Physics Letters A 375 (2011) 2782–2785

Contents lists available at ScienceDirect

Physics Letters A

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Rogue wave triplets

Adrian Ankiewicz *, David J. Kedziora, Nail Akhmediev

Optical Sciences Group, Research School of Physical Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia

ARTICLE INFO

Article history:
Received 17 May 2011
Accepted 20 May 2011
Available online 27 May 2011
Communicated by V.M. Agranovich

ABSTRACT

We consider the family of 2nd order rogue wave rational solutions of the nonlinear Schrödinger equation (NLSE) with two free parameters. Surprisingly, these solutions describe a formation consisting of 3 separate first order rogue waves, rather than just two. We show that the 3 components of the triplet are located on an equilateral triangle, thus maintaining a certain symmetry in the solution, even in its decomposed form. The two free parameters of the family define the size and orientation of the triangle on the (x,t) plane.

Rogue waves are in their emerging state of research [1–3]. They appear not only in oceanic conditions [4] but also in optics [5,6], superfluids [7], Bose–Einstein condensates [8] and in the form of capillary waves [9]. Common features and differences between them in various fields of physics are under intense discussion [10]. New studies of rogue waves in any of these disciplines enrich the concept and lead to progress towards a comprehensive understanding of this still mysterious phenomenon.

One of the formal ways to describe rogue waves mathematically is the so-called Peregrine solution [11,12]. This is a solution of the nonlinear Schrödinger equation (NLSE) which is localised both in t and x, thus describing a unique wave event. This solution is also unique in a mathematical sense, as it is written in terms of rational functions, in contrast to many other known solutions of the NLSE. Recently, the existence of these solutions has been proven in optical experiments [13]. The experimental observation of them in a water tank [14] may be an indication that they can describe rogue waves in oceanic conditions.

The Peregrine solution is not the only one localised both in t and x [15,16]. In fact, there is an infinite hierarchy of rational solutions which have the same property [16–20]. These have higher amplitudes and may serve as prototypes of even bigger waves at the water surface. Symmetric solutions of this type, with a single maximum, have been studied in several recent publications [17]. Recently, Dubard et al. [18] and Gaillard [19] from Matveev's group have presented general formulations for rational solutions of arbitrary order in terms of Wronskian determinants. Despite the explicit form of the solutions, their reduction to the simplest and most accessible structure is still cumbersome [18,19].

One advantage of the general formulation [18,19] is that the final expression contains a few arbitrary parameters that allow us to split the symmetric form solution into a multi-peaked solution, with the distances between the peaks being dependent on the free parameters. A surprising part of such analysis is that the second-order rational solution can be decomposed into three solutions of the first order. Our efforts to find a rational solution that can be decomposed into two solutions of the first order have failed. Taking into account that operating with polynomials of the fourth order in the relevant expressions is a straightforward exercise, we come to the conclusion that such solutions do not exist. This result shows that higher-order rational solutions are rather involved and deserve further careful study.

The mathematics behind the decomposition revolve around the fact that solutions obtained by Dubard et al. [18] and Gaillard [19] have free real parameters, thus showing that these solutions are families rather than isolated solutions. This new feature of the higher-order rational solutions requires special attention, as the process of splitting is indeed unique and does not have obvious analogues.

We start the analysis by writing the focusing NLSE. Namely:

$$i\frac{\partial\psi}{\partial x}+\frac{1}{2}\frac{\partial^2\psi}{\partial t^2}+|\psi|^2\psi=0. \tag{1}$$

Earlier [17], we described rational solutions of the NLSE with the following basic structure:

$$\psi_n(x,t)=\left[(-1)^n+\frac{G_n(x,t)+iK_n(x,t)}{D_n(x,t)}\right]e^{ix} \tag{2}$$

where the polynomials G_n and K_n have orders $m(n)$ that are related to the order of the solution n and are lower than that of D_n. The denominator D_n should have no zeros to ensure that the solution is finite everywhere. The "first order" ($n=1$) solution,

$$\psi_1(x,t)=\left[-1+4\frac{1+2ix}{1+4x^2+4t^2}\right]e^{ix} \tag{3}$$

is also known as the Peregrine soliton. It is shown in Fig. 1a.

