General Theory of Solitons

Nail Akhmediev

Australian Photonics CRC, Optical Sciences Centre,
Research School of Physical Sciences and Engineering,
The Australian National University, Canberra, ACT 0200, Australia;

I. INTRODUCTION

A soliton is a concept which describes various physical phenomena ranging from solitary waves on a water surface to ultra-short optical pulses in an optical fiber. The main feature of solitons is that they can propagate long distances without visible changes. From a mathematical point of view, a soliton is a localized solution of a partial differential equation describing the evolution of a nonlinear system with an infinite number of degrees of freedom. Solitons are usually attributed to integrable systems. In this instance, solitons remain unchanged during interactions, apart from a phase shift. They can be viewed as ‘modes’ of the system, and, along with radiation modes, they can be used to solve initial-value problems using a nonlinear superposition of the modes [1]. However, in the recent years, the notion of solitons has been extended to various systems which are not necessarily integrable. Following this new trend, we extend the notion of solitons and include a wider range of systems in our treatment. These include dissipative systems, Hamiltonian systems and a particular case of them, viz. integrable systems.

Let us consider a trivial example from classical mechanics - a system with one degree of freedom, namely, a pendulum [2]. A transition to a higher number of degrees of freedom could be made by taking several identical coupled pendula. Then we can make the following classification (see Fig.1). When the amplitude of the oscillations is small, the system can be approximated by a linear oscillator (Fig.1a). If we had several coupled oscillators, the general solutions could be written as a linear superposition of normal modes. If the amplitude of the oscillations is not small, then the oscillations are nonlinear (Fig.1b). The exact solution for this case does exist and it can be written in terms of elliptic Jacobi functions. However, for a coupled set of equations, the solution cannot be written as a linear superposition of modes. Finally, when losses are included (Fig.1c), the system becomes dissipative. The oscillations are undamped only if there is an external force pumping energy into the pendulum. For a system of coupled equations, the solution can only be found numerically in most cases.

A similar classification can be made in the case of systems with an infinite number of degrees of freedom. To be definite, we will mainly (but not exclusively) consider an equation which is widely-known as the complex cubic-quintic Ginzburg-Landau equation (CGLE):

$$i\psi_t + \frac{D}{2} \psi_{xx} + |\psi|^2 \psi = i\delta \psi + ic|\psi|^2 \psi + i\beta \psi_{xx} + i\mu |\psi|^4 \psi - \nu |\psi|^4 \psi,$$

(1)
FIG. 1. A classification of a dynamical system with one degree of freedom (a pendulum). (a) The amplitude is small (linear oscillations). (b) The amplitude is not small (nonlinear oscillations). The system is Hamiltonian. (c) Dissipative system. The pendulum is in a liquid. To maintain the oscillations, the pendulum has to have an external periodic force.

The physical meaning of the various terms in this equation can be found elsewhere [12]. Here, it is enough to know that, by equating various terms on the right-hand-side of (1) to zero, we can have an equation which describes, as particular cases, integrable, Hamiltonian or dissipative systems. In particular, the system is Hamiltonian when $\delta = \epsilon = \beta = \mu = 0$. The system is integrable, when, in addition, $\nu = 0$. Then it follows that Hamiltonian systems can be considered as a subclass of dissipative ones, while integrable systems can be viewed as a subclass of Hamiltonian ones. This classification is illustrated in Fig. 2.

FIG. 2. A rough classification of nonlinear systems with an infinite number of degrees of freedom admitting soliton solutions.

II. INTEGRABLE SYSTEMS

Let us start our analysis from the integrable case. The classical nonlinear equation found to be integrable [1] is the nonlinear Schrödinger equation (NLSE):
\[ i\psi_{\xi} + \frac{1}{2}\psi_{\tau\tau} + |\psi|^2\psi = 0, \]  

(2)

This is the standard way of writing this equation in the mathematical literature. The actual meaning of the independent variables \( \xi \) and \( \tau \) differs for different problems. For spatial solitons, both variables are spatial, so no confusion can arise. For problems related to dispersive waves, and, in particular, to optical fibers, \( \xi \) is the distance along the fiber, and \( \tau \) is the retarded time, i.e. a co-ordinate moving with the group velocity of the pulse. We will use this notation throughout this paper, bearing in mind that the solutions relate to various physically different situations. This form of the NLSE is for anomalous dispersion in the temporal (fiber) case and for self-focusing in the spatial case. In this section, we shall briefly describe methods for solving the NLSE and linear equations related to it. Clearly, we cannot cover the whole body of knowledge which exists today, and we refer the reader to existing books [3–7] for details.

