Chapter 8

WAVES IN ISOTROPIC PLASMA

8.1 Introduction

In this chapter, we use the fluid equations and Maxwell's equations to study wave propagation in cold and "warm" isotropic (unmagnetized) plasmas. For "cold" plasmas we simply ignore the restoring forces due to plasma particle kinetic energy (pressure). This is valid provided the wave phase veloicty is much greater than the average particle thermal velocity. It is often a good approximation, though as we have seen, pressure is important for low phase-velocity MHD waves.

8.2 Basic Equations

For each component of the plasma we have Maxwell's equations and the fluid equations

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\boldsymbol{u}) = 0 \tag{8.1}$$

$$m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - m\nu\boldsymbol{u}. \tag{8.2}$$

As in the previous chapter, we linearize about our uniform and time-independent background and assume harmonic plane wave solutions. Series expansion about equilibrium values n_0, $\boldsymbol{u}_0$, $\boldsymbol{E}_0$, $\boldsymbol{B}_0$ and keep only first order perturbations $\boldsymbol{u}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ etc. The perturbations are the wave-related quantities. Since we are assuming the plasma is unmagnetized, a considerable simplification is obtained by setting $\boldsymbol{B} = 0$ so that

$$n = n_0 + n_1 \quad \boldsymbol{B} = \boldsymbol{B}_1 \quad \boldsymbol{E} = \boldsymbol{E}_1 \quad \boldsymbol{u} = \boldsymbol{u}_1. \tag{8.3}$$

The three important equations for a cold unmagnetized electron gas (immobile ions) become (show this)

$$-\mathrm{i}\omega m_\mathrm{e}\boldsymbol{u} = -e\boldsymbol{E} - m_\mathrm{e}\nu\boldsymbol{u} \tag{8.4}$$

$$\boldsymbol{k}\times\boldsymbol{E} = \omega\boldsymbol{B}_1 \tag{8.5}$$

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{B}_1 = \mu_0(-en_0\boldsymbol{u} - \mathrm{i}\omega\varepsilon_0)\boldsymbol{E}) \tag{8.6}$$

The equation of state is unimportant because we have ignored pressure effects.

8.3 Isotropic electron gas

Equation (8.4) immediately gives

$$\boldsymbol{u} = \frac{e}{m(\nu - \mathrm{i}\omega)}\boldsymbol{E}. \tag{8.7}$$

Combining with Eq. (8.5) and Eq. (8.6) we find

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) = -\frac{\mathrm{i}\omega\mu_0 e^2 n_0}{m_\mathrm{e}(\nu - \mathrm{i}\omega)}\boldsymbol{E} - \frac{\omega^2}{c^2}\boldsymbol{E}. \tag{8.8}$$

We now separate $\boldsymbol{E}$ into its "longitudinal" component parallel to $\boldsymbol{k}$ (the direction of propagation) and the "transverse" component in the plane normal to $\boldsymbol{k}$:

$$\boldsymbol{E} = \boldsymbol{E}_l + \boldsymbol{E}_t \tag{8.9}$$

This decomposition in shown in Fig. 8.1. From the figure, we have

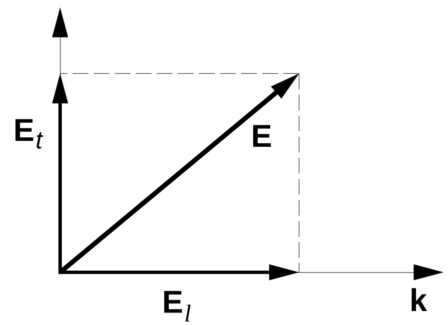

Figure 8.1: The longitudinal and transverse electric field perturbations for waves in a cold electron plasma are decoupled

$$\boldsymbol{k}\times\boldsymbol{E}_l = 0 \tag{8.10}$$

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_t) = -k^2\boldsymbol{E}_t \tag{8.11}$$

and Eq. (8.8) becomes

$$-k^2\boldsymbol{E}_t = -\left[\frac{\mathrm{i}\omega\mu_0 e^2 n_0}{m_\mathrm{e}(\nu - \mathrm{i}\omega)} + \frac{\omega^2}{c^2}\right](\boldsymbol{E}_l + \boldsymbol{E}_t) \tag{8.12}$$

which separates into an equation for the longitudinal component

$$\left[\frac{\omega_\mathrm{pe}^2}{c^2(1+\mathrm{i}\nu/\omega)} - \frac{\omega^2}{c^2}\right]\boldsymbol{E}_l = 0 \tag{8.13}$$

and the transverse component

$$-k^2\boldsymbol{E}_t = \left[\frac{\omega_\mathrm{pe}^2}{c^2(1+\mathrm{i}\nu/\omega)} - \frac{\omega^2}{c^2}\right]\boldsymbol{E}_t. \tag{8.14}$$

