Chapter 9

WAVES IN COLD MAGNETIZED PLASMA

9.1 Introduction

For this treatment, we will regard the plasma as a cold magnetofluid with an associated dielectric constant. We then derive a wave equation using Maxwell's equations. Assuming a harmonic solution will then give a dispersion relation that describes all possible propoagating wave modes consistent with our assumptions. An alternative approach that is more tedious but which can offer greater physical insights is to solve the fluid equations of motion together with Maxwell's equations as we did for low frequency ion waves. In this chapter, we'll take the former approach, pointing out the important physics where appropriate.

9.2 The Wave Equation

We start with Maxwell's equations, which we write as

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{9.1}$$

$$\begin{aligned} \nabla \times \boldsymbol{B} &= \mu_0 \left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \\ &= \mathrm{i}\mu_0 \omega \varepsilon_0 \left(\frac{\boldsymbol{j}}{\mathrm{i}\omega\varepsilon_0} - \boldsymbol{E} \right) \\ &= -\mathrm{i}\mu_0 \omega \varepsilon_0 \left(\overleftrightarrow{I} + \frac{\mathrm{i}\overleftrightarrow{\sigma}}{\varepsilon_0 \omega} \right) \boldsymbol{E} \\ &= -\mathrm{i}\mu_0 \omega \overleftrightarrow{\epsilon}\, \boldsymbol{E} \end{aligned} \tag{9.2}$$

where we have used the relation $\boldsymbol{j} = \overleftrightarrow{\sigma}\ \boldsymbol{E}$ (Ohm's law for high frequency behaviour) and the definition of the dielectric tensor given by Eq. (5.30). The plasma physics is all contained in the dielectric tensor — see Chapters 4 and 5.

To generate the wave equation, we take the curl of Eq. (9.1) and substitute from Eq. (9.2):

$$\begin{aligned} \nabla\times\nabla\times\boldsymbol{E} &= -\frac{\partial}{\partial t}\nabla\times\boldsymbol{B} \\ &= \mu_0\omega^2\varepsilon_0\ \overleftrightarrow{K}\ .\boldsymbol{E} \\ &= \frac{\omega^2}{c^2}\ \overleftrightarrow{K}\ .\boldsymbol{E} \end{aligned} \tag{9.3}$$

where

$$\overleftrightarrow{K} \equiv \overleftrightarrow{\epsilon}\ /\varepsilon_0 = \text{relative susceptibility tensor}$$

For plane wave solutions, the wave equation gives

$$\mathrm{i}\boldsymbol{k}\times(\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}) - \frac{\omega^2}{c^2}\ \overleftrightarrow{K}\ .\boldsymbol{E} = 0 \tag{9.4}$$

or

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{E}) + \overleftrightarrow{K}\ .\boldsymbol{E} = 0 \tag{9.5}$$

where

$$\boldsymbol{n} = \frac{c}{\omega}\boldsymbol{k} \tag{9.6}$$

is the refractive index vector. Note that $n = |\boldsymbol{n}| = ck/\omega = c/v_\phi$ or

$$\frac{\omega}{k} \equiv v_\phi = \frac{c}{n}. \tag{9.7}$$

Since wave frequency doesn't change, this implies that $\lambda = \lambda_0/n$ where λ_0 is the free space wavelength.

9.3 The Dielectric Susceptibility Tensor

From Eq. (5.31) and Eq. (5.34) we can write the dielectric susceptibility tensor in the form

$$\overleftrightarrow{K} = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix} \tag{9.8}$$

where the reasons for using the nomenclature S, P, D as opposed to ϵ_1,ϵ_2 and ϵ_3 will be apparent later. The components are given explicitly by the formulae [see

Eq. (5.34)]

$$S = 1 + \frac{\mathrm{i}}{\omega\varepsilon_0}\sigma_\perp \tag{9.9}$$

$$D = \mp\frac{1}{\omega\varepsilon_0}\sigma_H \tag{9.10}$$

$$P = 1 + \frac{\mathrm{i}}{\omega\varepsilon_0}\sigma_0. \tag{9.11}$$

where the minus sign is for ions and positive for electrons and where we use the conductivity components for high frequency electric fields given by Eq. (5.27) with ν replaced by $-\mathrm{i}\omega$ as described in Sec. cond-tvf or directly using Eq. (4.53). Thus for S we obtain (for electrons)

$$\begin{aligned} S &= 1 + \frac{i}{\omega\varepsilon_0}\frac{\mathrm{i}ne^2}{m_\mathrm{e}\omega}\frac{\omega^2}{\omega^2-\omega_\mathrm{ce}^2} \\ &= 1 + \left(\frac{ne^2}{m_\mathrm{e}\varepsilon_0}\right)\frac{1}{\omega^2-\omega_\mathrm{ce}^2} \\ &= 1 - \frac{\omega_\mathrm{pe}^2}{\omega^2-\omega_\mathrm{ce}^2}. \end{aligned}$$