The NLSE admits an infinite number of exact solutions. A complicated solution can be represented as a nonlinear superposition of simple ones. The structure of the superposition is defined by the spectrum of the inverse scattering technique (IST). One of the main points of interest is that (2) admits the soliton solution

\[ \psi = \frac{e^{i\xi/2}}{cosh \tau}. \]  

(3)

This simple solution can be extended to include more parameters by using the symmetries of the NLSE. For example, if we know a solution of the NLSE, \( \psi(\tau, \xi) \), then we can obtain a one-parameter family of solutions by the simple transformation

\[ \psi'(\tau, \xi|q) = q \psi(q\tau, q^2\xi), \]  

(4)

where \( q \) is a real parameter. This transformation can be applied to an arbitrary solution of the NLSE. We can use it to extend the one-soliton solution, (3), from a fixed solution to a one-parameter family of solutions:

\[ \psi'(\tau, \xi|q) = \frac{q}{cosh \frac{q\tau}{q}} e^{i\omega\xi} \]  

(5)

where \( \omega = q^2/2 \). Clearly, the parameter \( q \) gives the amplitude of the soliton. It also determines the width of the soliton and its period in \( \xi \).

A. Basics of integrability

The inverse scattering technique is the main tool for solving initial value problems related to integrable equations, including the NLSE. For the NLSE, the method has been developed by Zakharov and Shabat [1]. The inverse scattering technique is based upon the fact that the NLSE can be represented in the form of a compatibility condition between the linear equations of the following set:
\[ R_\tau = -J R A + U R, \]
\[ R_\xi = -J R A^2 + U R A - \frac{1}{2} (J U^2 - J U_\tau) R, \]  
where \( R, J \) and \( U \) are the following matrices:
\[ R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & \psi \\ \phi & 0 \end{bmatrix}, \quad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \]
and \( \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) is an arbitrary diagonal complex matrix. The compatibility condition for these differential equations in \( R \), with non-constant coefficients depending on \( \psi(\tau, \xi) \) and \( \phi(\tau, \xi) \), is the equation
\[ U_\xi - \frac{1}{2} J U_{\tau\tau} + J U^3 = 0, \]
which is called 'split NLSE' because it defines a set of two nonlinear equations involving the two functions \( \psi \) and \( \phi \):
\[ i \phi_\xi - \frac{1}{2} \phi_{\tau\tau} + \phi^2 \psi = 0, \quad i \psi_\xi + \frac{1}{2} \psi_{\tau\tau} - \psi^2 \phi = 0. \]
The matrix partial differential equations (6) are simultaneously satisfied for all \( \Lambda \) if and only if \( \psi(\tau, \xi) \) and \( \phi(\tau, \xi) \) solve (9). Equations (9) reduce to a single NLSE in \( \psi \) for a special choice of functions involved in the transformation, e.g. \( \phi = \pm \psi^* \). We set \( r_{21} = s, r_{12} = s^* \) and \( \lambda_2 = \lambda_1 \). In the case of anomalous dispersion \( \psi = -\phi^*, \ r_{22} = -r_{11}^* (\equiv -r^*) \) and equations (9) reduce to the NLSE for the function \( \psi \). For the case of normal dispersion \( \psi = \phi^*, \ r_{22} = r_{11}^* (\equiv r^*) \) and equations (9) reduce to the equation
\[ i \psi_\xi + \frac{1}{2} \psi_{\tau\tau} - |\psi|^2 \psi = 0 \]
and its complex conjugate.