In the *absence* of collisions, Eq. (8.13) gives

$$\omega^2 = \omega_\mathrm{pe}^2 \tag{8.15}$$

which is the expression describing electron plasma oscillations encountered in earlier chapters. As we have seen, these oscillations are elctrostatic in character and non-propagating, since ω is independent of k.

For the transverse component, Eq. (8.14) reduces to

$$v_\phi = \frac{\omega}{k} = \frac{c}{(1-\omega_\mathrm{pe}^2/\omega^2)^{1/2}}. \tag{8.16}$$

When $\omega < \omega_\mathrm{pe}$, v_ϕ is imaginary and this wave is evanescent (transports no energy). For $\omega > \omega_\mathrm{pe}$, $v_\phi > c$ and the wave group velocity is

$$v_g = \frac{\partial\omega}{\partial k} = \frac{c^2}{v_\phi} \tag{8.17}$$

and we find

$$v_\phi v_g = c^2. \tag{8.18}$$

As seen in Fig. 8.2 the transverse wave is dispersive, and for $\omega \gg \omega_\mathrm{ce}$ $v_\phi \to c$. At such high frequencies, even electrons cannot respond and the plasma appears transparent.

8.3.1 Poynting flux

Closely related to the concept of group velocity is the *Poynting flux* which is the power carried by an electromagnetic wave (the Poynting vector) averaged over a wave cycle:

$$\langle\boldsymbol{S}\rangle \equiv \frac{1}{2\mu_0}\Re(\boldsymbol{E}\times\boldsymbol{B}^*). \tag{8.19}$$

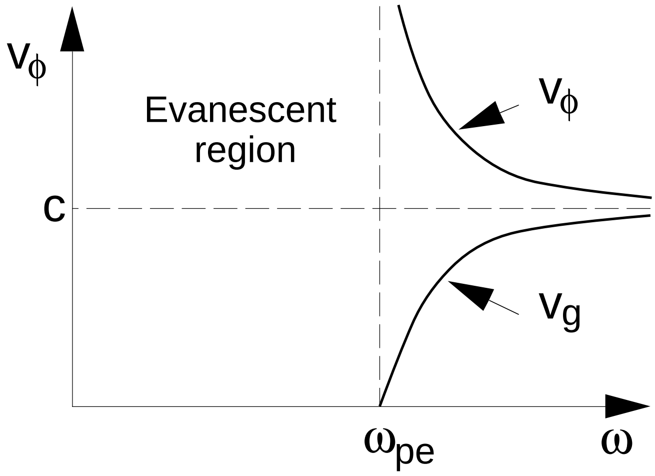

Figure 8.2: Phase velocity versus oscillation frequency for the transverse electron plasma wave. Note reciprocal behaviour of v_g and v_ϕ and the region of nonpropagation.

The transverse wave has magnetic component [see Eq. (8.5)]

$$\boldsymbol{B}_1 = (\boldsymbol{k}\times\boldsymbol{E})/\omega \tag{8.20}$$

where $\boldsymbol{E} \equiv \boldsymbol{E}_1$ so that

$$\begin{aligned}\langle\boldsymbol{S}\rangle &= \frac{1}{2\mu_0\omega}\Re[\boldsymbol{E}\times(\boldsymbol{k}^*\times\boldsymbol{E}^*)] \\ &= \hat{\boldsymbol{n}}\frac{1}{2\mu_0\omega}\Re[\boldsymbol{k}^* E(\boldsymbol{r},t)E^*(\boldsymbol{r},t)] \\ &= \hat{\boldsymbol{n}}\frac{E^2}{2\mu_0\omega}\Re\{\boldsymbol{k}^*\exp[\mathrm{i}(k-k^*)z]\}\end{aligned} \tag{8.21}$$

where $\hat{\boldsymbol{n}}$ is the unit vector in the direction $\boldsymbol{E}\times\boldsymbol{B}$ and, for transverse electron plasma waves, k is real depending on $\omega > \omega_{\rm pe}$ or $\omega < \omega_{\rm pe}$.