Including both ions and electrons gives

$$S = 1 - \sum_{\mathrm{i,e}}\frac{\omega_\mathrm{p}^2}{\omega^2-\omega_\mathrm{c}^2}. \tag{9.12}$$

In a similar way, we obtain

$$D = \sum_{\mathrm{i,e}} \pm\frac{\omega_\mathrm{p}^2\omega_\mathrm{c}}{\omega(\omega^2-\omega_\mathrm{c}^2)} \tag{9.13}$$

$$P = 1 - \sum_{\mathrm{i,e}}\frac{\omega_\mathrm{p}^2}{\omega^2} \tag{9.14}$$

where the plus sign is for ions and minus for electrons.

9.4 The Dispersion Relation

We have so far dealt with the second term of Eq. (9.5). Let us now simplify the first term and so develop a dispersion relation for waves characterized by the tensor $\overleftrightarrow{K}$. We assume the wave is propagating at an angle θ to the ambient magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{k}}$ and, without loss of generality, that the propagation vector lies in the x–z plane. The geometry is shown in Fig. 9.1.

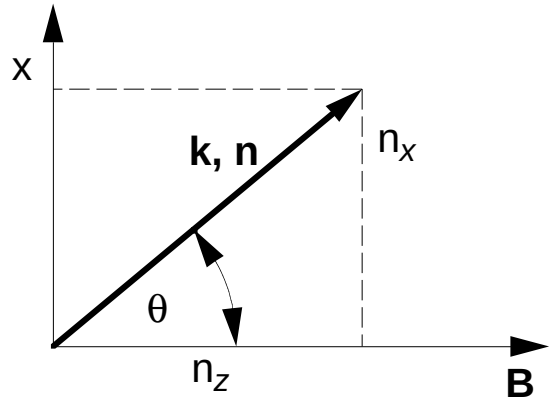

Figure 9.1: The geometry for analysis of plane waves in cold magnetized plasma.

With this setup, we obtain

$$\begin{aligned} \boldsymbol{n}\times\boldsymbol{E} &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ n_x & 0 & n_z \\ E_x & E_y & E_z \end{vmatrix} \\ &= \hat{\boldsymbol{i}}(-n_z E_y) - \hat{\boldsymbol{j}}(n_x E_z - n_z E_x) + \hat{\boldsymbol{k}}(n_x E_y) \end{aligned} \tag{9.15}$$

and

$$\begin{aligned} \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{E}) &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ n_x & 0 & n_z \\ -n_z E_y & n_z E_x - n_x E_z & n_x E_y \end{vmatrix} \\ &= \hat{\boldsymbol{i}} n_z(n_x E_z - n_z E_x) - \hat{\boldsymbol{j}}(n_x^2 E_y + n_z^2 E_y) + \hat{\boldsymbol{k}} n_x(n_z E_x - n_x E_z) \\ &= \begin{pmatrix} -n_z^2 & 0 & n_x n_z \\ 0 & -(n_x^2 + n_z^2) & 0 \\ n_x n_z & 0 & -n_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \end{aligned} \tag{9.16}$$

Combining Eq. (9.8) and Eq. (9.16) finally allows Eq. (9.5) to be expressed in matrix form:

$$\begin{pmatrix} S - n_z^2 & -\mathrm{i}D & n_x n_z \\ \mathrm{i}D & S - n_x^2 - n_z^2 & 0 \\ n_x n_z & 0 & P - n_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{9.17}$$