We define \( r = r_{11} \) and \( s = r_{21} \). Then, for the anomalous dispersion case we can preserve the full information content of matrix (6) with the following vector equation expressions:
\[ R_\tau = L \ R, \quad R_\xi = B \ R \]
where
\[ L = \begin{bmatrix} -i \lambda & \psi \\ -\psi^* & i \lambda \end{bmatrix}, \quad B = \begin{bmatrix} -i \lambda^2 + i \frac{1}{2} |\psi|^2 & \lambda \psi + i \frac{1}{2} \psi_\tau \\ -\lambda \psi^* + i \frac{1}{2} \psi_\tau^* & i \lambda^2 - i \frac{1}{2} |\psi|^2 \end{bmatrix}, \quad R = \begin{pmatrix} r \\ s \end{pmatrix}. \]
Thus \( R \) is the first column of the matrix \( R \). The operators \( L \) and \( B \) form a Lax pair for the NLSE.

For each solution, \( \psi(\tau, \xi) \), of (2), there is a basis of two functions, \( r \) and \( s \), parametrized by \( \lambda \), which solve the set of linear equations (11). The compatibility condition in this case takes the form

\[
L_\xi - B_\tau = BL - LB,
\]

and its consequence is the NLSE. In fact, (13) should hold for each \( \lambda \). It represents a cubic polynomial in \( \lambda \). The coefficients of \( \lambda, \lambda^2 \) and \( \lambda^3 \) vanish identically. The vanishing of the constant term is equivalent to the NLSE in \( \psi \). Equation (13) is sometimes called the zero curvature condition, as it has deep roots in differential geometry [9].

The above equations establish the relation between the nonlinear equation and the set of linear differential equations. This representation allows us to solve the initial value problem. Thus, having an arbitrary pulse-like (zero at infinity) initial condition

\[
\psi(\tau, \xi = 0) = f(\tau),
\]

we can, first, solve the eigenvalue problem (11) with \( \psi = f(\tau) \). The linear eigenfunctions, \( R \), evolve (in \( \xi \)) in accordance with (11), but the eigenvalues do not change during this process. Knowledge of the asymptotics of the eigenfunctions for arbitrary \( \xi \) allows us to reconstruct the function \( \psi \) at arbitrary \( \xi \).

Let us start with the eigenvalue problem (11) at \( \xi = 0 \). In explicit form,

\[
\tau_r - i\lambda r = i\psi^* s, \quad s_\tau + i\lambda s = i\psi r.
\]

At infinity \( (\tau \to \pm \infty) \), where \( \psi \to 0 \), the solutions of this eigenvalue problem are obvious. We construct the solutions of (11), \( u(\tau; \lambda) \), \( v(\tau; \lambda) \) and \( \bar{v}(\tau; \lambda) \), which, for real \( \lambda \), satisfy the boundary conditions \( u(\tau; \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(i\lambda \tau) \) at \( \tau \to \infty \),

\[
v(\tau; \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(-i\lambda \tau) \text{ at } \tau \to -\infty, \text{ and } \bar{v}(\tau; \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(i\lambda \tau) \text{ at } \tau \to -\infty.
\]

Here, the function \( \bar{v} \) is the adjoint of \( v \). By definition, if \( v \) is the solution of (11) for \( \lambda_1 \) then \( \bar{v} = \begin{pmatrix} s^* \\ -s^*_{-\tau} \end{pmatrix} \) is the solution of (11) for \( \lambda = \lambda_1^* \). These two functions, \( v(\tau; \lambda) \) and \( \bar{v}(\tau; \lambda) \), comprise a complete set of solutions. Hence, we can write \( u(\tau; \lambda) \) in terms of \( v(\tau; \lambda) \) and \( \bar{v}(\tau; \lambda) \):

\[
u(\tau; \lambda) = a(\lambda)\bar{v}(\tau; \lambda) + b(\lambda)v(\tau; \lambda).
\]

The coefficients \( a(\lambda) \) and \( b(\lambda) \) are transmission and reflection coefficients for the given initial condition, (14), for a given real \( \lambda \). They satisfy

\[
|a(\lambda)|^2 + |b(\lambda)|^2 = 1.
\]

The functions \( u \) and \( v \) may be analytically continued to the upper half-plane of \( \lambda \). Hence, the function \( a(\lambda) \) also admits this continuation. The zeros of \( a(\lambda) \), viz.
\( \lambda_k \ (k = 1, 2, \ldots, N) \), in the upper half-plane of \( \lambda \) determine the set of the discrete eigenvalues of (11). The imaginary parts of \( \lambda \) define the soliton amplitudes while the real parts of \( \lambda \) define their velocities. At these points,

\[
u(\tau; \lambda_j) = c_j \nu(\tau; \lambda_j), \quad j = 1, 2, \ldots, N.
\]

