Chapter 2

KINETIC THEORY

2.1 Distribution Functions

A plasma is an ensemble of particles electrons e, ions i and neutrals n with different positions $\boldsymbol{r}$ and velocities $\boldsymbol{v}$ which move under the influence of external forces (electromagnetic fields, gravity) and internal collision processes (ionization, Coulomb, charge exchange etc.)

However, what we observe is some "average" macroscopic plasma parameters such as $\boldsymbol{j}$ - current density, n_{e} - electron density, P - pressure, T_{i} - ion temperature etc.

These parameters are macrsocopic averages over the distribution of particle velocities and/or positions.

In this lecture we

- Introduce the concept of the distribution function $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ for a given plasma species;

- Derive the force balance equation (Boltzmann equation) that drives the temporal evolution of $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$;

- Show that low order velocity moments of $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ give various important macroscopic parameters;

- Consider the role of collision processes in coupling the charged and neutral speicies dynamics in a plasma and

- Show that low order velocity moments of the Boltzmann equation give "fluid" equations for the evolution of the macroscopic quantities.

2.2 Phase Space

Consider a single particle of species α. It can be described by a position vector

$$\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$

in *configuration space* and a velocity vector

$$\boldsymbol{v} = v_x\hat{\boldsymbol{i}} + v_y\hat{\boldsymbol{j}} + v_z\hat{\boldsymbol{k}}$$

in *velocity space*. The coordinates $(\boldsymbol{r}, \boldsymbol{v})$ define the particle position in *phase space*.

For multi-particle systems, we introduce the *distribution function* $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ for species α defined such that

$$f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{v} = \mathrm{d}N(\boldsymbol{r}, \boldsymbol{v}, t) \tag{2.1}$$

is the number of particles in the element of volume $\mathrm{d}V = \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{r}$ in phase space. Here, $\mathrm{d}\boldsymbol{r} \equiv \mathrm{d}^3 r \equiv \mathrm{d}x\mathrm{d}y\mathrm{d}z$ and $\mathrm{d}\boldsymbol{v} \equiv \mathrm{d}^3 v \equiv \mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$ $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ is a *positive* finite function that decreases to zero as $|\boldsymbol{v}|$ becomes large.

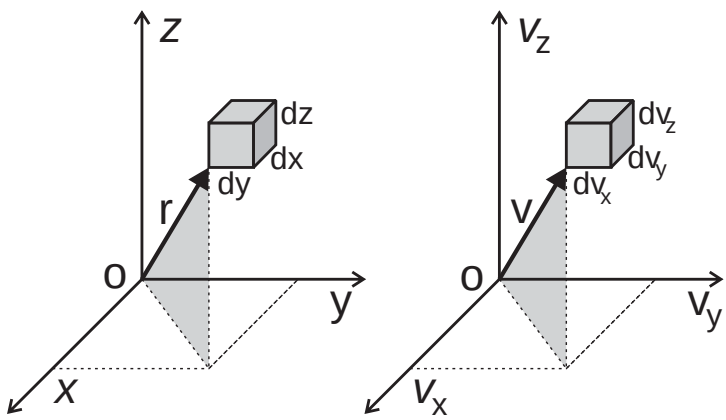

Figure 2.1: Left: A configuration space volume element $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ at spatial position $\boldsymbol{r}$. Right: The equivalent velocity space element. Together these two elements constitute a volume element $\mathrm{d}V = \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$ at position $(\boldsymbol{r}, \boldsymbol{v})$ in phase space.

The element $\mathrm{d}\boldsymbol{r}$ must not be so small that it doesn't contain a statistically significant number of particles. This allows $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ to be approximated by

a continuous function. For example, for typical densities in H-1NF 10^{12} m^{-3}, $\mathrm{d}\boldsymbol{r} \approx 10^{-12}$ m^{-3} $\Longrightarrow \mathrm{d}\boldsymbol{r} \int f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v} \sim 10^6$ particles.

Some defnitions:

- If f_α depends on $\boldsymbol{r}$, the distribution is *inhomogeneous*
- If f_α is independent of $\boldsymbol{r}$, the distribution is *homogeneous*
- If f_α depends on the direction of $\boldsymbol{v}$, the distribution is *anisotropic*
- If f_α is independent of the direction of $\boldsymbol{v}$, the distribution is *isotropic*
- A plasma in *thermal equilibrium* is characterized by a *homogeneous*, *isotropic* and *time-independent* distribution function

2.3 The Boltzmann Equation

As we have seen, the distribution of particles is a function of both time and the *phase space* coordinates $(\boldsymbol{r}, \boldsymbol{v})$. The Boltzmann equation describes the time evolution of f under the action of external forces and internal collisions. The remainder of this section draws on the derivation given in [2].

$f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ changes because of the flux of particles across the surface bounding the elemental volume $\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$ in phase space. This can arise continuously due to particle velocity and external forces (accelerations) or discontinuously through collisions. The collisional contribution to the rate of change $\partial f/\partial t$ of the distribution function is written as $(\partial f/\partial t)_{\mathrm{coll}}$.

To account for continuous phase space flow we use the divergence theorem

$$\int_S \boldsymbol{E}.\mathrm{d}\boldsymbol{s} = \int_{\Delta V} \nabla.\boldsymbol{E}\, \mathrm{d}V \tag{2.2}$$

applied to the six dimensional phase space surface S bounding the phase space volume ΔV enclosing the six dimensional vector field $\boldsymbol{E}$.

Now note that conservation of particles requires that the rate of particle flow over the surface $\mathrm{d}\boldsymbol{s}$ bounding the element ΔV plus those generated by collisions be equal to the rate at which particle phase space density changes with time. If we let $\boldsymbol{V} = (\boldsymbol{v}, \boldsymbol{a})$ be the generalized "velocity" vector for our mathematical phase space $(\boldsymbol{r}, \boldsymbol{v})$, then the rate of flow over S *into* the volume element is

$$-\int_S \mathrm{d}\boldsymbol{s}.\,[\boldsymbol{V}\, f]$$

Compare $\boldsymbol{V}f$ with the definition of particle flux $\boldsymbol{\Gamma}(\boldsymbol{r}, t)$ - a configuration space quantity [Eq. (2.14)]. Using the divergence theorem, this contribution can be written

$$-\int_{\Delta V} \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, [\boldsymbol{\nabla}_r.(\boldsymbol{v}f) + \boldsymbol{\nabla}_v.(\boldsymbol{a}f)]$$

where $\boldsymbol{\nabla}_r\boldsymbol{\cdot}$ is the divergence with respect to $\boldsymbol{r}$ and $\boldsymbol{\nabla}_v\boldsymbol{\cdot}$ is the divergence operator with respect to $\boldsymbol{v}$. For example

$$\boldsymbol{\nabla}_r\boldsymbol{\cdot} \equiv \hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}$$

Since ΔV is an elemental volume, the integrand changes negligibly and can be removed from the integral. The inclusion of the continuous phase space flow term and the discontinuous collision term then gives the result

$$\frac{\partial f}{\partial t} = -\boldsymbol{\nabla}_r\boldsymbol{\cdot}(\boldsymbol{v}f) - \boldsymbol{\nabla}_v\boldsymbol{\cdot}(\boldsymbol{a}f) + \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}. \tag{2.3}$$

The acceleration can be written in terms of the force $\boldsymbol{F} = m\boldsymbol{a}$ applied or acting on the particle species. For plasmas, the dominant force is electromagnetic, the Lorentz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \equiv m\boldsymbol{a}. \tag{2.4}$$

Using the vector identity (product rule)

$$\nabla\boldsymbol{\cdot}(f\boldsymbol{V}) = \nabla f\boldsymbol{\cdot}\boldsymbol{V} + f\nabla\boldsymbol{\cdot}\boldsymbol{V}$$

we can write

$$\boldsymbol{\nabla}_v\boldsymbol{\cdot}(\boldsymbol{v}\times\boldsymbol{B})f = (\boldsymbol{v}\times\boldsymbol{B})\boldsymbol{\cdot}\nabla_v f \qquad \text{(why?)}$$

Moreover, since $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent coordinates, we finally obtain the *Boltzmann equation*

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\nabla_r f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\boldsymbol{\cdot}\nabla_v f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{2.5}$$

Since f is a function of seven variables $\boldsymbol{r}$, $\boldsymbol{v}$, t its total derivative with respect to time is

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} \\ &+ \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &+ \frac{\partial f}{\partial v_x}\frac{\mathrm{d}v_x}{\mathrm{d}t} + \frac{\partial f}{\partial v_y}\frac{\mathrm{d}v_y}{\mathrm{d}t} + \frac{\partial f}{\partial v_z}\frac{\mathrm{d}v_z}{\mathrm{d}t}\end{aligned} \tag{2.6}$$

The second line is just the expansion of the term $\boldsymbol{v}\boldsymbol{\cdot}\nabla_r f$ and the third, the term $\boldsymbol{a}\boldsymbol{\cdot}\nabla_v f$. Thus Eq. (2.3) is simply saying that $\mathrm{d}f/\mathrm{d}t = 0$ unless there are collisions! When there are no collisions, $(\partial f/\partial t)_{\text{coll}} = 0$ and Eq. (2.5) is called the *Vlasov* equation.

The quantity $\mathrm{d}f/\mathrm{d}t$ is called the *convective derivative*. It contains terms due to the explicit variation of f (or any other configuration or phase space quantity)

with time $(\partial f/\partial t)$ and an additional term $\boldsymbol{v}.\nabla f$ that accounts for variations in f due to the change in location in time (*convection*) of f to a position where f has a different value. As an example, consider water flow at the outlet of a tap. The water is moving, but the flow pattern is not changing explicitly with time $(\partial f/\partial t) = 0$. However, the velocity of a fluid element changes on leaving the tap due to, for example, the force of gravity. The flow velocity changes in time as the water falls because the term $(\boldsymbol{v}.\nabla)\boldsymbol{v}$ is nonzero. When the tap is turned off, the flow slows and stops due to $(\partial f/\partial t) \neq 0$.

What does $\mathrm{d}f/\mathrm{d}t = 0$ mean? With reference to Fig 2.2, consider a group of particles at point A with 2-D phase space density $f(x, v)$. As time passes, the particles will move to B as a result of their velocity at A and their velocity will change as a result of forces acting. Since $F = F(x, v)$, all particles at A will be accelerated the same amount. Since the particles move together, the density at B will be the same as at A, f can only be changed by collisions.

It is important to understand that x and v are independent variables or co-ordinates, even though $v = \mathrm{d}x/\mathrm{d}t$ for a given particle. The reason is that the particle velocity is not a function of position - it can have any velocity at any posiiton.