This represents a set of three simultaneous equations for the components of $\boldsymbol{E}$ It has a non-trivial solution ($\boldsymbol{E} \neq 0$) if and only if the determinant vanishes. In evaluating this determinant, we let $n_x = n\sin\theta$, $n_z = n\cos\theta$ and $n_x^2 + n_z^2 = n^2$ to obtain the dispersion relation for waves propagating in a cold, magnetized,

collisionless plasma (in the sense that the propagation frequencies greatly exceed colision frequencies):

$$(S-n^2\cos^2\theta)(S-n^2)(P-n^2\sin^2\theta)-D^2(P-n^2\sin^2\theta)-n^4\sin^2\theta\cos^2\theta(S-n^2)=0. \tag{9.18}$$

We now make the following definitions

$$\begin{aligned} R &= S+D \quad \text{Right} && (9.19)\\ L &= S-D \quad \text{Left} && (9.20)\\ S &= (R+L)/2 \quad \text{Sum} && (9.21)\\ D &= (R-L)/2 \quad \text{Difference} && (9.22)\\ P &\quad \text{Plasma} && (9.23) \end{aligned}$$

To solve Eq. (9.18) we first eliminate cosine terms in favour of sine terms and isolate $\sin^2\theta$ on the left side. Similarly we express the result with $\cos^2\theta$ as the argument of the equation. We then divide the two expressions to finally obtain the cold wave dispersion relation (show this)

$$\tan^2\theta = \frac{P(n^2-L)(n^2-R)}{(n^2-P)(RL-n^2S)} \tag{9.24}$$

9.5 Propagation Parallel to $\boldsymbol{B}$

In this case, $\theta = 0$ and we immediately obtain the two separate dispersion relations $n^2 = L$ and $n^2 = R$. To investigate the nature of the wave (i.e. its polarization properties), we see that Eq. (9.17) gives (using $n_x = 0$)

$$\begin{aligned} (S-n^2)E_x - \mathrm{i}DE_y &= 0\\ \mathrm{i}DE_x + (S-n^2)E_y &= 0\\ PE_z &= 0. \end{aligned} \tag{9.25}$$

Since $P = 0$ represents a simple plasma oscillation and not a wave motion, we have $E_z = 0$. In other words, the propagating mode must be transverse electric. The ratio of E_x and E_y is obtained from the first two equations as

$$\begin{aligned} \frac{\mathrm{i}E_x}{E_y} &= \frac{n^2-S}{D} = \frac{D}{n^2-S}\\ \Rightarrow (n^2-S)^2 &= D^2\\ \Rightarrow (n^2-S) &= \pm D \end{aligned} \tag{9.26}$$

or $n^2 = S \pm D = R$ or L as before. Thus

$$\frac{\mathrm{i}E_x}{E_y} = \pm 1 \tag{9.27}$$

so that E_x and E_y are ninety degrees out of phase and the R and L waves are respectively right and left hand circularly polarized. These polarizations match the cyclotron orbits of the charged plasma particles.

For concreteness, let us consider the left hand wave.

$$n^2 = \frac{c^2}{v_\phi^2} = L = 1 - \sum_{\rm i,e} \frac{\omega_{\rm p}^2}{\omega(\omega \mp \omega_{\rm c})} \tag{9.28}$$

where the minus sign is for ions and the positive for electrons. The signs reverse for the right handed wave. For a single ion species we obtain

$$\frac{c^2}{v_\phi^2} = 1 - \frac{\omega_{\rm pe}^2}{\omega(\omega + \omega_{\rm ce})} - \frac{\omega_{\rm pi}^2}{\omega(\omega - \omega_{\rm ci})}. \tag{9.29}$$

9.5.1 Resonances, cutoffs and limiting behaviour

A wave resonance occurs when the phase velocity vanishes (its wavenumber becomes infinite). At a wave resonance, energy is deposited by the wave and the wave is absorbed. For the left hand wave this occurs at the ion cyclotron frquency $\omega_{\rm ci}$. For the right hand wave, the resonance is at $\omega_{\rm ce}$.