The eigenfunctions of this eigenvalue problem change according to (11), but the eigenvalues \( \lambda_j \) are constant. If, at \( \xi = 0 \), the function \( R \) is the solution of (11) (i.e. it is the initial condition) then the solution of (11) at arbitrary \( \xi \) satisfies (11) with the same \( \lambda \). In particular, \( a(\lambda) \) does not depend on \( \xi \). The evolution of the coefficients \( b(\lambda) \) and \( c_j \) is described by \( b(\lambda, \xi) = b(\lambda, 0) \exp(i\lambda^2 \xi) \) and \( c_j(\xi) = c_j(0) \exp(i\lambda^2 \xi) \). Knowledge of these coefficients at any \( \xi \) allows us to reconstruct the 'potential', \( \psi(\tau, \xi) \), at any \( \xi \). The solution of the NLSE is given by Zakharov and Shabat [1].

The final step in these calculations is highly non-trivial, and solutions have been found in analytical form in a limited number of special cases. The radiative part of the solution diffracts, and, in any constant velocity frame, its amplitude goes to zero as \( \xi \to \infty \). Hence, asymptotically, the solution consists of a finite number of solitons. For an arbitrary initial condition, the scheme can be programmed on a computer, so that the eigenvalues can be found numerically [41,42]. When \( b(\lambda) = 0 \), there is no radiation component and

\[
a(\lambda) = \prod_{i=1}^{N} \frac{\lambda - \lambda_i}{\lambda - \lambda_i^*}.
\]

The solution, which then consists of \( N \) solitons, can be written analytically, provided that the eigenvalues \( \lambda_j \) and the constants \( c_j \) are known.

**B. More complicated cases**

One more integrable system is described by a set of \( N \) coupled nonlinear Schrödinger equations (NLSEs). In some special cases, these equations are found to be integrable [5,47,54,55]. Then, in analogy with the single (scalar) NLSE [1] (where the number of equations, \( N \), is 1) and the Manakov case [44] \( (N = 2) \), the total solution consists of a finite number \( (N) \) of solitons plus small amplitude radiation waves. The former is defined by the discrete spectrum of linear \((L, A)\) operators [1,44], while the latter are defined by the continuous spectrum. Most applications deal with the soliton part of the solution, since it contains the most important features of the problem.

The number of solitons can be arbitrary. Additionally, we assume that the components have independent phases. When the phases are independent, the soliton solution is a multi-parameter family. It can be called a multi-soliton complex (MSC). The notion of MCS can be applied to various physical problems [48]. These include an important recent development: incoherent solitons [49–53].

The set of equations describing the propagation of \( N \) self-trapped, mutually-incoherent wave packets in a medium with a Kerr-like nonlinearity is
\[ i \frac{\partial \psi_i}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \psi_i}{\partial \tau^2} + \alpha \delta n(I) \psi_i = 0, \]

(15)

where \( \psi_i \) denotes the \( i \)-th component of the beam, \( \alpha \) is a coefficient representing the strength of nonlinearity, \( \tau \) is the transverse co-ordinate (a moving time frame in fibers), \( \xi \) is the co-ordinate along the direction of propagation, and \( \delta n(I) = f \left( \sum_{i=1}^{N} |\psi_i|^2 \right) \) is the change in refractive index profile created by all the incoherent components of the light beam. The response time of the nonlinearity is assumed to be long compared to temporal variations of the mutual phases of all the components, so the medium responds to the average light intensity, and this is just a simple sum of modal intensities.

Interestingly enough, the set of equations (15) has \( N \) quantities \( Q_i = \int_{-\infty}^{\infty} |\psi_i|^2 \, d\tau \) which are conserved separately from the conservation of the total energy \( Q = \sum_{i=1}^{N} Q_i \). This occurs because there is no energy transfer mechanism between the components. In fact, this is the main difference from the phase-dependent components case, where only the total energy is conserved.