* Corresponding author. Tel.: +612 612 50191; fax: +612 612 59476.
E-mail address: ana124@physics.anu.edu.au (A. Ankiewicz).

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doi:10.1016/j.physleta.2011.05.047

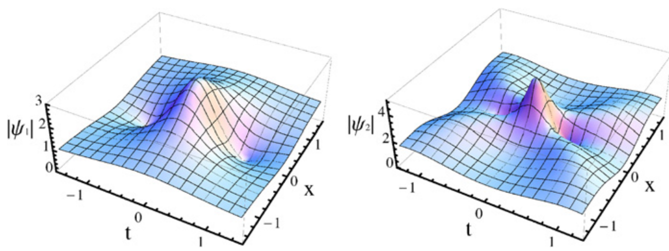

Fig. 1. (a) First order rogue wave. (b) Second order rogue wave with zero offsets $\gamma = 0$ and $\beta = 0$.

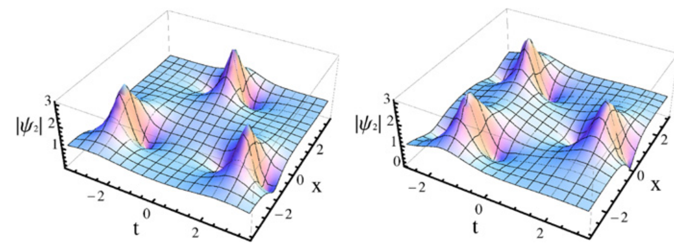

Fig. 2. Rogue wave triplets. Parameters (a) $\gamma = 200$ and $\beta = 0$; (b) $\gamma = 0$ and $\beta = 100$.

The second-order solution without free parameters has been presented earlier in [15–17]. Solutions which include offset parameters can be derived from Eq. (3.87) of the book [21] by taking the limit $\kappa \to 0$. This solution is essentially the one derived by Dubard et al. [18] and Gaillard [19] apart from the point that the coefficients are adapted to the NLSE in a different form, i.e. with different scaling coefficients.

The 2nd order rogue wave can be written as the $n = 2$ version of Eq. (2), where

$$G_2 = 12\big[3 - 16t^4 - 24t^2(4x^2+1) - 4\beta t - 80x^4 - 72x^2 + 4\gamma x\big], \tag{4}$$

$$K_2 = 24\left[x(15 - 16t^4 + 24t^2 - 4\beta t) - 8(4t^2+1)x^3 - 16x^5 + \gamma\left(2x^2 - 2t^2 - \frac{1}{2}\right)\right], \tag{5}$$

and

$$D_2 = 64t^6 + 48t^4(4x^2+1) + 12t^2(3-4x^2)^2 + 64x^6 + 432x^4 + 396x^2 + 9 + \beta\big[\beta + 4t(12x^2 - 4t^2 + 3)\big] + \gamma\big[\gamma + 4x(12t^2 - 4x^2 - 9)\big], \tag{6}$$

where β and γ are arbitrary real constants. For this solution, D_2 consists of polynomials of 6th order, while G_2 and K_2 are of 4th and 5th orders, respectively. The coefficients are different from those in [18] because of the particular form of the NLSE that we use here. These expressions have free parameters γ and β that distinguish them from those we obtained earlier [17]. When $\gamma = 0$ and $\beta = 0$, these solutions coincide with the expressions obtained in [17]. Non-zero parameters add nontrivial changes to the shape of the higher-order rogue wave.

Generally, we can add several free parameters to any solution of the NLSE [21]. The simplest ones are translations along the t and x axes. These shift the solutions located at the origin to an arbitrary position on the (x,t)-plane. These have been omitted in all previous analyses. The parameters γ and β are different from these trivial translations. They describe the relative positions of the first order solutions in the triplet. Finding these relative positions in terms of γ and β is the subject of our analysis below.