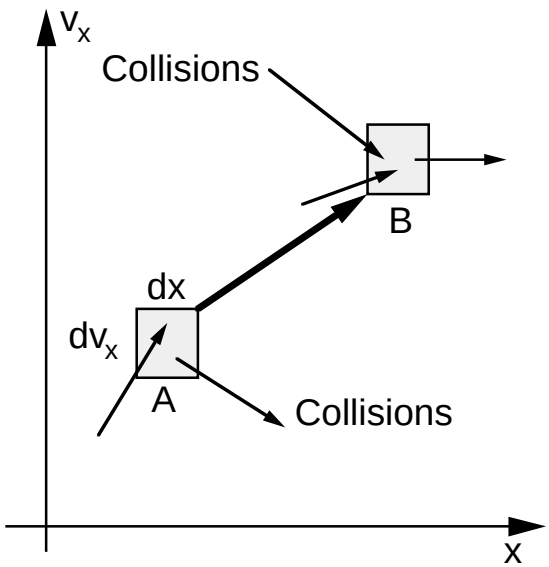

Figure 2.2: Evolution of phase space volume element under collisions

2.4 Plasma Macroscopic Variables

Macroscopic variables (measurable quantities) are obtained by taking appropriate *velocity moments* of the distribution function. It is perhaps not surprising that the dynamical evolution of these quantities can be described by equations obtained by taking the corresponding velocity moments of the Boltzmann equation. These equations (the *fluid equations*) are derived later in this chapter. For simplicity, and where possible, we drop the α subscript denoting the particle species for the remainder of this analysis.

2.4.1 Number density

Integration over all velocities gives the *particle number density*

$$n_\alpha = \int f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v} \tag{2.7}$$

This is the *zeroth order velocity moment.*

Multiply by mass or charge to obtain

$$\text{mass density :} \qquad \rho_{\alpha m} = m_\alpha \int f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v} \tag{2.8}$$

$$\text{charge density :} \qquad \rho_{\alpha q} = q_\alpha \int f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v} \tag{2.9}$$

2.4.2 Taking averages using the distribution function

The quantity

$$\hat{f}(\boldsymbol{r}, \boldsymbol{v}, t) = \frac{f(\boldsymbol{r}, \boldsymbol{v}, t)}{\int f(\boldsymbol{r}, \boldsymbol{v}, t)\mathrm{d}\boldsymbol{v}} = \frac{f(\boldsymbol{r}, \boldsymbol{v}, t)}{n(\boldsymbol{r}, t)} \tag{2.10}$$

is the *probability* of finding one particle in a unit volume of ordinary (configuration) space centred at $\boldsymbol{r}$ and in a unit volume of velocity space centred at $\boldsymbol{v}$ at time t. The *distribution average* of some quantity $g(\boldsymbol{r}, \boldsymbol{v}, t)$ is then the value of g at $(\boldsymbol{r}, \boldsymbol{v}, t)$ times the probability that there is a particle with these coodinates

$$\overline{g(\boldsymbol{r}, t)} = \int \mathrm{d}\boldsymbol{v}\, g(\boldsymbol{r}, \boldsymbol{v}, t)\, \hat{f}(\boldsymbol{r}, \boldsymbol{v}, t) \tag{2.11}$$

or

$$\overline{g(\boldsymbol{r}, t)} = \frac{1}{n(\boldsymbol{r}, t)} \int \mathrm{d}\boldsymbol{v}\, g(\boldsymbol{r}, \boldsymbol{v}, t)\, f(\boldsymbol{r}, \boldsymbol{v}, t) \tag{2.12}$$

2.4.3 First velocity moment

The *average velocity* is obtained from the first velocity moment of f.

$$\overline{\boldsymbol{v}(\boldsymbol{r}, t)} = \frac{1}{n(\boldsymbol{r}, t)} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v}\, f(\boldsymbol{r}, \boldsymbol{v}, t) \tag{2.13}$$

Note that $\overline{\boldsymbol{v}(\boldsymbol{r},t)}$ can be a function of $\boldsymbol{r}$ and t though $\boldsymbol{v}$ (a coordinate) is not. This is because f varies with $\boldsymbol{r}$ and t.

Multiplying the average velocity by the particle number density gives the *particle flux*

$$\boldsymbol{\Gamma} = n\bar{\boldsymbol{v}} = \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{v},t) \tag{2.14}$$

In a similar fashion, multiplying the flux by the charge density gives the charge flux density or *current density* for a given species

$$\boldsymbol{j} = qn\bar{\boldsymbol{v}} = q\boldsymbol{\Gamma} \tag{2.15}$$

2.4.4 Second velocity moment

Taking the second central moment with respect to average velocity $\bar{\boldsymbol{v}}$ gives the *pressure tensor* (or kinetic stress tensor)

$$\overleftrightarrow{P}\,(\boldsymbol{r},t) = m\int \mathrm{d}\boldsymbol{v}\,(\boldsymbol{v}-\bar{\boldsymbol{v}})(\boldsymbol{v}-\bar{\boldsymbol{v}})\, f(\boldsymbol{r},\boldsymbol{v},t) \tag{2.16}$$

We define $\boldsymbol{v}_r = \boldsymbol{v} - \bar{\boldsymbol{v}}$, the random thermal velocity of the particles where $\bar{\boldsymbol{v}}$ is the mean drift velocity. Then $\boldsymbol{v}_r\boldsymbol{v}_r$ is the tensor or dyadic

$$\boldsymbol{v}_r\boldsymbol{v}_r = \begin{pmatrix} v_xv_x & v_xv_y & v_xv_z \\ v_yv_x & v_yv_y & v_yv_z \\ v_zv_x & v_zv_y & v_zv_z \end{pmatrix} \tag{2.17}$$

and clearly $(\boldsymbol{v}_r\boldsymbol{v}_r)_{\alpha\beta} = (\boldsymbol{v}_r\boldsymbol{v}_r)_{\beta\alpha}$ so that $\overleftrightarrow{P}$ has only six independent components which we write as

$$\begin{aligned} P_{\alpha\beta} &= m\int \mathrm{d}\boldsymbol{v}\, v_{r\alpha}v_{r\beta}\, f(\boldsymbol{r},\boldsymbol{v},t) \\ &= mn\overline{v_{r\alpha}v_{r\beta}} \end{aligned} \tag{2.18}$$

Now consider the distribution function f to be isotropic in the system drifting with velocity $\bar{\boldsymbol{v}}(\boldsymbol{r},t)$ (see Fig 2.3). Because f is isotropic in this system, $\overline{\boldsymbol{v}_r} = 0$ and

$$\overline{v_{r\alpha}v_{r\beta}} = 0 \qquad \alpha \neq \beta \tag{2.19}$$

$$= \overline{v_{r\alpha}^2} = \overline{v_{r\beta}^2} = \overline{v_r^2}/3 \tag{2.20}$$

(show this result) and

$$\overleftrightarrow{P} \rightarrow p = nm\overline{v_r^2}/3. \tag{2.21}$$

p is the rate of transfer of particle momentum in any direction. It has dimensions force/area or pressure. $\overline{v_r^2}$ (or simply $\overline{v^2}$ when the plasma is not drifting) is the

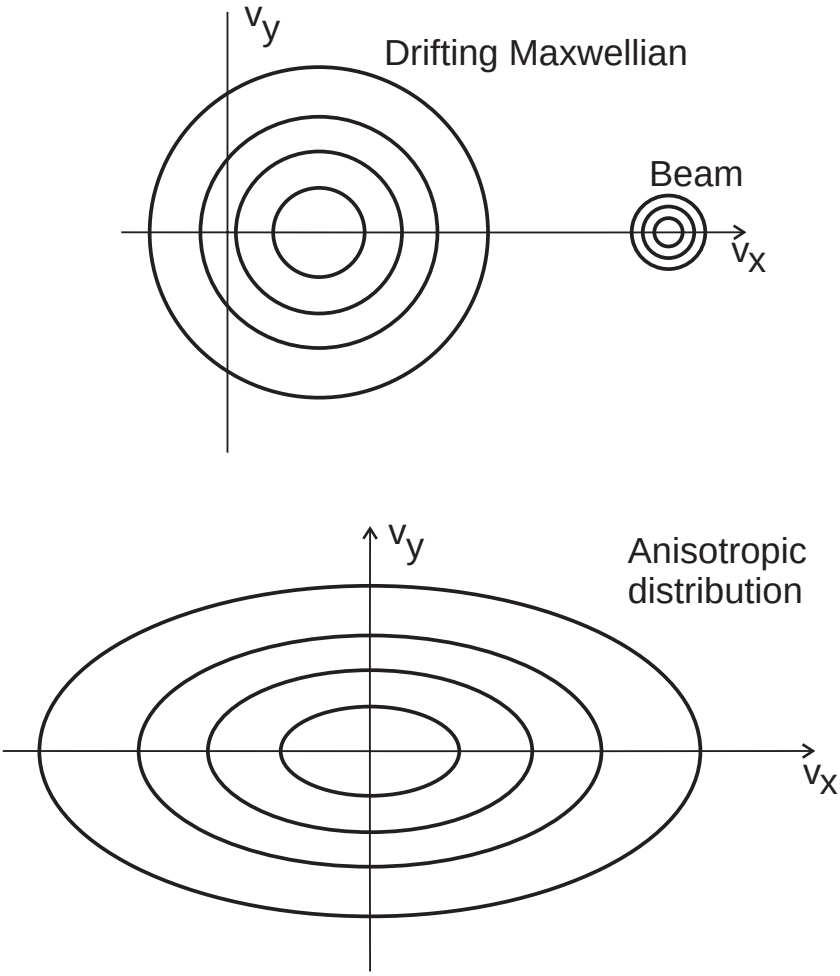

Figure 2.3: Examples of velocity distribution functions

mean square thermal speed and $v_{\rm rms} = \sqrt{\overline{v^2}}$ is the *rms thermal speed*. Note that the average kinetic energy is given from Eq. (2.12) by

$$\bar{U}_r \equiv \frac{1}{n} \int \mathrm{d}\boldsymbol{v}_r \, \frac{1}{2} m v_r^2 \, f(\boldsymbol{r}, \boldsymbol{v}, t) \tag{2.22}$$

and that this is therefore related to the pressure by

$$p = \frac{2}{3} n \bar{U}_r. \tag{2.23}$$

The average particle kinetic energy is also related to the second velocity moment of the distribution function.

2.5 Maxwell-Boltzmann Distribution

So far we have shown how low order velocity moments of the particle distribution function are related to macroscopic parameters such as current density, pressure etc. without looking at the details of the distribution function itself. A particularly important velocity distribution function is the *Maxwell-Boltzmann distribution*, or Maxwellian. It describes the spread of velocities for a gas which is in thermal equilibrium. Such a system can be described by a simple Gaussian spread of velocities, the half-width being related to the gas temperature. As always, the zeroth moment of such a distribution is the particle number density. Because a Gaussian has even symmetry, the odd moment will vanish unless the distribution happens to be drifting with respect to the laboratory frame. The second velocity moment is related to the temperature and also, not surprisingly, reveals a link between the gas pressure and temperature. In this section, we introduce this important distribution and its variants and discuss some of its physical ramifications.

2.5.1 The concept of temperature

For a gas in *thermal equilibrium* the most probable distribution of velocities can be calculated using statistical mechanics to be the Maxwellian distribution. In 1-D

$$f_{\mathrm{M}}(v) = A \exp\left(-\frac{mv^2/2}{k_{\mathrm{B}}T}\right) \tag{2.24}$$

This is homogeneous and isotropic. Using Eq. (2.7), this can be written in terms of the density as

$$f_{\mathrm{M}}(v) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{1/2} \exp\left(-\frac{mv^2/2}{k_{\mathrm{B}}T}\right) \tag{2.25}$$

If we define

$$v_{\mathrm{th}} = \left(\frac{2k_{\mathrm{B}}T}{m}\right)^{1/2} \tag{2.26}$$

then

$$f_{\mathrm{M}}(v) = \frac{n}{\sqrt{\pi}v_{\mathrm{th}}} \exp\left(-\frac{v^2}{v_{\mathrm{th}}^2}\right). \tag{2.27}$$

If we follow this analysis through in 3-D we find

$$f_{\mathrm{M}}(\boldsymbol{v}) = \frac{n}{(\sqrt{\pi}v_{\mathrm{th}})^3} \exp\left(-\frac{\boldsymbol{v}^2}{v_{\mathrm{th}}^2}\right) \tag{2.28}$$

The function f_{M} describes a 3-D distribution of velocities. It is useful to also calculate the distribution of particle speeds $g_{\mathrm{M}}(v)$ by averaging the Maxwellian

over the velocity directions. The result obtained is

$$g_{\mathrm{M}}(v) = 4\pi n \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{1/2} v^2 \exp\left(-v^2/v_{\mathrm{th}}^2\right) \tag{2.29}$$

It can be shown by differentiation that the peak of this distribution occurs at $v = v_{\mathrm{th}}$.