If the function \( \delta n(y) = y \), then the set of equations (15) is a generalized Manakov set, which has been shown to be integrable [47]. This means that all solutions, in principle, can be written in analytical form.

An MSC is a stationary solution of (15). If it is moving with a certain velocity, then it can be made stationary by using a Galilean transformation [12]. Stationary solutions of (15) are given by

\[ \psi_i(\tau, \xi) = \frac{1}{\sqrt{\alpha}} \, u_i(\tau) \exp \left( \frac{i k_i^2}{2} \xi \right), \]

(16)

with real functions \( u_i \), so that the set of Eqs.(15) reduces to the set of ODEs:

\[ \frac{d^2 u_i}{d\tau^2} + 2 \sum_{j=1}^{N} u_j^2 \, u_i = k_i^2 u_i, \]

(17)

which is also completely integrable for an arbitrary set of real \( k_i \). Various solutions to these equations, including soliton solutions [45,46,57] and periodic solutions [58–60] have been found, especially for \( N = 2 \) [61,62]. Examples of explicit solutions for \( N > 2 \) are still rare [58,63]. It is worth noting that ODEs obtained from the coupled NLSEs might be integrable in a wider range of parameters than the initial set of NLSEs. Examples of additional integrability of ODEs for the case of \( N = 2 \) are given in [64]. A substantial review of integrability of Hamiltonian systems with two degrees of freedom is given in [65]. It follows that stationary solutions can be obtained in analytic form even if the initial set of NLSEs is not integrable.

It can be shown, using Poisson brackets, that the set of ODEs (17) has \( N \) conserved quantities, namely the Hamiltonian \( H \):
\[ H = \sum_{i=1}^{N} (\dot{u}_i^2 - k_i^2 u_i^2) + \left( \sum_{i=1}^{N} u_i^2 \right)^2 = \text{const.}, \]  

(18)

and \( N - 1 \) additional integrals, \( I_p \) (\( p = 1, \ldots, N - 1 \)) =

\[ \sum_{i \neq j} (\dot{u}_i u_j - u_i \dot{u}_j)^2 - \sum_{i \neq p} \Delta k_{pi} \left( \dot{u}_i^2 + u_p^2 \sum_{m \neq i} u_m^2 + u_i^4 - k_i^2 u_i^2 \right) = \text{const.}, \]  

(19)

where \( \Delta k_{pi} = (k_p^2 - k_i^2) \), and each dot over \( u_i \) denotes a derivative with respect to \( x \). For zero background solutions, the integrals must be equal to zero (\( H = 0 \) and \( I_p = 0 \)).

We note, from (17), that the constants \( k_i \) have a dual physical meaning. Firstly, they can be considered as the amplitudes of partial fundamental solitons in the multisoliton complex. Secondly, if we consider \( \sum |u_i|^2 \) as a given self-induced refractive index profile, then each \( k_i \) is an eigenvalue related to a certain mode of the self-induced waveguide. The number of linear eigenvalues, \( N \), equals the number of fundamental solitons in the multisoliton complex.

It can also be shown [66–68] that solutions of (17) can be found from the linear set of algebraic equations:

\[ \sum_{i=1}^{N} \frac{\exp[k_i \tau]}{k_j + k_i} \frac{\exp[k_j \tau]}{\sqrt{2}k_i} \frac{u_i(\tau)}{\sqrt{2}k_j} + \frac{u_j(\tau)}{\sqrt{2}k_j} = -\exp[k_j \tau], \]  

(20)

which can be written in the following matrix form

\[ D_{j,m} \frac{u_m(\tau)}{\sqrt{2}k_m} = -e_j \]  

(21)

where the elements of matrix \( D \) are

\[ D_{j,m} = \delta_{j,m} + \frac{e_j e_m}{k_j + k_m}, \]  

(22)

with \( e_j = \exp(k_j \tau) \).