As mentioned, when $\gamma = 0$ and $\beta = 0$, the solution is symmetric and has a single maximum. It is shown in Fig. 1b. The amplitude of the second order solution is 5/3 times higher than that of the Peregrine solution. In the absence of translations, the maximum is located at the origin. In this case, the expressions (4)–(6) coincide with our formulae in [17]. Non-zero parameters split the single maximum solution into three parts, where each can be approximately expressed as a shifted first-order solution. Two examples are shown in Fig. 2. As we can see, there are sets of offsets that locate the three maxima on the plane relative to each other. How are they related to the parameters γ and β?

To start with, we consider the simple case $\beta = 0$. At the same time, we assume γ is large but arbitrary. In this case, the three peaks of the solution are located at the corners of a triangle. In order to find their approximate locations, we start with a little numerical analysis. From numerical calculations, we find that the offsets of the peaks are approximately:

$$x_1 \approx 0.4997\gamma^{1/3}, \tag{7}$$

$$x_2 = x_3 \approx -0.253\gamma^{1/3}, \tag{8}$$

and

$$t_2 = -t_3 \approx 0.436\gamma^{1/3}, \tag{9}$$

where the indices denote the peak numbering. The approximation works well when separations are notably larger than the width of each first order solution.

This forms a triplet consisting of one peak at $(x_1, 0)$ and two others at $(x_2, \pm t_2)$, as shown in Fig. 2a. Thus, we can attempt to find the separations analytically by setting

$$x_1 = 1/p, \qquad x_2 = -f_1/p, \qquad t_2 = f_2/p,$$

and assuming p to be small, with $t_1 = 0$, $x_3 = x_2$ and $t_3 = -t_2$. We find the overall denominator, $D = d_1 d_2 d_3$, as a 6th order polynomial. By equating this with $D_2(x,t)$, we find $f_1 = \frac{1}{2}$, $f_2 = \frac{\sqrt{3}}{2}$, while $p = 2/\gamma^{1/3}$. Thus $x_1 = \frac{1}{2}\gamma^{1/3}$, $x_2 = -\frac{1}{4}\gamma^{1/3}$, while $t_2 = \sqrt{3}|x_2| = \frac{\sqrt{3}}{4}\gamma^{1/3} \approx 0.433\gamma^{1/3}$, which agree with the above estimates. For each of the 3 peaks, the radial distance from the origin is $R = 1/p = \frac{1}{2}\gamma^{1/3}$. So the relative offsets are

$$\frac{x_2}{x_1} = -\frac{1}{2} \quad \text{and} \quad \frac{t_2}{x_1} = \frac{\sqrt{3}}{2}. \tag{10}$$

If $\gamma = 0$, then a similar procedure shows that there is one peak at

$$x_1 = 0, \qquad t_1 = \frac{1}{2}\beta^{1/3}$$

and two others at

$$x_2 = -x_3 = \frac{\sqrt{3}}{4}\beta^{1/3}, \qquad t_2 = -\frac{1}{4}\beta^{1/3},$$

as shown in Fig. 2b. To put it simply, the triplet is rotated around the origin by 90° relative to the case $\beta = 0$, while the orientation of each of the first order solutions in the (x,t)-plane remains the same. Here, the radial distance of each first order solution from the origin is $R = \frac{1}{2}\beta^{1/3}$.

After these simple estimates, we turn to the case when both γ and β are non-zero. When γ and β are not very small, the solution consists basically of 3 well-separated fundamental ($n = 1$) rogue waves on a unit background. This means that, roughly, we can approximate the solution as the sum of three first-order solutions:

$$\psi_2(x,t) \approx e^{ix}\left[-1 + 4\sum_{j=1}^{3}\frac{1 + 2i(x - x_j)}{d_j}\right] \tag{11}$$