Several other important characteristic velocities can be related to temperature in the case of a Maxwellian distribution. For example, the mean particle speed is calculated in Sec. 2.5.2. Perhaps most important is the rms thermal speed

$$v_{\mathrm{rms}} = \sqrt{\frac{3k_{\mathrm{B}}T}{m}}. \tag{2.30}$$

Relation to kinetic energy

The average particle kinetic energy can be calculated from Eq. (2.22) as

$$\bar{U}_r \equiv E_{\mathrm{av}} = \frac{1}{\sqrt{\pi} v_{\mathrm{th}}} \int_{-\infty}^{\infty} \frac{1}{2} m v^2 \, \exp\left(-v^2/v_{\mathrm{th}}^2\right) \mathrm{d}v. \tag{2.31}$$

It is a standard result that

$$\int_{-\infty}^{\infty} v^2 \exp\left(-av^2\right) \mathrm{d}v = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \tag{2.32}$$

so that we may simplify to obtain

$$E_{\mathrm{av}} = \frac{1}{4} m v_{\mathrm{th}}^2 = \frac{1}{2} k_{\mathrm{B}} T \tag{2.33}$$

In three dimensions, it can be shown that

$$E_{\mathrm{av}} = \frac{3}{2} k_{\mathrm{B}} T, \tag{2.34}$$

implying that the plasma particles possess $\frac{1}{2}k_{\mathrm{B}}T$ of energy per degree of freedom.

In plasma physics it is customary to give temperatures in units of energy $k_{\mathrm{B}}T$. This energy is expressed in terms of the potential V through which an electron must fall to acquire that energy. Thus, $eV = k_{\mathrm{B}}T$ and the temperature equivalent to one electron volt is

$$T = e/k = 11600\mathrm{K} \tag{2.35}$$

Thus, for a 2eV plasma, $k_{\mathrm{B}}T = 2\,\mathrm{eV}$ and the average particle energy $E_{\mathrm{av}} = \frac{3}{2}k_{\mathrm{B}}T = 3\,\mathrm{eV}$ For a 100 eV plasma (H-1NF conditions), $T \sim 10^6\mathrm{K}$ and

$$\begin{aligned} v_{\mathrm{thi}} &\sim 1.6 \times 10^5 \mathrm{m/s} \\ v_{\mathrm{the}} &\sim 6.7 \times 10^6 \mathrm{m/s}. \end{aligned}$$

These velocities are in the ratio of the square root of the particle masses.

Relation to pressure

The pressure is also a second moment quantity, so that it wil not be surprising to find it related to temperature in the case of a Maxwellian distribution. It is instructive to review the concept of plasma pressure using a simple minded model and show how the result relates to that obtained using distribution functions.

Consider the imaginary plasma container shown in Fig. 2.4. The number of particles hitting the wall from a given direction per second is $nAv/2$ (only half contribute). The momentum imparted to the wall per collision is $mv - (-mv) = 2mv$ so that the force (rate of change of momentum) is $F = (2mv)(nAv/2)$ and the pressure is $p = F/A = nmv^2$ per velocity component. The energy per degree of freedom is $\frac{1}{2}mv^2 = \frac{1}{2}k_{\mathrm{B}}T$ so that $p = nk_{\mathrm{B}}T$

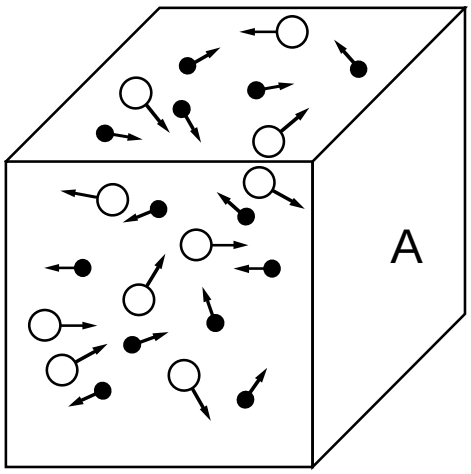

Figure 2.4: Imaginary box containing plasma at temperature T

If we sum over all plasma species we obtain

$$p = \sum_{\alpha} n_{\alpha} k T_{\alpha} \tag{2.36}$$

That is, the electrons and ions contribute equally to the plasma pressure!
Explanation:
By equipartition, $\frac{1}{2}m_{\mathrm{i}}v_{\mathrm{thi}}^2 = \frac{1}{2}m_{\mathrm{e}}v_{\mathrm{the}}^2$ so that $v_{\mathrm{the}} = v_{\mathrm{thi}}\sqrt{m_{\mathrm{i}}/m_{\mathrm{e}}}$ whereupon the momenta imparted per collision are related by $m_{\mathrm{e}}v_{\mathrm{the}} = m_{\mathrm{i}}v_{\mathrm{thi}}\sqrt{m_{\mathrm{e}}/m_{\mathrm{i}}}$. Note however, that the collison rate for electrons is $\sqrt{m_{\mathrm{i}}/m_{\mathrm{e}}}$ faster.

If we use distribution functions, we obtain for a Maxwellian

$$\begin{aligned} \bar{U}_r = E_{\mathrm{av}} &= \frac{3}{2}k_{\mathrm{B}}T \qquad \text{[Eq. (2.34)]} \\ p &= \frac{2}{3}n\bar{U}_r \qquad \text{[Eq (2.22)]} \end{aligned}$$

Combining these two recovers Eq. (2.36)

2.5.2 Thermal particle flux to a wall

Ignoring the effects of electric fields in the plasma sheath, it is useful to calculate the *random particle flux* to a surface immersed in a plasma in thermal equilibrium. This flux is defined as

$$\Gamma_s = n\langle v_s \rangle = \int \mathrm{d}\boldsymbol{v}\, f_{\mathrm{M}}(\boldsymbol{r}, \boldsymbol{v}, t)\, \boldsymbol{v}.\hat{\boldsymbol{s}} \tag{2.37}$$

where $\hat{\boldsymbol{s}}$ is the normal to the surface under consideration. We assume f is Maxwellian and consider the flux in only one direction, since in both directions the nett flux crossing a plane vanishes (why?). In spherical polar coordinates, $\mathrm{d}\boldsymbol{v} = v^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}v$ and $\boldsymbol{v}.\hat{\boldsymbol{s}} = v\cos\theta$ and we have

$$\begin{aligned}
\Gamma_s &= \int_0^\infty \mathrm{d}v\, v^3 f_{\mathrm{M}}(v) \int_0^{\pi/2} \mathrm{d}\theta\, \sin\theta\cos\theta \int_0^{2\pi} \mathrm{d}\phi \\
&= \pi \int_0^\infty \mathrm{d}v\, v^3 f_{\mathrm{M}}(v) \\
&= \pi n \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \int_0^\infty \mathrm{d}v\, v^3 \exp\left(-\frac{mv^2}{2k_{\mathrm{B}}T}\right) \\
&= n \left(\frac{k_{\mathrm{B}}T}{2\pi m}\right)^{1/2} \\
&\equiv n\langle v \rangle/4
\end{aligned} \tag{2.38}$$

where

$$\begin{aligned}
\langle v \rangle &\equiv \frac{1}{n}\int_0^\infty \mathrm{d}\boldsymbol{v}\ |\boldsymbol{v}|\ f_{\mathrm{M}}(\boldsymbol{v}) \\
&= \left(\frac{8k_{\mathrm{B}}T}{\pi m}\right)^{1/2} \qquad \text{(Maxwellian)}
\end{aligned} \tag{2.39}$$

is the *mean speed* of the particles. The flux in a given direction is thus non-zero. Question: Why doesn't the plasma vessel melt?

1. Low energy flux ($\Gamma_E = E_{\mathrm{av}}\Gamma_s$)
2. Magnetic bottle (thermal insulation)
3. Pulsed experiments (short duration)

But the heat flux is a problem for fusion reactors.

2.5.3 Local Maxwellian distribution

We define the local Maxwellian velocity distribution as

$$f_{\mathrm{LM}}(\boldsymbol{v}) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{m\boldsymbol{v}^2}{2k_{\mathrm{B}}T}\right) \tag{2.40}$$

where now

$$\begin{aligned} n &\equiv n(\boldsymbol{r},t) \\ T &\equiv T(\boldsymbol{r},t) \end{aligned}$$

depend on the spatial coordinates and time. f_{LM} can be shown to satisfy the Boltzmann equation. Though each species in a plasma is characterised by its own distribution function, the time evolution of these distributions are coupled by collisions between the species and the self-consistent electric field that arises through particle diffusion.

2.5.4 The effect of an electric field

Till now we have implicitly ignored the influence of electric fields upon the distribution function. When more than a single particle species (for example ions and electrons) are considered, however, such fields can arise spontaneously in the plasma due to differences in the mobility or transport of the two species. The electric field in turn affects the spatial distribution of particle velocities through the Lorentz force (we here ignore complications due to magnetic fields). This effect is important for the electrons owing to their much greater mobility. The spatial variation of the electric potential ($\boldsymbol{E} = -\nabla\phi$) suggests the use of the concept of a local Maxwellian.

In equilibrium ($\partial/\partial t = 0$) and for $\boldsymbol{B} = 0$, the Boltzmann equation for electrons becomes

$$\boldsymbol{v}.\nabla_r f_{\mathrm{e}} - \frac{e}{m}\boldsymbol{E}.\nabla_v f_{\mathrm{e}} = 0. \tag{2.41}$$

It can be verified by substitution that, in thermal equilibrium, the electron distribution function differs from Maxwellian by a factor related to the electric potential:

$$f_{\mathrm{e}} = f_{\mathrm{LM}}\exp\left(\frac{e\phi}{k_{\mathrm{B}}T_{\mathrm{e}}}\right). \tag{2.42}$$

The exponent is independent of v so that integration over v gives the *Boltzmann relation*

$$n_{\mathrm{e}} = n_{\mathrm{e0}}\exp\left(\frac{e\phi}{k_{\mathrm{B}}T_{\mathrm{e}}}\right) \tag{2.43}$$

where n_{e0} is the electron density in the zero potential region and the exponential term is known as the *Boltzmann factor*. Substituting from Eq. (2.28) we have

$$f_e(\boldsymbol{r}, \boldsymbol{v}) = A \exp\left(\frac{-mv^2/2 + e\phi}{k_B T_e}\right) = A \exp\left(-E/k_B T_e\right) \tag{2.44}$$

where E is the sum of the kinetic and potential components of the electron energy. The expression indicates that only the fastest electrons can overcome the potential barrier and enter regions of negative potential. In other words, the electron density (zeroth moment of f_{LM}) is depleted in regions where the electric potential is negative. We shall visit this concept again in relation to the electric sheath that is established at the boundary between a plasma and a wall in Chapter 3.