A wave cutoff occurs when the phase velocity becomes infinite (its wavenumber vanishes). At a wave cutoff, the incident wave is reflected from the plasma. Now $v_\phi = c/n \to \infty \Rightarrow n \to 0$. The behaviour of the wave field near a cutoff or resonance is shown in Fig. 9.2

From Eq. (9.24), $n = 0 \Rightarrow L = 0$, $R = 0$. The cutoff frequencies for the left and right waves are then given by (ignoring ion motions)

$$\begin{aligned} \omega_{0L} &= \left[-\omega_{\rm ce} + (\omega_{\rm ce}^2 + 4\omega_{\rm pe}^2)^{1/2}\right]/2 \\ &< \omega_{\rm ce} \qquad (\omega_{\rm pe} \gtrsim \omega_{\rm ce} \text{ typically}) \qquad (9.30) \\ \omega_{0R} &= \left[\omega_{\rm ce} + (\omega_{\rm ce}^2 + 4\omega_{\rm pe}^2)^{1/2}\right]/2 \qquad (9.31) \\ &= \omega_{0L} + \omega_{\rm ce} \\ &> \omega_{\rm ce}. \qquad (9.32) \end{aligned}$$

From the dispersion relation for the left hand wave we obtain

$$\begin{aligned} v_{\phi L} &= \left(\frac{\omega}{k}\right)_L = \frac{c(1+\omega_{\rm ce}/\omega)^{1/2}}{(1+\omega_{\rm ce}/\omega - \omega_{\rm pe}^2/\omega^2)^{1/2}} \qquad (9.33) \\ &= \text{real} \Rightarrow \text{propagating} \qquad \text{for } \omega > \omega_{0L} \\ &= \text{imaginary} \Rightarrow \text{evanescent} \qquad \text{for } \omega < \omega_{0L} \end{aligned}$$

Had we retained ion motions we would have found $v_{\phi L}$ to be real also in the frequency range $\omega < \omega_{\rm ci} (< \omega_{0L})$.

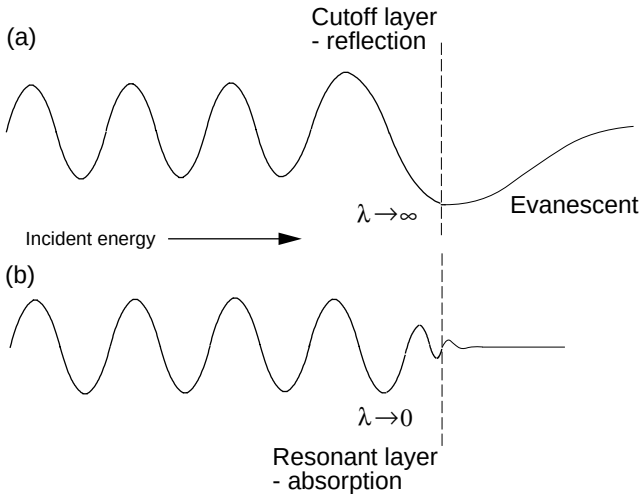

Figure 9.2: (a) Near a cutoff, the wave field swells, the wavelength increases and the wave is ultimately reflected. (b) near a resonance, the wavefield diminishes, the wavelength decreases and the wave enrgy is absorbed.

For the right hand wave,

$$v_{\phi R} = \left(\frac{\omega}{k}\right)_R = \frac{c(1-\omega_{\rm ce}/\omega)^{1/2}}{(1-\omega_{\rm ce}/\omega-\omega_{\rm pe}^2/\omega^2)^{1/2}} \tag{9.34}$$

This is real for $\omega > \omega_{0R}$ and $\omega < \omega_{\rm ce}$.

At low frequencies, the left hand and right hand waves merge to become the torsional Alfvén wave propagating along $\boldsymbol{B}$ at phase velocity V_A. This behaviour is shown in Fig. 9.3 which plots the phase velocity versus frequency for waves propagating parallel to the field.

At high frequencies $\omega \gg \omega_{\rm ce}$ Eq. (9.29)

$$\frac{c^2}{v_\phi^2} = 1 - \frac{\omega_{\rm pe}^2}{\omega^2}. \tag{9.35}$$

This is the dispersion relation for an electromagnetic wave in an unmagnetized plasma (as might be expected). Note that the phase velocity is greater than c for this wave. When the wave frequency is much greater than both magnetic and plasma frequencies, we obtain $v_\phi = c$ and the wave is an electromagnetic light wave that is insensitive to the presence of the conducting plasma.

