplifies to

$$\big(\gamma^2 m\,\mathrm{sn}^2(K/p;m) + A^2\big)\,\mathrm{cn}^2 = 0. \tag{53}$$

This is satisfied for all n provided

$$m = -\frac{A^2}{\gamma^2\,\mathrm{sn}^2(K/p;m)}. \tag{54}$$

From (54), (49) and (50), we obtain

$$m = \frac{4\nu^2A^2(A^2+\gamma^2)^2}{A^4+\gamma^2(A^2-4\nu^2(A^2+\gamma^2)^2)}. \tag{55}$$

In this case, the $p = 2$ solution simplifies and all expressions again reduce to hyperbolic functions. Then the full solution can be written explicitly. It is given by Eq. (48) with $m = 1 - \sinh^4(b)$. Thus we can write it explicitly:

$$\phi_n = A\big(\mathrm{mod}(n+1,2) + \sqrt{2}\,\mathrm{mod}(n,2)\,\mathrm{sech}(b)\big) \times \sin\left(n\frac{\pi}{4}\right),$$

where $b = \mathrm{arcsech}(r)$ and

$$r = \frac{1}{2D\sqrt{2}}\sqrt{\frac{-D}{\nu}}\big(\sqrt{1-8D\nu} - 1\big).$$

Using Eq. (54), the amplitude squared is

$$A^2 = \big(\sinh^2(b) - 1\big)\gamma^2.$$

Thus

$$\phi_n = A\big(0, \mathrm{sech}(b), 1, \mathrm{sech}(b), 0, -\mathrm{sech}(b), -1, -\mathrm{sech}(b), 0, \ldots\big),$$

and the period is 8.

4.2. Periodic solutions in Case 2

4.2.1. sn solution

To solve Eq. (30), we consider the periodic solution

$$\phi_n = A\,\mathrm{sn}(nK/p; m), \tag{56}$$

with the boundary condition $\phi_0 = 0$, $\phi_p = A$ and $\phi_{p-1} = \phi_{p+1}$ for any positive integer p. For $n = p$, Eq. (30) gives

$$\phi_{p-1} = -\frac{A}{2\nu(A^2 - y^2)}, \tag{57}$$

and (56) with $n = p - 1$ gives

$$\phi_{p-1} = A\,\mathrm{sn}\bigl(K - (K/p); m\bigr) = A\frac{\mathrm{cn}(K/p; m)}{\mathrm{dn}(K/p; m)}. \tag{58}$$

Using the formula

$$\mathrm{sn}(u+v) + \mathrm{sn}(u-v) = 2\frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v}{1 - m\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}, \tag{59}$$

and substituting (56) into (57), we obtain

$$\frac{2A\,\mathrm{sn}(nK/p; m)\,\mathrm{cn}(K/p; m)\,\mathrm{dn}(K/p; m)}{1 - m\,\mathrm{sn}^2(nK/p; m)\,\mathrm{sn}^2(K/p; m)} = -\frac{A\,\mathrm{sn}(nK/p; m)}{\nu(A^2\,\mathrm{sn}^2(nK/p; m) - y^2)}, \tag{60}$$

which simplifies to

$$\bigl(-y^2 m\,\mathrm{sn}^2(K/p; m) + A^2\bigr)\,\mathrm{cn}^2 = 0. \tag{61}$$

This is satisfied for all n provided

$$m = \frac{A^2}{y^2\,\mathrm{sn}^2(K/p; m)}. \tag{62}$$

From (62), (57) and (58), we obtain

$$m = \frac{4\nu^2 A^2 (A^2 - y^2)^2}{A^4 - \sqrt{-\frac{D}{2\nu}}(A^2 - 4\nu^2(A^2 - y^2)^2)}. \tag{63}$$

Periodic solutions in terms of elliptic Jacobi functions allow us to obtain variety of other particular periodic solutions. Their stability is still an open question.

5. Conclusions

In conclusion, we have studied, analytically, the discrete complex cubic–quintic Ginzburg–Landau equation with a non-local quintic term. We have found a set of exact solutions which includes, as particular cases, bright and dark soliton solutions, constant magnitude solutions, periodic solutions in terms of elliptic Jacobi functions in general form, and particular cases of periodic solutions. We have given the range of parameters where various of these exact solutions exist. Using numerical simulations, we have found that (some) soliton solutions of the discrete complex cubic–quintic Ginzburg–Landau equation are stable, in contrast to the soliton solutions of the discrete complex cubic Ginzburg–Landau equation.

A comparison between our discrete complex cubic–quintic Ginzburg–Landau equation and other models forms an interesting topic. In our model, exact solutions exist in a narrow region of parameter space. The study of (numerical) solutions outside the region of existence of exact solutions can also be a fruitful avenue. Indeed, this is a subject that deserves further investigation.

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