2.6 Quasi-neutrality

In Chapter 1 we introduced a number of fundamental concepts and criteria that characterise the plasma state. In this and the following two sections we apply the tools of kinetic theory to obtain a more rigorous treatment of Debye shielding, plasma oscillations and collision phenomena.

The potential at a distance r from an isolated positive test charge Q is

$$\phi = \frac{Q}{4\pi\varepsilon_0 r}. \tag{2.45}$$

When immersed in a plasma Q attracts negative charge and repels positive ions. As a result, the number density of ions and electrons will vary slightly around Q but will be in charge balance for large r (i.e. $n_e = n_i$ for $Z = 1$).

The effect of this screening will be to exponentially damp ϕ over a distance comparable to the Debye length

$$\tilde{\phi} = \phi \exp\left(-r/\lambda_D\right)$$

To show this we start with Poisson's equation

$$\nabla . \boldsymbol{E} = \rho/\varepsilon_0, \qquad \boldsymbol{E} = -\nabla\phi \qquad \Rightarrow \qquad \nabla^2\phi = -\rho/\varepsilon_0.$$

ρ is composed of ρ_e and ρ_i as given by Eq. (2.9) plus the contribution from $+Q$ at $r = 0$. Over time scales comparable with the plasma frequency, the ions are immobile and we have $\rho_i = en_i = en_0$ where n_0 is the charge density in the potential free (unperturbed) region, while for the electrons (c.f. Eq. (2.43))

$$\rho_e(\boldsymbol{r}) = -en_0 \exp\left[e\phi(\boldsymbol{r})/k_B T_e\right]$$

and Poisson's equation gives

$$\nabla^2\phi(\boldsymbol{r}) - \frac{en_0}{\varepsilon_0}\left[\exp\left(e\phi/k_\mathrm{B}T_\mathrm{e}\right) - 1\right] = -\frac{Q}{\varepsilon_0}\delta(\boldsymbol{r}).$$

If the electrostatic potential energy $e\phi \ll k_\mathrm{B}T_\mathrm{e}$ then $\exp\left(e\phi/k_\mathrm{B}T_\mathrm{e}\right) \approx 1 + e\phi/k_\mathrm{B}T_\mathrm{e}$ and

$$\nabla^2\phi(\boldsymbol{r}) - \frac{1}{\lambda_\mathrm{D}^2}\phi = -\frac{Q}{\varepsilon_0}\delta(\boldsymbol{r}).$$

For spherical symmetry, only the r variation remains and the equation becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) - \frac{\phi(r)}{\lambda_\mathrm{D}^2} = 0 \qquad (r \neq 0).$$

It can be verified by substitution that the solution is

$$\tilde{\phi} = \frac{Q}{4\pi\varepsilon_0 r}\exp\left(-r/\lambda_\mathrm{D}\right). \tag{2.46}$$

Noting that

$$\begin{aligned}\rho(\boldsymbol{r}) &\approx -n_0 e^2\phi(\boldsymbol{r})/k_\mathrm{B}T_\mathrm{e} + Q\delta(\boldsymbol{r}) \\ &= -\frac{Q}{4\pi r\lambda_\mathrm{D}^2}\exp\left(-r/\lambda_\mathrm{D}\right) + Q\delta(\boldsymbol{r}),\end{aligned}$$

the total charge is obtained by integration

$$\begin{aligned}Q_T &= \int \rho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = 4\pi\int \mathrm{d}r\, r^2\rho(r) \\ &= -\frac{Q}{\lambda_\mathrm{D}^2}\int_0^\infty \mathrm{d}r\, r\exp\left(-r/\lambda_\mathrm{D}\right) + Q\int\delta(\boldsymbol{r})\mathrm{d}\boldsymbol{r} \\ &= 0.\end{aligned} \tag{2.47}$$

Thus neutralization of the test particle takes place on account of the imbalance of charged particles in the Debye sphere.
<u>Check:</u> $e\phi/k_\mathrm{B}T_\mathrm{e} \ll 1$ (range of validity)

$$\begin{aligned}\frac{e\phi}{k_\mathrm{B}T_\mathrm{e}} &= \frac{e^2}{4\pi\varepsilon_0 r k_\mathrm{B}T_\mathrm{e}}\exp\left(-r/\lambda_\mathrm{D}\right) \\ &= \frac{\lambda_\mathrm{D}}{3\Lambda}\frac{\exp\left(-r/\lambda_\mathrm{D}\right)}{r} \\ &\ll 1 \qquad \text{for } r > \lambda_\mathrm{D}/\Lambda\end{aligned} \tag{2.48}$$

This condition is valid for most particles in the Debye sphere.

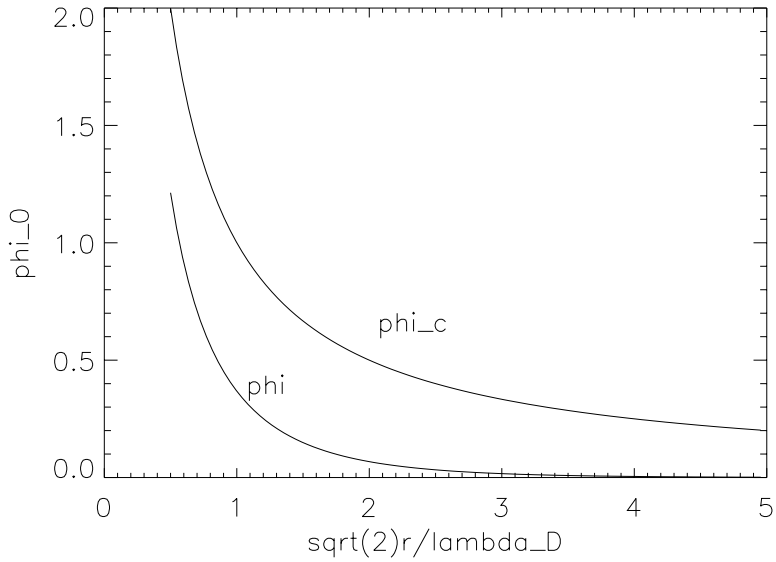

Figure 2.5: Electrostatic Coulomb potential and Debye potential as a function of distance from a test charge Q.

2.7 Kinetic Theory of Electron Waves

As another example of the importance of a kinetic theory, we consider the description of electron plasma oscillations (waves) in a warm plasma. When thermal effects (a distribution of velocities) are included in the treatment of plasma oscillations, it is found that the disturbances, which were found to be pure oscillations in Sec 1.1.2, do indeed propagate and are strongly damped by a linear collisionless mechanism known as *Landau damping* after the Soviet physicist who predicted it mathematically in 1946. The analysis presented here closely follows those given in [3, 4].

To describe these phenomena, we consider a simple one-dimensional system, ignore collisions and assume no magnetic field. Because of their much greater inertia, we regard the ions as providing a fixed uniform background at frequencies near $\omega_{\rm pe}$. The *Vlasov equation* can be written

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} = 0. \tag{2.49}$$

In the presence of a wave, we allow the equilibrium distribution function f_0 to be perturbed an amount f_1 with an associated small electric field $E = E_1$ due to the disturbance. Substituting $f = f_0 + f_1$ into the 1-D Valsov equation, expanding

and retaining terms up to first order in small quantities gives

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{qE}{m}\frac{\partial f_0}{\partial v} = 0. \tag{2.50}$$

We assume a plane harmonic form for the response $f_1 \sim \exp(\mathrm{i}kx - \mathrm{i}\omega t)$ to the initial one-dimensional perturbation, recognizing that ω may be complex in order to allow for wave damping. This allows the substitutions

$$\frac{\partial}{\partial t} \rightarrow -\mathrm{i}\omega \tag{2.51}$$

$$\frac{\partial}{\partial x} \rightarrow \mathrm{i}k \tag{2.52}$$

to obtain

$$-\mathrm{i}\omega f_1 + \mathrm{i}kv f_1 + \frac{qE}{m}\frac{\partial f_0}{\partial v} = 0. \tag{2.53}$$

(We have implicitly integrated f over the velocity variables orthogonal to v, so that f is one-dimensional.) Solving for the perturbation gives

$$f_1 = -\frac{qE}{\mathrm{i}m}\frac{\partial f_0/\partial v}{kv - \omega}. \tag{2.54}$$

By analogy with Eq. (2.7) the perturbed density is given by

$$n_1 = -\frac{qE}{\mathrm{i}m}\int_{-\infty}^{\infty} \mathrm{d}v\, \frac{\partial f_0/\partial v}{kv - \omega}. \tag{2.55}$$

This in turn gives rise to an electric potential fluctuation ϕ_1 that can be obtained using $E_1 = -\partial\phi_1/\partial x$ and Poisson's equation $\partial E/\partial x = qn_1/\varepsilon_0$. Substituting in Eq. (2.55) gives the dispersion relation (a relation linking the phase velocity of the wave to its frequency and wavenumber)

$$1 = -\frac{\omega_{\mathrm{pe}}^2}{k^2}\int_{-\infty}^{\infty} \mathrm{d}v\, \frac{\partial \hat{f}_0/\partial v}{(v - \omega/k)}. \tag{2.56}$$

We have normalized the unperturbed distribution function to the electron number density $\hat{f}_0 = f_0/n$ (see Eq. (2.10)) and used Eq. (1.14) for the plasma frequency. Note that the integrand contains a singularity at $v_{\mathrm{ph}} = \omega/k$ and the integral requires to be evaluated using contour integration in the complex plane. However, the correct result can also be obtained using the less rigorous approach described below.

Fig. 2.6 shows the distribution $\hat{f}_0(v)$ and the integrand. Assuming $v_{\mathrm{ph}} >> v_{\mathrm{the}}$, the latter can be conveniently separated into two parts

(i) the central region dominated by $\partial\hat{f}_0/\partial v$

(ii) the region near $v = v_{\mathrm{ph}}$ dominated by the singularity

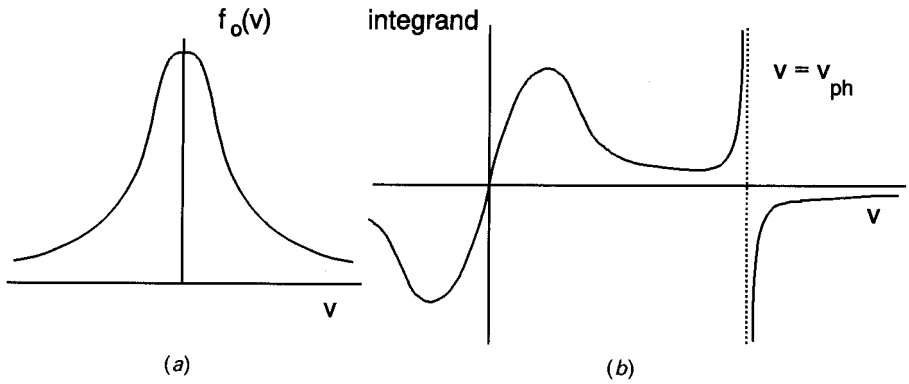

Figure 2.6: The contributions (i) and (ii) noted in the text [3]