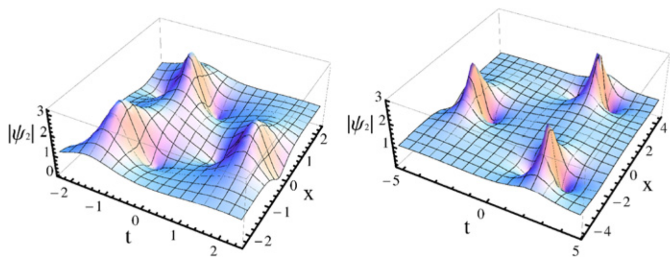

Fig. 3. Rogue wave triplets. Parameters (a) $\gamma = 20$ and $\beta = 40$; (b) $\gamma = 100$ and $\beta = -400$.

where

$$d_j = 1 + 4(x - x_j)^2 + 4(t - t_j)^2, \tag{12}$$

$j = 1, 2, 3$. In general, all six offsets (x_j, t_j) are non-zero. In each case, the 'centre-of-mass' is at the origin $(0, 0)$. This holds for any (γ, β) (see Fig. 3). The radial distance from the centre for arbitrary γ and β is

$$R \approx \frac{1}{2}\left(\beta^2 + \gamma^2\right)^{1/6}. \tag{13}$$

For the case $\gamma = 0$, one of the components is located on the x axis; we set this angle as zero. In the general case of non-zero γ, the angular offset from this zero angle is

$$\theta = -\frac{1}{3}\arctan\left(\frac{\beta}{\gamma}\right) = -\frac{1}{3}\arcsin\left(\frac{\beta}{8R^3}\right). \tag{14}$$

Then all 3 components are equally spaced around the circle. Consequently, the other 2 angles are $\theta \pm \frac{2\pi}{3}$. If $\gamma = 0$, then $\theta = -\frac{\pi}{6}$, while $\beta = 0$ leads to $\theta = 0$, agreeing with the above analysis. Thus, using a straightforward technique, for $R \gg 1$, we have found analytic values for the positions of the triplet which forms the rogue wave of the order of $n = 2$.

Our main result here is that the 3 first order components in the second order solution are located symmetrically on a circle, with 120 degrees between the peaks, independent of the values of γ and β. Note that, in [19], one axis has a factor of 1/2 in its normalization, so this causes the neat circular arrangement to be lost. When the radius of the circle is higher than the width of individual components, this circular location can be attributed to the maxima of each component. When they are located closer to each other, the three peaks join together, with the symmetric second order rogue wave emerging in the limit of $\gamma = 0$ and $\beta = 0$.

Now, the question is: if we have two Peregrine solitons nearby do we have to look for the third one? Our analysis tells us: yes, we do. Two Peregrine solitons do not exist without the third one unless they are so far from each other that they have no significant interaction. Moreover, all three have to be located on a circle on the (x, t)-plane.

Does our result have any consequences for the observation of rogue waves in the ocean? We are not in a position to answer this question now. However, the existence of the so-called "three sisters" i.e. three big waves on the water surface in a row is one of the facts discussed in the literature [22].

Generalisation to even higher order solutions is not trivial. However, we can see analogous symmetries from numerical results. The third order rational solution with non-zero offset parameters is shown in Fig. 4. As we can see, the solution remains highly symmetric with five components located symmetrically on a circle with the sixth one in the middle of the circle. The complete analysis of these cases is well beyond the simple concepts presented

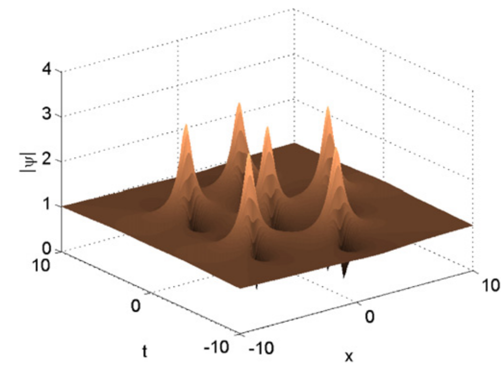

Fig. 4. Rogue wave sextet ($n = 3$).

here. Just to give a hint, we suppose that the general n-th order rogue wave can be approximated as follows:

$$\psi_n(x, t) \approx e^{ix}\left[-1 + 4\sum_{j=1}^{n(n+1)/2} \frac{1 + 2i(x - x_j)}{d_j}\right], \tag{15}$$

where the denominators are $d_j = 1 + 4(x - x_j)^2 + 4(t - t_j)^2$.