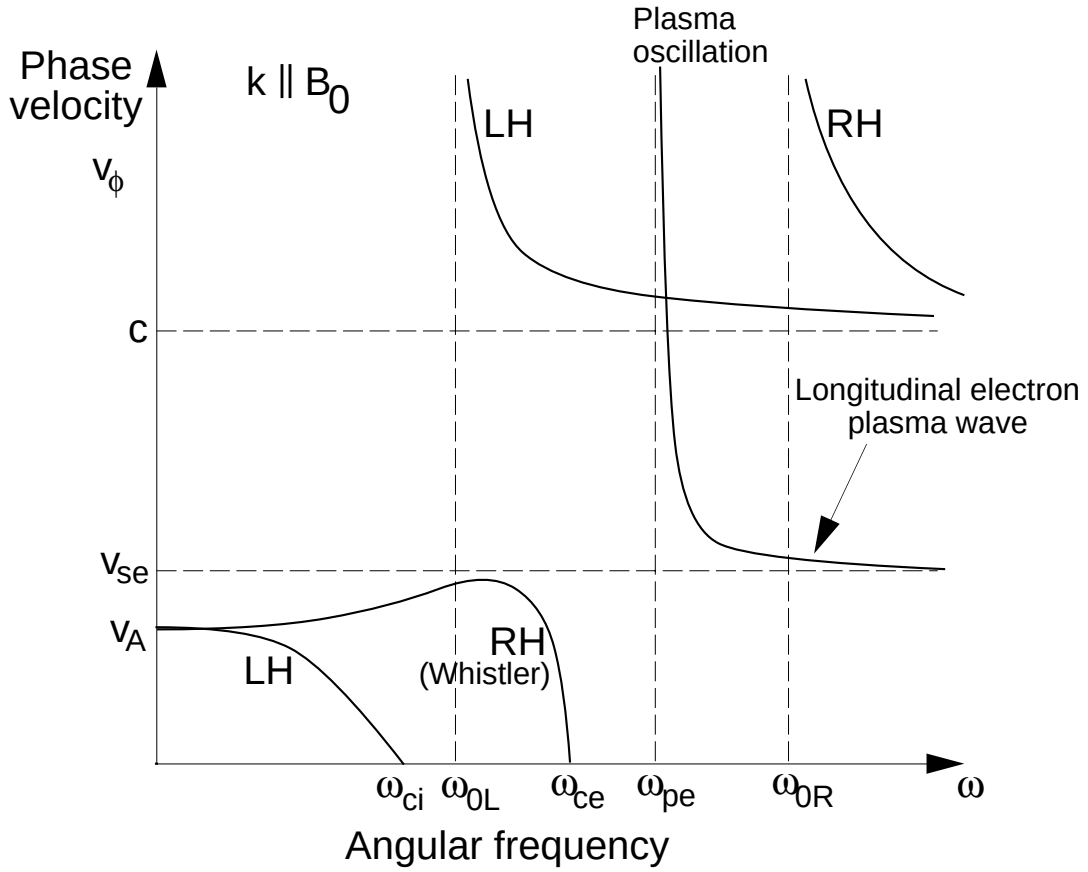

Figure 9.3: A plot of wave phase velocity versus frequency for waves propagating parallel to the magnetic field for a cold plasma.

9.5.2 Some examples

The Whistler wave

For $\omega < \omega_{\rm ce}$, the phase velocity for the right handed wave increases with frequency. It can also be shown that this is so for the group velocity v_g. A lightning strike in the atmosphere produces an electromagnetic pulse that can excite a broad spectrum of plasma waves in the ionisphere. Because of the dispersion, the higher frequency waves (10-15 kHz) will be guided by the earth's magnetic field to an observer more quickly than the lower frequency components resulting in a tone that descends by ∼10 kHz in a matter of seconds. The characteristic note was often heard by early investigators of ionispheric emissions.

Faraday rotation

Above ω_{0R} both left and right handed waves propagate, but at different phase velocities along the magnetic field. After crossing the plasma, the phase of the left hand wave has increased more than the right because of its lower phase velocity (it takes longer to get there). The resulting wave emerging from the plasma will have had its plane of polarization rotated as shown in Fig. 9.4. The total rotation angle ψ van be used to measure the internal magnetic field:

$$\psi \propto \int_0^L \mathrm{d}\ell \, n_\mathrm{e} B_\parallel \mathrm{d}\ell \tag{9.36}$$