It follows that

$$\begin{aligned}\langle\boldsymbol{S}\rangle &= 0 \qquad \omega < \omega_{\rm pe} \\ &= \hat{\boldsymbol{n}}\frac{1}{2}\varepsilon_0 E^2 v_g \qquad \omega > \omega_{\rm pe}\end{aligned} \tag{8.22}$$

and power is transported at v_g.

At $\omega = \omega_{\rm pe}$, $v_\phi \to \infty$, $\lambda = 2\pi/k \to \infty$ and the index of refraction $n = ck/\omega \to 0$. This is called a cutoff. An electromagnetic wave launched into a plasma that encounters a cutoff (for example due to a changing electron density profile) is reflected. This fact can be used to measure the plasma refractive index profile using a technique known as *reflectometry*.

Example

Consider the ionisphere: $n_e \sim 1 \times 10^{11}\,\mathrm{m}^{-3}$, $\omega_{pe}/2\pi = 3$ MHz. Only electromagnetic waves with frequency greater than 3MHz can penetrate the ionisphere (hence we can see the stars). Radio waves can travel around the world, making successive bounces from the ionispheric layer (but not TV). For typical lab plasmas $n_e \sim 1 \times 10^{19}\,\mathrm{m}^{-3}$, $\omega_{pe}/2\pi = 28$ GHz.

8.3.2 Effect of collisons

If collisons are included, the longitudinal dispersion becomes

$$\omega^2 + i\nu\omega + \omega_{pe}^2 = 0 \tag{8.23}$$

or

$$\omega = [-i\nu \pm (4\omega_{pe}^2 - \nu^2)^{1/2}]/2. \tag{8.24}$$

For any ν, the imaginary part of $\omega = \omega_r + i\omega_i$ is negative so that the wave is exponentially damped.

For the transverse wave, the dispersion relation becomes

$$k^2c^2 = \omega^2 - \frac{\omega_{pe}^2 + i\omega_{pe}^2(\nu/\omega)}{1 + (\nu/\omega)^2}. \tag{8.25}$$

It is straightforward to show that for this wave, the imaginary part of k is positive (see Fig. 8.3) and the wave is damped.

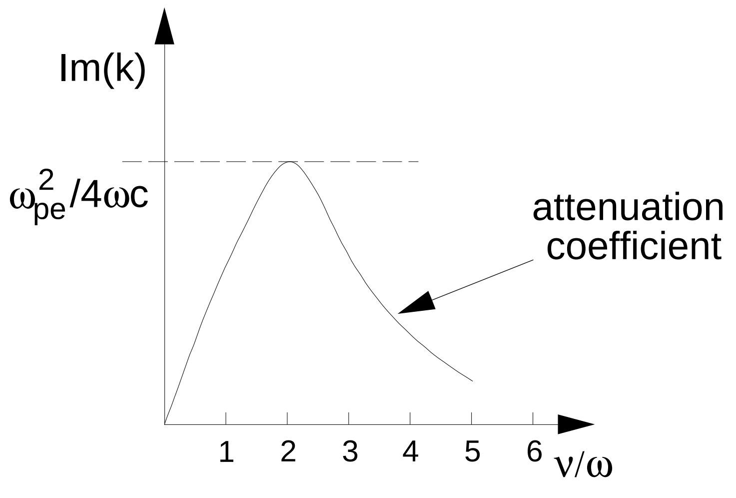

Figure 8.3: The form of the complex wavenumber for transverse electron plasma waves.

8.4 Inclusion of Plasma Pressure

It is very instructive to study the modifications to the electron plasma wave dispersion when finite pressure effects are included. The linearized electron equation of motion now becomes

$$n_0 m_{\mathrm{e}} \frac{\partial \boldsymbol{u}}{\partial t} = -n_0 e \boldsymbol{E} - \nabla p \tag{8.26}$$

where we now ignore collisions and use the equation of state to link the pressure and the density:

$$\nabla p = \frac{\gamma p}{\rho} \nabla \rho. \tag{8.27}$$

Since $v_{\mathrm{th}} \ll \omega/k$, the electrons cannot dissipate energy through collisions and the wave motion is adiabatic. In one dimension $N = 1$, the ratio of specific heats becomes $\gamma = (2+N)/N = 3$ so that

$$\nabla p = \frac{\gamma n k_{\mathrm{B}} T_{\mathrm{e}}}{n m_{\mathrm{e}}} \nabla (n m_{\mathrm{e}}) = 3 k_{\mathrm{B}} T_{\mathrm{e}} \nabla n_1 \tag{8.28}$$

where we have used $\nabla n_0 = 0$ (uniform background).