Henceforth, we will replace the \( e_j \) functions with more general ones, namely

\[ e_j = \sqrt{2} k_j a_j \exp(k_j \bar{\tau}_j) \]  

(23)

where \( \bar{\tau}_j = \tau - \tau_j \) and the parameters \( \tau_j \) are shifts for each fundamental soliton. These are parameters which contribute nontrivially to the shape of the MSC. The new functions also give a solution for each \( u_j \). The new feature of the functions \( e_j \) here is the addition, not only of shifts \( \tau_j \), but also of arbitrary coefficients \( a_j \). We could absorb the \( \tau_j \) into the \( a_j \), but we keep both coefficients \( a_j \) and \( \tau_j \) as independent parameters. The reason is that the coefficients \( a_j \) define the specific choice needed to achieve symmetry in the presentation of the solution [69] and the \( \tau_j \) define fundamental soliton locations in the multisoliton complex.
We arrange the eigenvalues required in decreasing order \((k_1 > k_2 > k_3 > \ldots)\) and define the positive coefficient \(c_{ij} = \frac{k_{i+1} + k_i}{k_i - k_j}\). We have found [69] that the choice \(a_i = \prod_{j \neq i} c_{ij}\) is the one which allows us to obtain the above-mentioned symmetry, provided that all \(\tau_i = 0\). Note that each \(a_i > 0\). For example, if there are 4 eigenvalues, then \(a_2 = \prod_{j \neq 2} c_{2j} = c_{21} c_{23} c_{24} = c_{12} c_{23} c_{24}\). If, on the other hand, the \(x_i\)'s remain arbitrary parameters, then the solution is asymmetric, but is represented in the same compact and convenient form.

The solution components themselves can be written in a simple form:

\[
u_i(\tau) = -\sqrt{2k_i} D_{i,j}^{-1} e_j.
\] (24)

Although the inversion of the matrix \(D\) is a standard technique, it requires some effort to present the solution in a compact and simple form.

The solution is a multi-parameter family. It contains \(N\) soliton parameters, \(k_i\), as well as \(N\) shifts, \(\tau_i\). Admitting translational symmetry of the solution as a whole, we can define all shifts relative to one of them, so that the total solution then contains \(2N - 1\) free parameters. These parameters give a huge diversity of MSC shapes. For \(N = 1\), we define \(D_1 = \cosh(k_1 \tau_1)\), so the fundamental NLSE soliton is \(u_1(\tau) = \frac{k_1}{D_1} = k_1 \text{sech}(k_1 \tau_1)\).

**The solution for \(N = 2\).** The matrix elements are given by

\[
D_{11} = 1 + a_1 \exp(2 k_1 \tau_1),
\]

\[
D_{22} = 1 + a_2 \exp(2 k_2 \tau_2),
\]

\[
D_{12} = D_{21} = \frac{2 \sqrt{a_1 a_2}}{k_1 + k_2} \sqrt{k_1 k_2} \exp(k_1 \tau_1 + k_2 \tau_2).
\]

The specific choice needed to achieve symmetry is \(a_2 = a_1 = c_{12} = \frac{k_1 + k_2}{k_1 - k_2}\).

Choosing these special coefficients and inverting the matrix \(D\) gives, after some simple algebra,

\[
u_1 = \pm \frac{2k_1 \sqrt{a_1}}{D_2} \cosh(k_2 \tau_2),
\]

\[
u_2 = \pm \frac{2k_2 \sqrt{a_2}}{D_2} \sinh(k_1 \tau_1),
\] (25)

where \(D_2 = \cosh(k_1 \tau_1 + k_2 \tau_2) + c_{12} \cosh(k_1 \tau_1 - k_2 \tau_2)\). This form of the solution is convenient for generalizations when \(N > 2\) and can be viewed as the standard form. Other forms has been used in the presentation of this solution in [12,45,46,56,57].

The solution is asymmetric, in general, for arbitrary \(k_1\) and \(k_2\), but becomes symmetric for the special choice of \(\Delta \tau_12 = \tau_2 - \tau_1 = 0\). Then \(u_1\) and \(u_2\) are, respectively, the even and odd modes of a symmetric self-induced waveguide. If \(k_1/k_2 = 2\) with \(k_2\) arbitrary, then \(D_2\) reduces to \(4 \cosh^2(k_2 \tau)\) and \(u_1^2 + u_2^2\) is simply \(3k_2^2 \text{sech}^2(k_2 \tau)\). Fig.3 shows the two modes, as well as the intensity profile for two different separations, \(\Delta x_{12}\). Note that the intensity profile for the symmetric solution is not necessarily limited to having a single maximum. When \(k_1\) and \(k_2\) are close to each other, the solution may exhibit two peaks in its intensity profile.
An example of a double peak structure of a symmetric MSC with $N = 2$ is shown in Fig. 3.