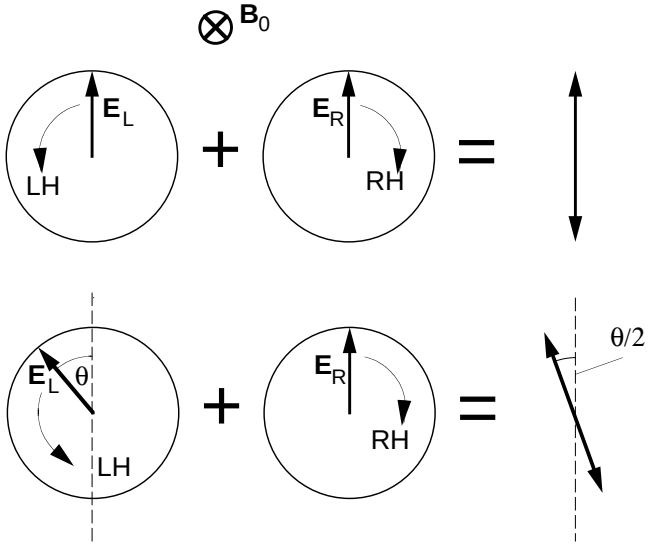

Figure 9.4: The principle of Faraday rotation for an initially plane polarized wave propagating parallel to the magnetic field.

9.6 Propagation Perpendicular to B

In this case, the solutions to Eq. (9.24) are simply

$$\begin{aligned} n^2 &= P \\ n^2 &= RL/S. \end{aligned} \tag{9.37}$$

We examine the two solutions in turn.

9.6.1 The ordinary wave

For this wave, we have $n^2 = P$ which reads

$$\begin{aligned} n^2 = \frac{c^2}{v_\phi^2} &= 1 - \sum_{\mathrm{i,e}} \frac{\omega_\mathrm{p}^2}{\omega^2} \\ &= 1 - \frac{\omega_\mathrm{pe}^2}{\omega^2} - \frac{\omega_\mathrm{pi}^2}{\omega^2} \\ &\approx 1 - \frac{\omega_\mathrm{pe}^2}{\omega^2}. \end{aligned} \tag{9.38}$$

Because the propagation vector is perpendicular to $\boldsymbol{B}$, we have $n_x = n$ and $n_z = 0$ and Eq. (9.17) becomes

$$\begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S - n^2 & 0 \\ 0 & 0 & P - n^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \tag{9.39}$$

For the ordinary mode, $(P - n^2)E_z = 0$ requiring $E_z \neq 0$ for a non-trivial solution. The wave field E_z is parallel to $\boldsymbol{B}$ so that electron motions are unimpeded by the magnetic Lorentz force. This dispersion relation is thus the same as for an unmagnetized plasma. The alignment of the wave fields are shown in Fig. 9.5.

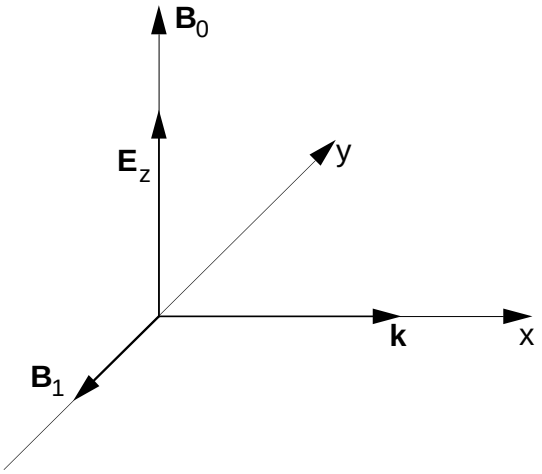

Figure 9.5: The ordinary wave is a transverse electromagnetic wave having its electric vector parallel to $\boldsymbol{B}$.

9.6.2 The extraordinary wave

For the extraordinary wave, Eq. (9.39) gives

$$SE_x - \mathrm{i}DE_y = 0$$

$$\begin{aligned} \mathrm{i}DE_x + (S - n^2)E_y &= 0 \\ \Rightarrow \frac{E_x}{E_y} &= \frac{\mathrm{i}D}{S}. \end{aligned} \tag{9.40}$$

This wave is elliptically polarized with an electric field component in the direction of $\boldsymbol{k}$. Since we have a component $E_x \parallel \boldsymbol{k}$, this wave is partly electrostatic and partly electromagnetic. The nature of the wave field is displayed in Fig. 9.6