Once again, assuming harmonic plane wave propagation, Eq. (8.26) becomes

$$-\mathrm{i}\omega n_0 m_{\mathrm{e}} u = -n_0 e E - \mathrm{i} 3 k_{\mathrm{B}} T_{\mathrm{e}} k n_1$$

where vector notation is no longer required (one dimensional wave). By restricting attention to one dimension, this treatment looks only at the longitudinal electron plasma wave.

To proceed further, we can eliminate the wave electric field E using Poisson's equation

$$\nabla . \boldsymbol{E} = \frac{e(n_{\mathrm{i}} - n_{\mathrm{e}})}{\varepsilon_0} \;\rightarrow\; \mathrm{i}kE = -\frac{e n_1}{\varepsilon_0} \tag{8.29}$$

to obtain

$$-\mathrm{i}\omega n_0 m_{\mathrm{e}} u = -n_0 e \left(-\frac{e n_1}{\varepsilon_0 \mathrm{i} k} \right) - \mathrm{i} 3 k_{\mathrm{B}} T_{\mathrm{e}} k n_1.$$

The linearized continuity equation allows to eliminate n_1

$$\frac{\partial n_1}{\partial t} + n_0 \nabla . \boldsymbol{u} = 0 \;\rightarrow\; \mathrm{i}\omega n_1 = \mathrm{i} k n_0 u \tag{8.30}$$

with the result

$$-\mathrm{i}\omega n_0 m_{\mathrm{e}} u = \left(\frac{n_0 e^2}{\mathrm{i} k \varepsilon_0} - \mathrm{i} k 3 k_{\mathrm{B}} T_{\mathrm{e}} \right) \frac{k n_0 u}{\omega}$$

which can be solved for ω^2:

$$\omega^2 = \omega_{\mathrm{pe}}^2 + \left(\frac{3 k_{\mathrm{B}} T_{\mathrm{e}}}{m_{\mathrm{e}}} \right) k^2. \tag{8.31}$$

This is just the Bohm-Gross dispersion relation that we encountered in Sec. 2.7 We can express this in an alternative way as

$$\begin{aligned} \omega^2 &= \omega_{\rm pe}^2 + k^2 V_{S\rm e}^2 \\ V_{S\rm e} &\equiv \left(\frac{3k_{\rm B}T_{\rm e}}{m_{\rm e}}\right)^{1/2} \end{aligned} \tag{8.32}$$

and $V_{S\rm e}$ is the electron sound speed.

The longitudinal electron plasma wave is now a propagating wave with group velocity

$$v_g = \frac{\partial \omega}{\partial k} = \frac{k}{\omega} V_{S\rm e}^2 \tag{8.33}$$

and

$$v_g v_\phi = V_{S\rm e}^2. \tag{8.34}$$

Had we included ion moions, we would need to replace $\omega_{\rm pe}^2$ in Eq. (8.31) with $\omega_{\rm pe}^2 + \omega_{\rm pi}^2$. We would also find that the corresponding ion longitudinal oscillation would have the same phase velocity as the ion acoustic wave obtained using the MHD equations. At very low frequencies, the ion acoustic wave has phase velocity $v_\phi \equiv V_{Sp}$ where the *plasma sound speed* is given by

$$V_{Sp} = \left(\frac{\gamma_{\rm e} k_{\rm B} T_{\rm e} + \gamma_{\rm i} k_{\rm B} T_{\rm i}}{m_{\rm i}}\right)^{1/2}. \tag{8.35}$$

Usually one takes $\gamma_{\rm e} = 1$ (very low frequency - isothermal electrons) and $\gamma_{\rm i} = 3$ (adiabatic ions).

The combined dispersion relations for the electron and ion acoustic wave modes in a warm isotropic (unmagnetized) plasma are shown in Fig. 8.4. Their properties are summarized in the table below.