Contribution of the central region

In this region, $v \ll v_{\rm ph}$, allowing the integrand to be expanded in powers of $v/v_{\rm ph}$ to give for the integral:

$$\begin{aligned} -\frac{1}{v_{\rm ph}}\int_{-\infty}^{\infty} {\rm d}v\, \frac{\partial \hat{f}_0/\partial v}{1-v/v_{\rm ph}} &= -\frac{1}{v_{\rm ph}}\int_{-\infty}^{\infty} {\rm d}v\, \frac{\partial \hat{f}_0}{\partial v}\left[1+\frac{v}{v_{\rm ph}}+\left(\frac{v}{v_{\rm ph}}\right)^2+\left(\frac{v}{v_{\rm ph}}\right)^3+\ldots\right] \\ &\approx -\frac{1}{v_{\rm ph}^2}\int_{-\infty}^{\infty} {\rm d}v\, v\frac{\partial \hat{f}_0}{\partial v} - \frac{1}{v_{\rm ph}^4}\int_{-\infty}^{\infty} {\rm d}v\, v^3\frac{\partial \hat{f}_0}{\partial v} \end{aligned} \tag{2.57}$$

where we have truncated the series after the two most significant terms (the terms containing even powers of v are zero by the antisymmetry of the integrand). These integrals can be evaluated by parts and substituted into Eq. (2.56) to obtain

$$1 = \frac{\omega_{\rm pe}^2}{\omega^2}\left(1+\frac{3k^2}{\omega^2}\overline{v^2}\right)$$

where we have made use of the definition of the mean square particle speed and ignored the contribution from the singularity. For $\hat{f}_0$ Maxwellian, we have

$$\frac{1}{2}m\overline{v^2} = \frac{1}{2}k_{\rm B}T$$

and the above can be solved for ω^2 in terms of k^2 to obtain (show this)

$$\omega^2 \approx \omega_{\rm pe}^2 + \frac{3k_{\rm B}T}{m}k^2. \tag{2.58}$$

This is the so-called *Bohm-Gross dispersion relation* for electron plasma waves (or Langmuir waves). The inclusion of finite electron temperature has contributed

a propagating component (k-dependent) to the dispersion relation. This result can also be obtained using the fluid equations (Chapter **??**) which describe the evolution of distribution averaged quantities. Some important non-equilibrium physical effects (such as Landau damping) are lost in this averaging process. It is the remaining integral over the singularity that reveals this.

Contribution of the singularity

Within a narrow range of velocity around the singularity, we may regard $\partial \hat{f}_0/\partial v$ as constant and remove it from the integral. With $u = v - \omega/k$, the integral can be written

$$\frac{\omega_{\mathrm{pe}}^2}{k^2}\left(\frac{\partial \hat{f}_0}{\partial v}\right)_{\omega/k}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{u} \tag{2.59}$$

Now

$$\begin{aligned}\int_{-a}^{a}\frac{\mathrm{d}u}{u} = \ln a - \ln -a &= \ln a - \ln\left[a\exp\left(\mathrm{i}\pi + \mathrm{i}2\pi n\right)\right] \\ &= \ln a - \ln a - (\mathrm{i}\pi + \mathrm{i}2\pi n) \\ &= -\mathrm{i}\pi \end{aligned} \tag{2.60}$$

where we have taken $n = 0$. Thus the singularity component becomes

$$-\mathrm{i}\pi\frac{\omega_{\mathrm{pe}}^2}{k^2}\left(\frac{\partial \hat{f}_0}{\partial v}\right)_{\omega/k} \tag{2.61}$$

and the full dispersion relation for electron plasma waves is

$$1 = \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\overline{v^2}\right) - \mathrm{i}\pi\frac{\omega_{\mathrm{pe}}^2}{k^2}\left(\frac{\partial \hat{f}_0}{\partial v}\right)_{\omega/k} \tag{2.62}$$

Treating the imaginary part (damping term) as small, and neglecting the thermal correction, we can use Taylor's theorem and rearrange to obtain (Prove this result)

$$\omega = \omega_{\mathrm{pe}}\left[1 + \mathrm{i}\frac{\pi}{2}\frac{\omega_{\mathrm{pe}}^2}{k^2}\left(\frac{\partial \hat{f}_0}{\partial v}\right)_{\omega/k}\right] \tag{2.63}$$

It is left as an exercise to show that if $\hat{f}_0$ is a one-dimensional Maxwellian, the damping is finally given by

$$\mathrm{Im}\left(\frac{\omega}{\omega_{\mathrm{pe}}}\right) = -0.22\sqrt{\pi}\left(\frac{\omega_{\mathrm{pe}}}{kv_{\mathrm{th}}}\right)^2\exp\left(\frac{-1}{2k^2\lambda_{\mathrm{D}}^2}\right) \tag{2.64}$$

where λ_{D} is given by Eq. (1.15). Since $\mathrm{Im}(\omega)$ is negative, the wave is damped, the effect becoming important for waves of wavelength smaller or comparable with the Debye length.

2.7.1 The physics of Landau damping

We have noted the appearance of a small electric field associated with the electron wave and an accompanying disturbance of the distribution from thermal equilibrium. The Boltzmann relation requires that the density varies with the electric potential according to Eq. (2.43). This is shown schematically in Fig. 2.7 where is plotted the potential energy $q\phi = -e\phi$ of an electron in an electron plasma wave. Thus, at the wave potential energy peaks, the electron density is lower than in the troughs.

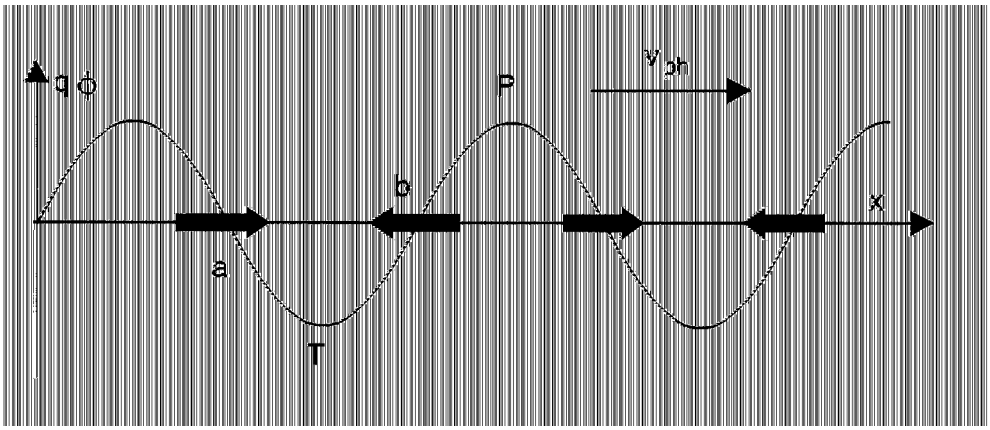

Figure 2.7: Diagram showing the electron potential energy $-e\phi$ in an electron plasma wave. Regions labelled A accelerate the electron while in B, the electron is decelerated [3].

Now consider electrons at the potential energy peak moving with velocity slightly faster than the phase velocity of the wave. These electrons will be accelerate, sacrificing potential for kinetic energy. Those particles in the trough will be decelerated. However, there are more particles in the trough than at the peak, so the net effect is deceleration and the electrons give energy to the wave.

This argument can be inverted, however, by considering electrons moving slightly slower than the wave. The net effect is opposite: the wave gives energy to the electrons. There is remaining, however, a fundamental asymmetry in that there are more particles moving slower than the wave velocity $v_{\rm ph}$ than going faster due to the slope of the distribution function. Thus, overall, there is a net loss of energy by the wave which is damped and a corresponding modification of the distribution function around $v = v_{\rm ph}$. There is no dissipation inherent in the process. However, collisions will cause the perturbed distribution to relax back to Maxwellian in the absence of a source for the wave excitation which could maintain the perturbation.

2.8 Collision Processes

Collisions mediate the transfer of energy and momentum between various species in a plasma, and as we shall see later, allow a treatment of highly ionized plasma as a single conducting fluid with resistivity determined by electron-ion collisions.

Collisions which conserve total kinetic energy are called *elastic*. Some examples are atom-atom, electron-atom, ion-atom (charge exchange) etc. In *inelastic collisions*, there is some exchange between potential and kinetic energies of the system. Examples are electron-impact ionization/excitation, collisions with surfaces etc. In this section, we consider both types of collision process, with particular emphasis on Coulomb collisions between charged particles, this being the dominant process in a plasma.

A simple expression for the collision term on the right of Eq. (2.5) is the Krook collision term

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\left(\frac{f - f_{\text{M}}}{\tau}\right) \tag{2.65}$$

where f_{M} is the Maxwellian towards which the system is tending and τ is the mean (constant) collision time. The model integrates to give

$$f(t) = f_{\text{M}} + (f(0) - f_{\text{M}}) \exp(-t/\tau). \tag{2.66}$$

The model is not very good if the masses of the species are very different.

2.8.1 Fokker-Planck equation

2.8.2 Mean free path and cross-section

To properly treat the physics of collisions we need to introduce the concept of mean free path - a measure of the likelihood of a collision event. Imagine electrons impinging on a box of neutral gas of cross-sectional area A (see Fig 2.8).

If there are n_{n} atoms m^{-3} in volume element $A\mathrm{d}x$ the total area of atoms in the volume (viewed along the x-axis) is $n_{\text{n}}A\mathrm{d}x\sigma$ where σ is the cross-section and $n_{\text{n}}A\mathrm{d}x\sigma \ll A$ so that there is no "shadowing". The fraction of particles making a collision is thus $n_{\text{n}}A\mathrm{d}x\sigma/A$ - the fraction of the cross-section blocked by atoms. If Γ is the incident particle flux, the emerging flux is $\Gamma' = \Gamma(1 - n_{\text{n}}\sigma\mathrm{d}x)$ and the change of Γ with distance is described by

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x} = -n_{\text{n}}\sigma\Gamma. \tag{2.67}$$

The solution is

$$\begin{aligned} \Gamma &= \Gamma_0 \exp(-x/\lambda_{\text{mfp}}) \\ \lambda_{\text{mfp}} &= \frac{1}{n_{\text{n}}\sigma} \end{aligned} \tag{2.68}$$

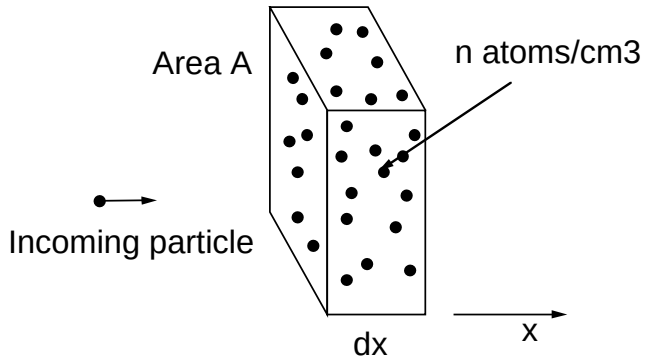

Figure 2.8: Particles moving from the left and impinging on a gas undergo collisions.

where λ_{mfp} is the *mean free path* for collisions characterised by the cross-section σ. The physics of the interaction is carried by σ, the rest is geometry.

The mean free time between collisions, or *collision time* for particles of velocity v is $\tau = \lambda_{\mathrm{mfp}}/v$ and the *collision frequency* is $\nu = \tau^{-1} = v/\lambda_{\mathrm{mfp}} = n_{\mathrm{n}}\sigma v$. Averaging over all of the velocities in the distribution gives the average collision frequency

$$\nu = n_{\mathrm{n}}\overline{\sigma v} \tag{2.69}$$

where we have allowed for the fact that σ can be energy dependent as we shall see below.