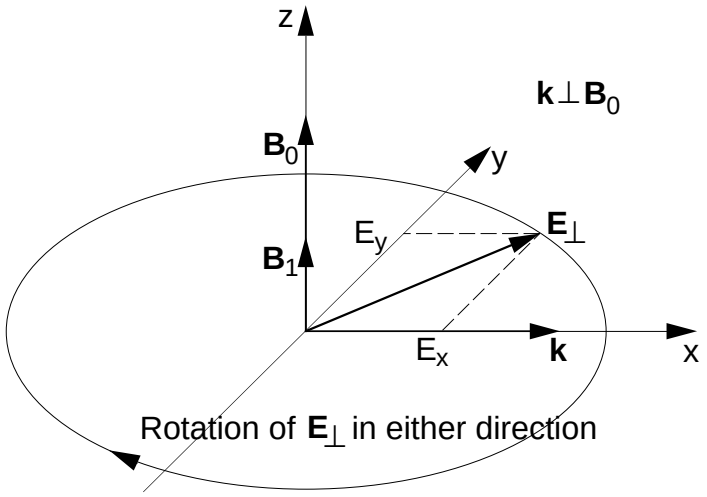

Figure 9.6: The relationship between the propagation vector, magnetic field and wave components for the extraordinary wave. The wave exhibits an electric field in the direction of motion and so is partly electrostatic in character.

9.6.3 Resonances, cutoffs and limiting behaviour

It is clear that the ordinary wave is cutoff ($n^2 = 0$) for frequencies $\omega < \omega_{\mathrm{pe}}$. At these frequencies, electrons are free to move along the magnetic lines of force to shield out the electric field of the incident wave which is then refelcted.

The extraordinary wave exhibits cutoffs at $R = 0$ ($\omega = \omega_{0R}$) and $L = 0$ ($\omega = \omega_{0L}$). It also shows a resonance at $S = 0$ ($n \to \infty$). That is, when

$$S = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2 - \omega_{\mathrm{ce}}^2} - \frac{\omega_{\mathrm{pi}}^2}{\omega^2 - \omega_{\mathrm{ci}}^2} = 0. \tag{9.41}$$

For $\omega_{\mathrm{pi}} \ll \omega_{\mathrm{pe}}$ we see that the resonance occurs at the so-called *upper-hybrid frequency*

$$\omega_{UH} = (\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{ce}}^2)^{1/2}. \tag{9.42}$$

Had ion motions been included we would have found another resonance at a lower frequency, the *lower hybrid frequency*

$$\omega_{LH} \approx (\omega_{\rm ci}\omega_{\rm ce})^{1/2}. \tag{9.43}$$

To highlight the role of the various resonances etc., the explicit dispersion relation for the extraordinary wave can be written as (no ion motions)

$$\begin{aligned} n^2 = \frac{c^2}{v_\phi^2} &= \frac{(\omega^2-\omega_{0L}^2)(\omega^2-\omega_{0R}^2)}{\omega^2(\omega^2-\omega_{UH}^2)} \qquad (9.44)\\ &= \text{real for} \quad \omega > \omega_{0R} \\ &= \text{real for} \quad \omega_{0L} < \omega < \omega_{UH} \qquad (9.45) \end{aligned}$$

Figure 9.7 shows the various branches for waves propagating perpendicular to the magnetic field. Oscillations at almost constant frequency occuring at $\omega_{UH} = (\omega_{\rm pe}^2 + \omega_{\rm ce}^2)^{1/2}$ are the equivalent of plasma oscillations at $\omega_{\rm pe}$ in the absence of a magnetic field. The magnetic field gives an additional restoring force to the usual electric force and hence an increase in the oscillation frequency results. Of course $\omega \to \omega_{\rm pe}$ as the B-field vanishes.

Problems

Problem 9.1 Consider electromagnetic wave propagation parallel to the magnetic field. Show that the group velocities for the left and right hand circularly polarized waves are given by

$$v_{gL} = \left(\frac{\partial\omega}{\partial k}\right)_L = \frac{2c(\omega+\omega_{\rm ce})^{3/2}[\omega(\omega^2+\omega\omega_{\rm ce}-\omega_{\rm pe}^2)]^{1/2}}{2\omega(\omega+\omega_{\rm ce})^2-\omega_{\rm ce}\omega_{\rm pe}^2}$$

and

$$v_{gR} = \left(\frac{\partial\omega}{\partial k}\right)_R = \frac{2c(\omega-\omega_{\rm ce})^{3/2}[\omega(\omega^2+\omega\omega_{\rm ce}-\omega_{\rm pe}^2)]^{1/2}}{2\omega(\omega-\omega_{\rm ce})^2+\omega_{\rm ce}\omega_{\rm pe}^2}$$

Make a plot of phase velocity and group velocity for the right hand wave as a function of frequency, showing resonance and cutoff frequencies. Show that the group velocity vanishes at these frequencies.