There are $n(n + 1)/2$ terms which produce several components in the complete solution. (When $n = 3$, the number of peaks is 6, as shown in Fig. 4.) The denominator product then can be presented in the form,

$$\prod_{j=1}^{n(n+1)/2} d_j = d_1 d_2 \cdots d_{n(n+1)/2}.$$

This polynomial has order $n(n + 1)$, which equals the order of the denominator in the exact solution of order n [17]. Further steps can be carried out in analogy with the calculations above.

In conclusion, we have found that the second order rational solution of the NLSE splits into three, rather than two, Peregrine solutions when the offsets are non-zero. The individual components are located on a circle with 120 degrees angular separation between the peaks. There is no way to construct an "intermediate" solution which would contain only two Peregrine solutions. Thus, from a physical point of view, we still encounter a mystery in the way higher order rogue waves are formed.

Acknowledgements

The authors acknowledge the support of the Australian Research Council (Discovery Project number DP110102068). N.A. and A.A. acknowledge the support from the VolkswagenStiftung. N.A. is a recipient of the Alexander von Humboldt Award.

References

[1] E. Pelinovsky, C. Kharif (Eds.), Extreme Ocean Waves, Springer, Berlin, 2008.
[2] A.R. Osborne, Nonlinear Ocean Waves and the Inverse Scattering Transform, Elsevier, 2010.
[3] K.B. Dysthe, K. Trulsen, Physica Scripta T82 (1999) 48.
[4] C. Garrett, J. Gemmrich, Rogue waves, Physics Today, June, 62–63 (2009).
[5] D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Nature 450 (2007) 1054.
[6] M. Erkintalo, G. Genty, J.M. Dudley, Optics Lett. 34 (2009) 2468.
[7] V.B. Efimov, A.N. Ganshin, G.V. Kolmakov, P.V.E. McClintock, L.P. Mezhov-Deglin, Eur. Phys. J. Special Topics 185 (2010) 181.
[8] Yu.V. Bludov, V.V. Konotop, N. Akhmediev, Phys. Rev. A 80 (2009) 033610.
[9] M. Shats, H. Punzmann, H. Xia, Phys. Rev. Lett. 104 (2010) 104503.
[10] N. Akhmediev, E. Pelinovsky (Eds.), Rogue Waves — Towards a Unifying Concept, Special Issue of Eur. Phys. J. Special Topics 185 (2010).
[11] D.H. Peregrine, J. Australian Math. Soc. Ser. B 25 (1983) 16.
[12] V.I. Shrira, V.V. Geogjaev, J. Eng. Math. (2009), doi:10.1007/s10665-009-9347-2.
[13] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, J.M. Dudley, Nature Physics 6 (2010) 790.

[14] A. Chabchoub, N.P. Hoffmann, N. Akhmediev, Phys. Rev. Lett. 106 (2011) 204502.
[15] N. Akhmediev, V.M. Eleonskii, N.E. Kulagin, Sov. Phys. JETP 62 (1985) 894.
[16] N. Akhmediev, A. Ankiewicz, M. Taki, Phys. Lett. A 373 (2009) 675.
[17] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Phys. Rev. E 80 (2009) 026601.
[18] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, Eur. Phys. J. Special Topics 185 (2010) 247.
[19] P. Gaillard, Families of quasi-rational solutions of the NLS equation as an extension of higher order Peregrine breathers, preprint hal-00573955.
[20] A. Ankiewicz, P.A. Clarkson, N. Akhmediev, J. Physics A: Mathematical and Theoretical 43 (2010) 122002.
[21] N. Akhmediev, A. Ankiewicz, Solitons: Nonlinear Pulses and Beams, Chapman and Hall, London, 1997.
[22] Julius F. Wolff, Lake Superior Shipwrecks, Lake Superior Marine Museum Association, Inc., Duluth, MN, USA, 1979, ISBN 0932212-18-8, p. 28.