2.8.3 Coulomb collisions

Coulomb collisions between free particles in a plasma is an elastic process. Let us consider the Coulomb force between two test charges q and Q

$$F = \frac{qQ}{4\pi\varepsilon_0 r^2} \equiv \frac{C}{r^2} \tag{2.70}$$

This is a long range force and the cross-section for inetraction of isolated charges is infinity! It is quite different from elastic "hard-sphere" encounters such as that which can occur between electrons and neutrals for example. In a plasma, however, the Debye shielding limits the range of the force so that an effective cross-section can be found. Nevertheless, because of the nature of the force, the most frequent Coulomb deflections result in only a small deviation of the particle path before it encounters another free charge. To produce an effective 90° scattering of the particle (and hence momentum transfer) requires an accumulation of many such glancing collisions. The collision cross-section is then calculated by the statistical analysis of many such small-angle encounters.

Consider the force Eq. (2.70) on an electron as it follows the unperturbed path shown in Fig. 2.9 In this picture, only the perpendicular component matters because the parallel compoent of the force reverses direction after q passes Q. Thus $F_\perp/F = b/r$ or

$$F_\perp = Cb/r^3 = \frac{cb}{(x^2+b^2)^{3/2}}$$

where b is the impact parameter for the interaction. Since x (the parallel coordinate) changes with time, we integrate along the path to obtain the net perpendicular impulse delivered to q

$$\delta(mv_\perp) = \int_{-\infty}^{\infty} F_\perp \mathrm{d}t = cb\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+b^2)^{3/2}}\frac{\mathrm{d}t}{\mathrm{d}x}$$

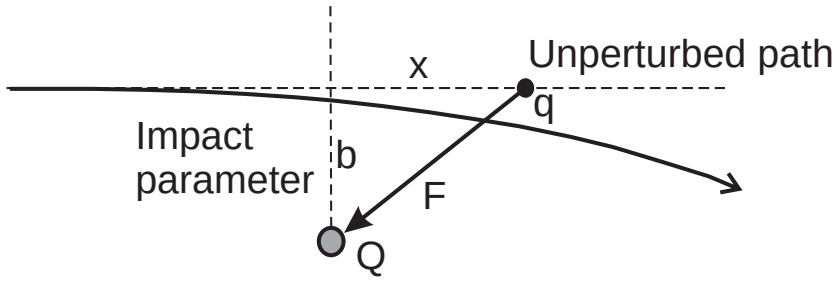

Figure 2.9: The trajectory taken by an electron as it makes a glancing impact with a massive test charge Q

Now

$$\begin{aligned}\int \frac{\mathrm{d}x}{(x^2+b^2)^{3/2}} &= \frac{x}{b^2(x^2+b^2)^{1/2}} \\ &\to -1/b^2 \qquad x \to -\infty \\ &\to 1/b^2 \qquad x \to \infty\end{aligned}$$

so that

$$\delta v_\perp = \frac{2c}{mvb}. \tag{2.71}$$

We consider a statistical average over a random distribution of such small angle collisions. For a random walk with step length δs, the total displacement after N steps is $\Delta s = \sqrt{N}\delta s$ (i.e. $(\Delta s)^2 = \sum(\delta s)^2$) where N is the number of steps. The total change in velocity is thus

$$(\Delta v_\perp)^2 = N(\delta v_\perp)^2. \tag{2.72}$$

Now integrate over the range of impact parameters b to estimate the number of glancing collisions in time t. Small angle colisions are much less likely because of the geometrical effect

$$N(b) = n(2\pi b \mathrm{d}b)vt. \tag{2.73}$$

We combine equations (2.71), (2.72) and (2.73) and integrate over impact parameter:

$$(\Delta v_\perp)^2 = \frac{8\pi n C^2 t}{m^2 v} \int_{b_{\min}}^{b_{\max}} \frac{\mathrm{d}b}{b}. \tag{2.74}$$

$b_{\min}$ is the closest approach that satisfies the small deflection hypothesis. We obtain this be setting $\delta v_\perp = v$

$$b_{\min} = \frac{2c}{mv^2} = \frac{2qQ}{4\pi\varepsilon_0 m v^2}. \tag{2.75}$$

Outside λ_{D} the charge Q is not felt. We thus take $b_{\max} = \lambda_{\mathrm{D}}$ and

$$\int_{b_{\min}}^{b_{\max}} \frac{\mathrm{d}b}{b} = \ln\left(\frac{\lambda_{\mathrm{D}}}{b_{\min}}\right) = \ln\Lambda.$$

$\ln\Lambda$ is called the Coulomb logarithm and is a slowly varying function of electron density and temperature. For fusion plasma $\ln\Lambda \sim 6 - 16$. One usually sets $\ln\Lambda = 10$ in quantitative estimations.

We are finally in a position to find the elapsed time necessary for a net 90° deflection i.e. $(\Delta v_\perp)^2 = v^2$. Using Eq. (2.74)

$$(\Delta v_\perp)^2 = v^2 = \frac{8\pi n c^2 t}{m^2 v} \ln\Lambda$$

and

$$\nu_{90} = 1/t = \frac{8\pi n q^2 Q^2 \ln\Lambda}{16\pi^2 \varepsilon_0^2 m^2 v^3}.$$

For electron-ion encounters, $q = -e$ and $Q = Ze$ so

$$\nu_{90\mathrm{ei}} \equiv \nu_{\mathrm{ei}} = \frac{n_{\mathrm{i}} Z^2 e^4 \ln\Lambda}{2\pi\varepsilon_0^2 m^2 v^3} \tag{2.76}$$

and $\nu_{\mathrm{ei}} = n_{\mathrm{i}}\sigma_{\mathrm{ei}} v$ implies that

$$\sigma_{\mathrm{ei}} = \frac{Z^2 e^4 \ln\Lambda}{2\pi\varepsilon_0^2 m^2 v^4} \tag{2.77}$$

Let's review this derivation

- Perpendicular impulse $\sim 1/vb$
- Angular deflection $\Delta\theta \sim \delta v_\perp / v \sim 1/v^2$

- Random walk to scatter one radian $(\Delta v/v = 1) \sim 1/(\Delta\theta)^2 \rightarrow \sigma \sim 1/v^4$
- Integrate over $b \rightarrow \ln\Lambda$.

The dependence $\sigma \sim 1/v^4$ has very important ramifications. A high temperature plasma is essentially collisionless. This means that plasma resistance decreases as temperature increases. In some circumstances populations of particles can be continually accelerated, losing energy only through synchrotron radiation (for example runaway electrons in a tokamak).

Energy transfer in electron-ion collisions

A pervading theme in plasma physics is $m_\mathrm{e} \ll m_\mathrm{i}$. This has consequences for collisions between the species in that we expect very little energy transfer between the species. To illustrate this, consider such a collision in the centre-of-mass frame (a direct hit).

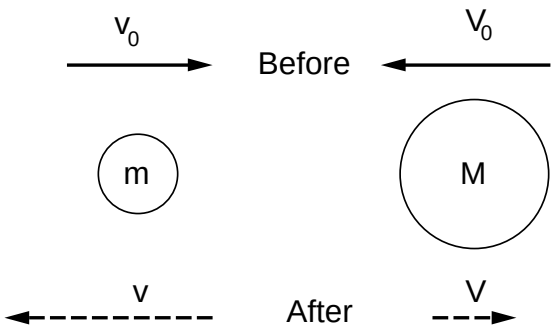

Figure 2.10: Collison between ion and electron in centre of mass frame.

$$\begin{aligned} mv_0 + MV_0 &= 0 = mv + MV \qquad \text{momentum} \\ mv_0^2 + MV_0^2 &= mv^2 + MV^2 \qquad \text{energy} \\ mv_0^2\left(1+\frac{m}{M}\right) &= mv^2\left(1+\frac{m}{M}\right) \qquad \text{eliminate V} \\ \Rightarrow v &= \pm v_0 \quad \text{and} \quad V = \pm V_0. \end{aligned} \tag{2.78}$$

Now translate to frame in which ion M is at rest (initially). In this frame the initial electron energy is

$$E_{\mathrm{e}0} = \frac{1}{2}m(v_0 - V_0)^2 = \frac{1}{2}mv_0^2\left(1+\frac{m}{M}\right)^2$$

and the final ion energy is

$$E_\mathrm{i} = \frac{1}{2}M(2V_0)^2 = 2M\left(\frac{m^2v_0^2}{M^2}\right).$$

Their ratio is given by

$$\begin{aligned}\frac{\text{ion (final)}}{\text{electron (initial)}} &= \frac{2m^2v_0^2}{\frac{1}{2}mv_0^2M}\left(1+\frac{m}{M}\right)^2 \\ &\approx \frac{4m}{M} \qquad (2.79)\end{aligned}$$

and thus

$$\text{e} \to \text{i} \qquad \Delta E \sim \frac{4m_\text{e}E_\text{e}}{m_\text{i}} \qquad (2.80)$$

$$\text{e} \to \text{e} \qquad \Delta E \sim E_\text{e} \qquad (2.81)$$

$$\text{i} \to \text{i} \qquad \Delta E \sim E_\text{i} \qquad (2.82)$$

For glancing collisions the energy transfer between ions and electrons is even less. Coulomb collisions result in very poor energy transfer between electrons and ions. The rate of energy transfer is roughly (m_e/m_i) slower than the e-i collision frequency . On the other hand, energy transfer rate and collision frequency are the same for collisions between ions $\nu_\text{ii} \sim \nu_\text{ei}\sqrt{m_\text{e}/m_\text{i}}$:

$$\frac{\nu_\text{ii}}{\nu_\text{ei}} = \frac{m_\text{e}^2v_\text{e}^3}{m_\text{i}^2v_\text{i}^3} = \left(\frac{m_\text{e}}{m_\text{i}}\right)^{1/2}\left(\frac{m_\text{e}v_\text{e}^2}{m_\text{i}v_\text{i}^2}\right)^{3/2} = \left(\frac{m_\text{e}}{m_\text{i}}\right)^{1/2} \qquad \text{for } T_\text{e} = T_\text{i}$$

2.8.4 Light scattering from charged particles

2.8.5 Nuclear fusion

This is an inelastic process where the final kinetic energy of the system is much greater than the initial kinetic energy. This excess energy can be harnessed to produce power. For example, Fig. 1.7 shows schematically the Deuterium-Tritium reaction upon which present-day fusion reactors are based. To overcome the Coulomb repulsion requires 400 keV of kinetic energy. Upon fusing, the two isotopes transmute to a fast neutron and a helium nucleus (α-particle). Even in a plasma whose temperature is only 10 keV, there are sufficient particles in the wings of the distribution to sustain a fusion reactor. The reason is that in a fusion reaction, the energy release is a huge 17.6 MeV - more than enough to account for the relatively small number of fusing ions.

Using energy and momentum conservation, we can estimate the relative energies of the reaction by-products. In the centre-of-mass frame we have

$$\begin{aligned} m_1v_1 &= m_2v_2 \\ \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_1v_1^2 &= 17.6 \text{ MeV}\end{aligned}$$

where subscripts 1 and 2 refer respectively to the α-particle and the neutron. Eliminating v_1 gives

$$\frac{1}{2}m_2 v_2^2 \left(1 + \frac{m_2}{m_1}\right) = 17.6 \text{ MeV}.$$

Substituting $m_2/m_1 = 1/4$ shows that 14.1 MeV is carried by the neutron which escapes the confining magnetic field to deliver useful energy elsewhere. The fast α (3.5 MeV) is confined by the magnetic field where it delivers its energy to the deuterium and tritium ions.