Problem 9.2 Show that the Faraday rotation angle, in degrees, of a linearly polarized transverse wave propagating along B_0 is given by

$$\theta = 1.5\times10^{-11}\lambda_0^2\int_0^L B_0(z)n_0(z)\,{\rm d}z$$

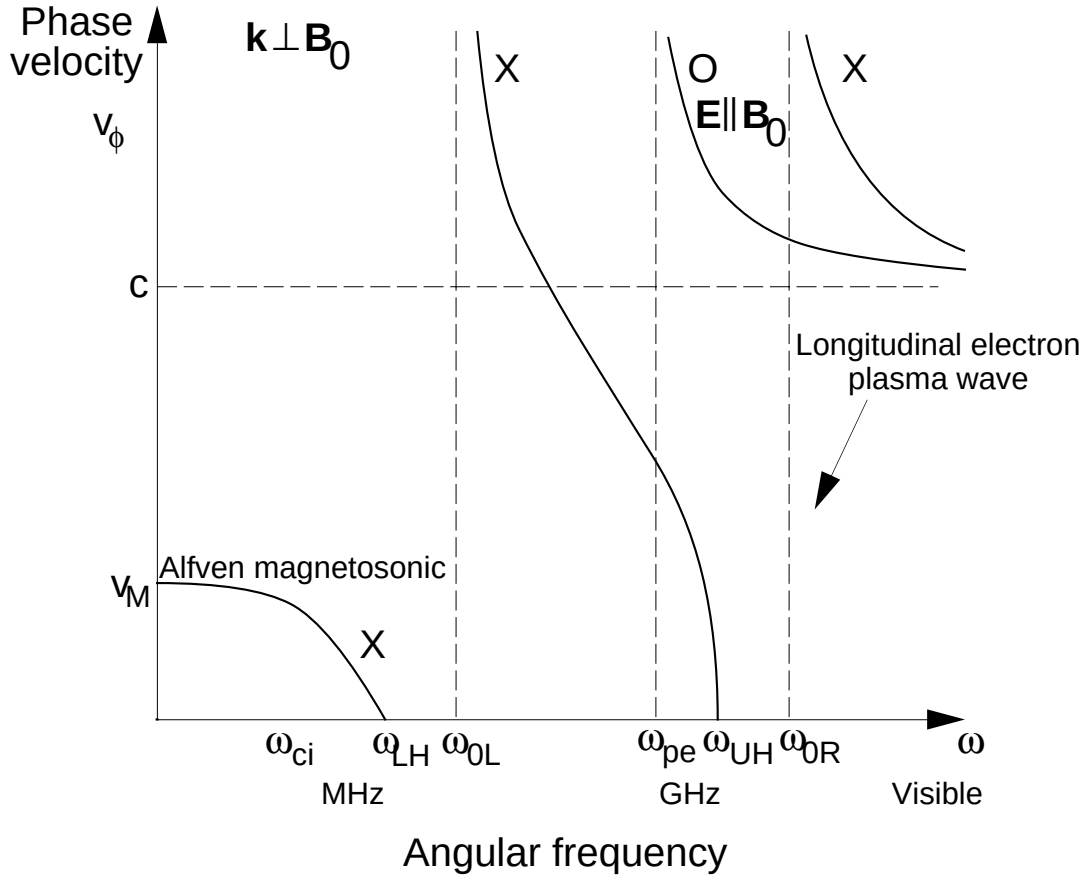

Figure 9.7: A plot of wave phase velocity versus frequency for waves propagating perpendicular to the magnetic field for a cold plasma.

where λ_0 is the free-space wavelength, n_0 is the plasma density and L is the path length in the plasma. Assume $\omega^2 \gg \omega_{\rm p}^2, \omega_{\rm c}^2$.

Problem 9.3 Derive Eq. (9.44).