	Low k	High k
electron acoustic	Constant frequency $\omega_{\rm pe}$	constant velocity $V_{S\rm e}$
ion acoustic	Constant velocity $V_{S\rm i}$	constant frequency $\omega_{\rm pi}$

Problems

Problem 8.1 An electromagnetic wave of angular frequency ω_0 passing through a cold isotropic plasma having $\omega_{\rm pe} \ll \omega_0$ has wavelength $\lambda = 2\pi/k$ in the plasma greater than the free space wavelength $\lambda_0 = 2\pi/k_0 = 2\pi c/\omega_0$. Assume that the wave propagates in the x direction in the plasma and that $k(x)$ is slowly varying in

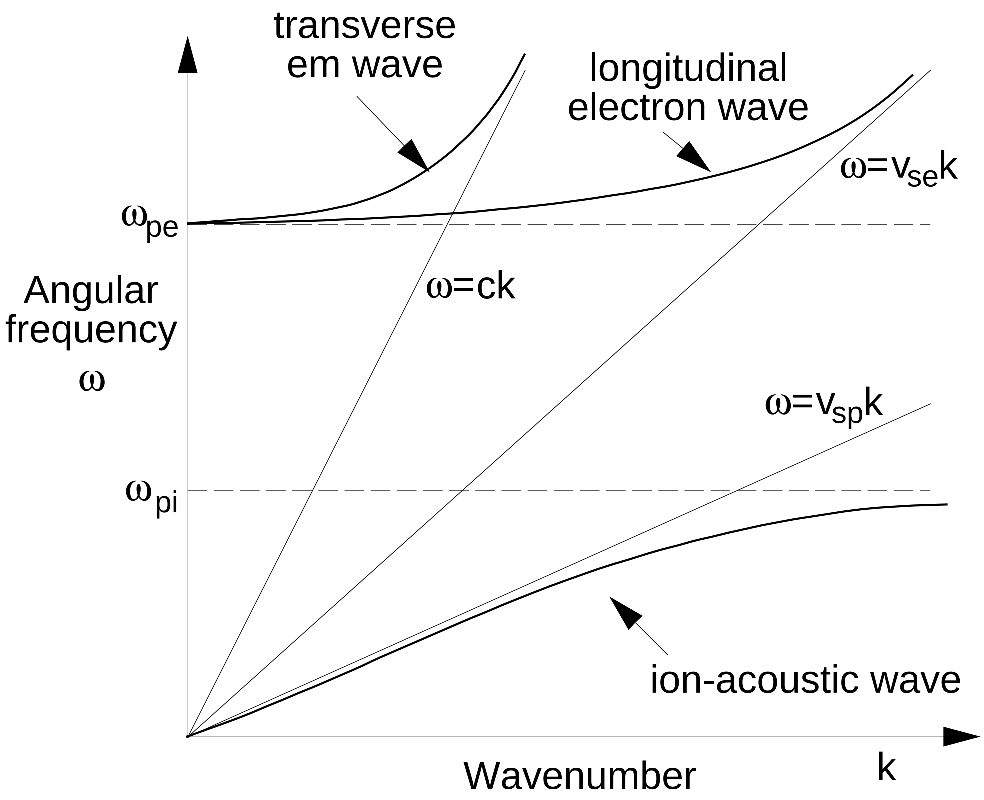

Figure 8.4: Dispersion relations for the three wave modes supported in an isotropic (unmagnetized) warm plasma.

space [due to variation in the plasma density profile $n_e(x)$]. The phase change of the wave as it moves from (x_1, t_1) to position (x_2, t_2) in the plasma is given by

$$\phi = \int_{x_1}^{x_2} \mathrm{d}x\, k(x)x \;\; - \mathrm{i}\omega_0(t_2 - t_1) \tag{8.36}$$

where we have taken $\phi = 0$ at point (x_1, t_1). Show that the difference $\delta\phi = \phi - \phi_0$ between the phase shift ϕ suffered by the wave in plasma and the phase $\phi_0 = k_0(x_2 - x_1) - \mathrm{i}\omega_0(t_2 - t_1)$ in vacuum is given by

$$\delta\phi = -r_e\lambda_0 \int_{x_1}^{x_2} n_e(x)\,\mathrm{d}x$$

where r_e is the classical electron radius.