For the D-T reaction in a high temperature plasma with $n \sim 1 \times 10^{20}$ m^{-3} and $\sigma \sim 10^{-29}$ m^{-2} at 100 keV where $v \sim 5 \times 10^6$ m/s, the mean free path for a fusion collision is

$$\lambda_{\text{mfp}} = \frac{1}{n\sigma} = 10^9 \text{ m}$$

and $\tau = \lambda_{\text{mfp}}/v \approx 200$ s. In other words, D and T have to be confined for 200 s and travel a million km without hitting the container walls! The situation is mitigated by the fact that 17.6 MeV $\gg$ 10 keV so that not all particles need to fuse in order to achieve a net energy gain.

2.8.6 Photoionization and excitation

To understand these processes, we must first look at the atomic physics of the H-atom. This is governed by the Coulomb force. Taking a semi-classical approach, we may equate the centripetal force with the Coulomb attraction

$$F = \frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}.$$

The kinetic energy of the electron is $\frac{1}{2}mv^2 = e^2/(8\pi\varepsilon_0 r)$ and the potential energy is $-e^2/(4\pi\varepsilon_0 r)$ so that the total energy of the atomic system is KE+PE $= -e^2/(8\pi\varepsilon_0 r)$ (it is a bound system).

Given that electrons are bound in states having de Broglie wavelength satisfying $n\lambda = 2\pi r$, we may obtain for the energies of the bound states

$$E_n = -13.6 \text{ eV}/n^2 \qquad n = 1, 2, 3, \ldots$$

(To see this, we write $n\hbar = \hbar kr = mvr$, or $r = n\hbar/mv$ and substitute into the expression for the kinetic energy to obtain $r = (n^2/Z)a_0$ where a_0 is the Bohr radius. This is finally subsituted into the expression for the total energy of the atom.) It is clear that the energy required to ionize the H atom is 13.6 eV and to excite from the ground state is $E_2 - E_1 = 10.2$ eV. Photons of these energies reside in the deep ultarviolet of the spectrum. Since the photon energy is used to disrupt a bound system, this is an inelastic process.

2.8.7 Electron impact ionization

The area of the hydrogen atom is $\pi a_0^2 = 0.88 \times 10^{-20}\text{m}^{-2}$. If an electron comes within a radius a_0 of the atom and has energy in excess of 13.6 eV, ionization will probably result. The behaviour of the electron impact ioization cross-section as a function of electron energy is shown in Fig. 2.11.

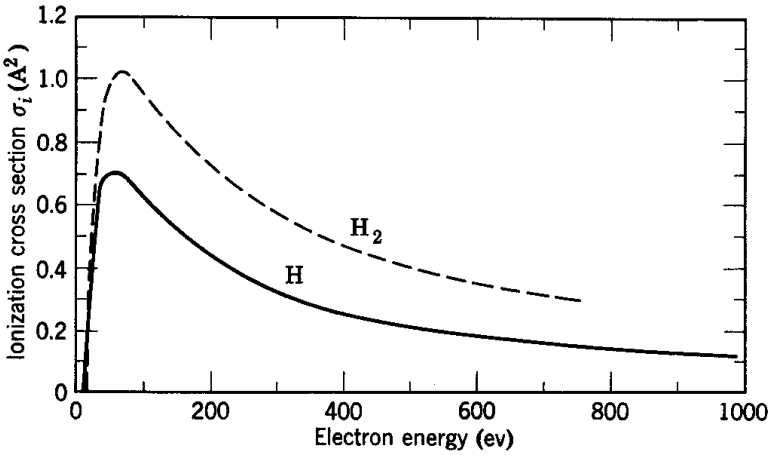

Figure 2.11: The electron impact ionization cross-section for hydrogen as a function of electron energy. Note the turn-on at 13.6 eV. [2]

The cross-section falls for energies greater than 100 eV since the electron does not spend much time in the vicinity of the atom. The cross section, averaged over the distribution of electron velocities, can be used to estimate the electron mean free path for a given neutral density. Again this is an inelastic process.

2.8.8 Collisions with surfaces

Reactions with surfaces are complex. Gases are readily adsorbed on a clean surface, sticking with probability 0.3 to 0.5 to form a "monolayer" surface. The binding energy is a strong function of the type of gas however, being smallest for the noble gases. Heating at temperatures $\gtrsim 2000°$ will usually produce an atomically clean surface in $\lesssim 1$ sec.

Secondary emission by electrons

Electrons striking a surface can, if their energy exceeds that of the surface work function, eject other electrons. The emission coefficient falls at high energies because the incident electron penetrates too deeply and secondaries cannot escape.

At low energies, the released electron energy is less than the work function and it cannot escape. At intermediate energies we can have $\delta > 1$ (gain). The shape of the plot of secondary emission coefficient δ (number of electrons emitted for each incident electron) versus incident energy is similar for all materials. Because of the mass difference, electrons have low probability of ejecting atoms.

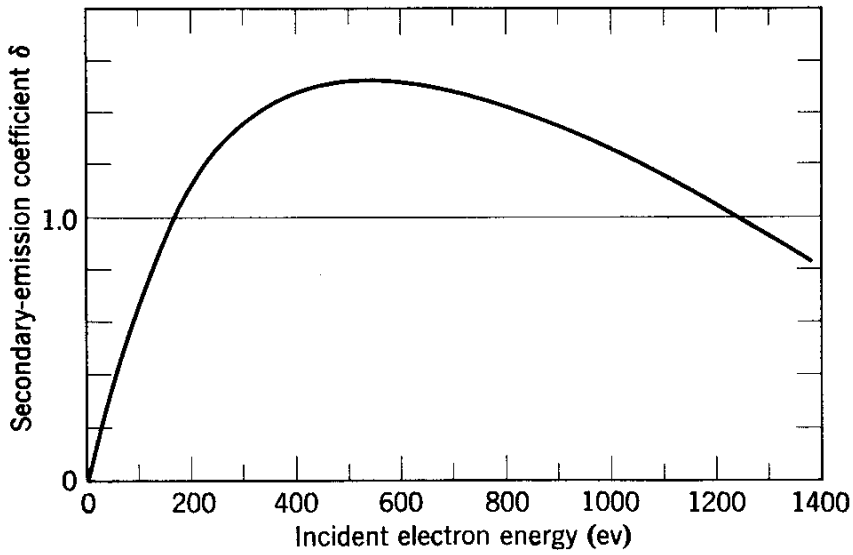

Figure 2.12: Electron secondary emission for normal incidence on a typical metal surface [2]

Secondary electron emission by ions

The number of secondary electrons emitted per positive ion is denoted γ_{i}. Typically $\gamma_{\mathrm{i}} < 0.3$ and is roughly independent of the ion speed.

2.9 Fluid Equations

Just as the velocity moments of the distribution function give important macroscopic variables, so the velocity moments of the plasma kinetic equation (Boltzmann equation Eq. (2.5)) give the equations that describe the time evolution of these macroscopic parameters. Because these equations (apart from the Lorentz force) are identical with the continuum hydrodynamic equations, the theories using the low-order moments are called *fluid theories*. Remember that the fluid equations can be applied separately to each of the species that constitute the plasma.

2.9.1 Continuity—particle conservation

The *equation of continuity* is derived from the zeroth velocity moment of the Boltzmann equation. Its analogue for the particle distribution function is just the number density. It is not surprising that continuity describes conservation of particle number. We shall derive the moment equations only for the zeroth moment and simply state the result for the higher order moments of the Boltzmann equation.

Since $\boldsymbol{v}^0 = 1$, the zeroth order moment implies a simple integration over velocity space to obtain

$$\int \frac{\partial f}{\partial t}\mathrm{d}\boldsymbol{v} + \int \boldsymbol{v}.\nabla_r f\mathrm{d}\boldsymbol{v} + \frac{q}{m}\int(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}).\nabla_v f\mathrm{d}\boldsymbol{v} = \int\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}}\mathrm{d}\boldsymbol{v} \tag{2.83}$$

The first term gives

$$\int \frac{\partial f}{\partial t}\mathrm{d}\boldsymbol{v} = \frac{\partial}{\partial t}\int f\mathrm{d}\boldsymbol{v} = \frac{\partial n}{\partial t}. \tag{2.84}$$

Since $\boldsymbol{v}$ is independent of $\boldsymbol{r}$ (independent variables) the second term gives

$$\int \boldsymbol{v}.\nabla_r f\mathrm{d}\boldsymbol{v} = \boldsymbol{\nabla}_r.\int \boldsymbol{v}f\mathrm{d}\boldsymbol{v} = \nabla.(n\bar{\boldsymbol{v}}) \equiv \nabla.(n\boldsymbol{u}) \tag{2.85}$$

where we have used Eq. (2.14) and defined $\boldsymbol{u}$ as the "fluid" velocity. The term involving $\boldsymbol{E}$ in the third integral vanishes:

$$\int \boldsymbol{E}.\frac{\partial f}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}}.(f\boldsymbol{E})\mathrm{d}\boldsymbol{v} = \int_{S_\infty} f\boldsymbol{E}.\mathrm{d}\boldsymbol{S} = 0 \tag{2.86}$$

where the surface S_∞ is the v-space surface at $v \to \infty$. The first step is valid because $\boldsymbol{E}$ is independent of $\boldsymbol{v}$ while the second is an application of Gauss' vector integral theorem. This last integral vanishes because f vanishes faster than v^{-2} (area varies as $4\pi v^2$) as $v \to \infty$ (i.e. finite energy system). The $\boldsymbol{v}\times\boldsymbol{B}$ component can be written

$$\int(\boldsymbol{v}\times\boldsymbol{B}).\frac{\partial f}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}}.(f\boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v} - \int f\frac{\partial}{\partial \boldsymbol{v}}\times(\boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v} = 0 \tag{2.87}$$

The second term can be transformed to a surface integral which again vaishes at infinity. Because $\partial/\partial\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{B}$ are mutually perpendicular, the second right hand term also vanishes. Finally, the 4th term of Eq. (2.83) vanishes because collisions cannot change the number of particles (at least for warm plasmas, where recombination can be ignored). The final result is

$$\frac{\partial n}{\partial t} + \nabla.(n\boldsymbol{u}) = 0 \tag{2.88}$$

which expresses conservation of particles . Multiplying by m or q yields conservation of mass or charge.

Note that taking the zeroth moment of the Boltzmann equation has introduced a higher order moment $\boldsymbol{u}$ — the first velocity moment of f. Its description will require taking the first order moment of the Boltzmann equation. As we shall see, this will in turn generate second order moments and so on. This is known as the closure problem. At some stage, the sequence must be terminated by some reasonable procedure. Usually this is effected by setting the third velocity moment of f (describing the thermal conductivity) to zero.

2.9.2 Fluid equation of motion

The first moment equation (multiply by $m\boldsymbol{v}$ the particle momentum and integrate) expresses conservation of momentum. The resulting *equation of motion* is given by

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla.\overleftrightarrow{P} + P_{\text{coll}} \tag{2.89}$$

where the left hand side will be recognized as proportional to the convective derivative of the average velocity $\boldsymbol{u}$. The right side includes the Lorentz force, the divergence of the pressure tensor (a second order quantity) and the exchange of momentum due to collisions with other species in the plasma:

$$P_{\text{coll}} = \int m\boldsymbol{v}\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \mathrm{d}\boldsymbol{v} = \left[\frac{\partial}{\partial t}\int m\boldsymbol{v} f \mathrm{d}\boldsymbol{v}\right]_{\text{coll}} \tag{2.90}$$

Collisions between like particles cannot produce a net change of momentum of that species.

Pressure tensor

The elements of the pressure tensor are defined by Eq. (2.16) such that $P_{ij} = mn\overline{v_i v_j}$. For an isotropic Maxwellian distribution,

$$\overleftrightarrow{P} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \tag{2.91}$$

and $\nabla.\overleftrightarrow{P} = \nabla P$ where $P = mn\overline{v_r^2}/3 = nkT$ is the scalar pressure. In the presence of a magnetic field, isotropy can be lost and the species may assume distict temperatures along and across the magnetic field so that $p_\parallel = nkT_\parallel$ and $p_\perp = nkT_\perp$ with

$$\overleftrightarrow{P} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix}$$

where it is customary to take the z direction to conincide with the direction of $\boldsymbol{B}$. Note that the pressure is still isotropic in the plane normal to $\boldsymbol{B}$.

The collision term

If a species α (for example electrons) is interacting with another via collisions, the momentum lost per collision will be proportional to the relative fluid velocity $\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta$ where β denotes the "background" fluid (for example ions or neutrals). If the mean free time between collisions τ is approximately constant, the resulting frictional drag term can be written as

$$P_{\mathrm{ei}} = -\frac{m_{\mathrm{e}} n_{\mathrm{e}} (\boldsymbol{u}_{\mathrm{e}} - \boldsymbol{u}_{\mathrm{i}})}{\tau_{\mathrm{ei}}} \tag{2.92}$$

where P_{ei} is the force felt by the electron fluid flowing with velocity $\boldsymbol{u}_{\mathrm{e}} - \boldsymbol{u}_{\mathrm{i}}$ relative to the background (in this case ion) fluid. By conservation of momentum, the momentum lost (gained) through collisions by the electrons on ions must equal the momentum gained (lost) by the ions. Thus

$$P_{\mathrm{ei}} = -P_{\mathrm{ie}}. \tag{2.93}$$

What does this say about ν_{ie} compared with ν_{ei}?

2.9.3 Equation of state

The second moment equation describes conservation of energy. It is derived by multiplying the Boltzmann equation by $mv^2/2$ and integrating over velocity. The result is

$$\frac{\partial}{\partial t}\left(\frac{nm}{2}v^2\right) + \boldsymbol{\nabla}_r \boldsymbol{.} \left(\frac{nm}{2}\overline{\boldsymbol{v}v^2}\right) = \boldsymbol{j}\boldsymbol{.}\boldsymbol{E} + \int \mathrm{d}\boldsymbol{v} \frac{nm}{v^2}\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}}. \tag{2.94}$$

The left hand side terms represent the change of energy density with time and the energy flow across the surface bounding the volume element of interest. The sum of these two terms equals the electric power dissipated in the plasma plus any energy that might be generated per unit time through collisions. The magnetic field does no work because it always acts in a direction perpendicular to the particle velocity.

The equation can be dramatically simplified under the assumption that the distribution function is a drifting Maxwellian ($\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}_r$ where $\boldsymbol{v}_r$ is the random or thermal velocity component) and provided collisions do not create new particles (ionization) or change the temperature (thermal conductivity = 0). This corresponds to a system undergoing *adiabatic changes* (no heat flow). The second moment equation then becomes the *equation of state*:

$$\frac{\mathrm{d}p}{\mathrm{d}n} = \frac{5p}{3n} \tag{2.95}$$

On integration this gives

$$p = p_0 n^{5/3}. \tag{2.96}$$

More generally, we can write

$$p = C\rho^{\gamma} \tag{2.97}$$

where C is a constant,

$$\gamma = (2+N)/N \tag{2.98}$$

is the ratio of specific heats (C_p/C_V) and N is the number of degrees of freedom. In 3-D, $N = 3$, $\gamma = 5/3$ and we recover Eq. (2.96). Equation (2.97) can also be written as

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n} \tag{2.99}$$

$$\text{or} \quad \nabla p = \gamma k_{\mathrm{B}} T \nabla n \tag{2.100}$$

For the *isothermal* case (T=constant, thermal conductivity infinite) we have

$$\nabla p = \nabla(n k_{\mathrm{B}} T) = k_{\mathrm{B}} T \nabla n \tag{2.101}$$

and $\gamma = 1$

2.9.4 The complete set of fluid equations

Let us define

$$\sigma = n_{\mathrm{i}} q_{\mathrm{i}} + n_{\mathrm{e}} q_{\mathrm{e}} \tag{2.102}$$

$$\boldsymbol{j} = n_{\mathrm{i}} q_{\mathrm{i}} \boldsymbol{u}_{\mathrm{i}} + n_{\mathrm{e}} q_{\mathrm{e}} \boldsymbol{u}_{\mathrm{e}} \tag{2.103}$$

as net charge and current densities respecticely. Ignoring collisions and viscosity (i.e. the pressure tensor $\rightarrow$ scalar pressure p) we have fluid equations for species $j = \mathrm{i}, \mathrm{e}$

$$\frac{\partial n_j}{\partial t} + \nabla.(n_j \boldsymbol{u}_j) = 0 \tag{2.104}$$

$$m_j n_j \left[\frac{\partial \boldsymbol{u}_j}{\partial t} + (\boldsymbol{u}_j.\nabla)\boldsymbol{u}_j \right] = q_j n_j (\boldsymbol{E} + \boldsymbol{u}_j \times \boldsymbol{B}) - \nabla p_j \tag{2.105}$$

$$p_j = C_j n_j^{\gamma_j} \tag{2.106}$$

supplemented by Maxwell's equations for the field

$$\nabla.\boldsymbol{E} = \frac{\sigma}{\varepsilon_0} \tag{2.107}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.108}$$

$$\nabla.\boldsymbol{B} = 0 \tag{2.109}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2.110}$$

These two sets of equations provide 16 independent equations for the 16 unknowns n_{i}, n_{e}, p_{i}, p_{e}, $\boldsymbol{u}_{\mathrm{i}}$, $\boldsymbol{u}_{\mathrm{e}}$, $\boldsymbol{E}$ and $\boldsymbol{B}$. Two of Maxwell's equations are redundant. In this course we shall examine and employ these equations to solve a range of simple plasma physics problems.

2.9.5 The plasma approximation

In many plasma applications it is usually possible to assume $n_\mathrm{i} = n_\mathrm{e}$ and $\nabla.\boldsymbol{E} \neq 0$ at the same time. On time scales slow enough for the ions and electrons to move, Poisson's equation can be replaced by $n_\mathrm{i} = n_\mathrm{e}$ (i.e. quasi-neutrality). Poisson's equation holds for electrostatic applications:

$$\frac{\partial \boldsymbol{B}}{\partial t} = 0 \;\Rightarrow\; \nabla\times\boldsymbol{E} = 0 \;\Rightarrow\; \boldsymbol{E} = -\nabla\phi \;\Rightarrow\; \nabla^2\phi = -\sigma/\varepsilon_0.$$

The electric field can be found from the equations of motion. If necessary, the charge density can then be found from Poisson's equation. but it is usually very small. This, so-called "plasma approximation" is only valid for low frequency applications. For high frequency electron waves, for example, where charges can separate, it is not a valid approximation.

Problems

Problem 2.1 A chamber of volume 0.5 m^3 is filled with hydrogen at a pressure of 10 Pa at room temperature ($20°$. Calculate

(a) The number of H_2 molecules in the chamber

(b) the mass of the gas in the chamber (proton mass $= 1.67 \times 10^{27}$ kg)

(c) the average kinetic energy of the molecules

(d) the RMS speed of the molecules

(e) the energy in Joules required to heat the gas to 100,000 K, given that it takes 2 eV to dissociate a H_2 molecule into H atoms and 13.6 eV to ionize each atom. Assume that $T_\mathrm{e} = T_\mathrm{i}$ and that the plasma is fully ionized.

(f) the total plasma pressure when $T_\mathrm{e} = T_\mathrm{i} = 10^5$K

(g) the RMS speeds of the ions and electrons when $T_\mathrm{e} = T_\mathrm{i} = 10^5$K

(h) The electron plasma frequency ω_{pe}, Debye length λ_D, number of particles in the Debye sphere N_D and the ion-electron collision frequency ν_ei.

Problem 2.2 In the two dimensional phase space (x, v), show the trajectory of a particle at rest, moving with constant velocity, accelerating at a constant rate, executing simple harmonic motion and damped simple harmonic motion.

Problem 2.3 For any distribution f which is isotropic about its drift velocity $\overline{\boldsymbol{v}}$, prove that

$$\overline{v_{r\alpha} v_{r\beta}} = v_r^2 \delta_{\alpha\beta}/3$$

where $\delta_{\alpha\beta}$ is the Kronecker delta

Problem 2.4 If a particle of mass m, velocity v, charge q approaches a massive particle ($M \to \infty$) of like charge q_0, show by energy considerations that the distance of closest approach is $2qq_0/(4\pi\varepsilon_0 m v^2)$.

Problem 2.5 Calculate the cross-section σ_{90} for an electron Coulomb collsion with an ion which results in at least a $90°$ deflection of the electron from its initial trajectory. (HINT: This cross-section is πb^2 where b is the maximum impact parameter for such a collision). Show that the ratio $\sigma_{ei}/\sigma_{90} = 2\log_e \Lambda$ where σ_{ei} is the collsion cross section calculated in the notes for multiple small angle Coulomb collisions. Comment on the significance of this result.

Problem 2.6 Consider the motion of charged particles, in one dimension only, in an electric potential $\phi(x)$. Show, by direct substitution, that a function of the form $f = f(mv^2/2 + q\phi)$ is a solution of the Boltzmann equation under steady state conditions.

Problem 2.7 Since the Maxwellian velocity distribution is isotropic (independent of velocity direction) it is of interest to define a distribution of speeds $v = \mid \boldsymbol{v} \mid$. That is, we want a new function $g(v)$ which describes the numer of particles with speeds between v and $v + dv$. Show that the desired distribution of speeds (in 3-d) is

$$g(v) = 4\pi v^2 n (\frac{m}{2\pi kT})^{3/2} \exp\left(-mv^2/2kT\right)$$

Plot this distribution and calculate the most probable particle speed, or in other words, the speed for which $g(v)$ is a maximum.

Problem 2.8 Consider the following two-dimensional Maxwellian distribution function:

$$f(v_x, v_y) = n_0 (\frac{m}{2\pi kT}) \exp\left[-m(v_x^2 + v_y^2)/2kT\right]$$

(a) Verify that n_0 represents correctly the particle number density, that is, the number of particles per unit area. (b) Draw, in 3-d, the surface for this distribution function, plotting f in terms of v_x and v_y. Sketch on this surface curves of constant v_x, constant v_y and constant f.

Problem 2.9 The entropy of a system can be expressed in terms of the distribution function as

$$S = -k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{v}\; f\, f$$

Show that the total time derivative of the entropy for a system which obeys the collisionless Boltzmann equtaion is zero.