![Graphs showing intensity and amplitude for different values of $\tau$](image)

**FIG. 3.** Transverse profiles and linear modes of the MSC for $N = 2$. Calculations use $k_1 = 1.0$, $k_2 = 0.5$. For the symmetric solution (a) $\Delta \tau_{12} = 0$, while for the asymmetric solution (b) $\Delta \tau_{12} = 2.0$.

### III. HAMILTONIAN SYSTEMS

Reductions to integrable systems are extreme simplifications of the complex systems existing in nature. They can be considered as a subclass of the more general Hamiltonian systems (see Fig. 2). Indeed, such a simplification allows us to analyse the systems quantitatively and to completely understand the behaviour of the solitons. Solitons in Hamiltonian (but nonintegrable) systems can also be regarded as nonlinear modes, but in the sense that they allow us to describe the behaviour of systems with an infinite number of degrees of freedom in terms of a few variables, thus allowing us effectively to reduce the number of degrees of freedom. Solitons in these systems collide inelastically and interact with radiation waves, thus showing that they are qualitatively different from those in integrable systems. However, as in the integrable case, the solitons are still a one- (or a few-) parameter family of solutions.

The Hamiltonian ($H$) is one of the fundamental notions in mechanics [2] and more generally in the theory of conservative dynamical systems with a finite (or even infinite) number of degrees of freedom. The Hamiltonian formalism has turned out to be one of the most universal in the theory of integrable systems [9] and nonlinear waves in general [8]. In the case of non-integrable systems, the Hamiltonian exists whenever the system is conservative, and it is useful for stability analysis [10,11]. It turns out that the most useful approach in soliton theory of
conservative non-integrable Hamiltonian systems is a representation on the plane of conserved quantities: Hamiltonian versus energy [12]. A three-dimensional plot (Hamiltonian - energy - momentum) is useful when dealing with two-parameter families of solutions [13].

Recently, Hamiltonian versus energy curves have been used effectively to study families of solitons and their properties, viz. range of existence, stability and general dynamics. Specific problems considered up to now include scalar solitons in non-Kerr media [12], vector solitons in birefringent waveguides [14], radiation phenomena from unstable soliton branches [15], optical couplers [16], general principles of coupled nonlinear Schrödinger equations [17,18], parametric solitons in quadratic media [19] and the theory of Bose-Einstein condensates [20]. Moreover, Hamiltonian-versus-energy curves are useful not only for studying single soliton solutions, but also for analysing the stability of bound states (when they exist) [21]. Other examples could be mentioned as well.

In most publications, soliton families have been studied using plots of energy versus propagation constant. These curves allow the soliton families to be presented graphically and, moreover, allow predictions of their stability properties. We believe that the first example of their application was presented in [22]. Kuzmarnsev [23] was the first person to understand the importance of projecting curves on the plane of conserved quantities. He applied catastrophe theory and a mapping technique to represent soliton families with diagrams and to show that the critical points on these diagrams define the bifurcations where the soliton stability changes. In [28], a direct approach to analyze the $H(Q)$ soliton curves has been presented and, additionally, the concept has been enhanced with a stability theorem. This theorem turns the employment of $H(Q)$ curves into a powerful tool for analyzing soliton solutions, their stability and their dynamics. In particular, a theorem which relates the concavity of the $H - Q$ curves to the stability of the solitons has been proved. The main advantages of this approach are its simplicity, clarity and the fact that it provides the possibility of predicting simple dynamics of evolution for solitons on unstable branches.

For simplicity, let us consider scalar wave fields $\psi(t, \xi)$. The nonlinear Schrödinger equation (NLSE) for a general nonlinearity law is [10–12]:

$$i\psi_t + \frac{1}{2}\psi_{\tau\tau} + N(|\psi|^2)\psi = 0.$$  \hspace{1cm} (26)

where $N$ is the nonlinearity law. It indicates that the change in refractive index depends on the local intensity. Localized solutions satisfy the ansatz

$$\psi(\tau, \xi) = f(\tau) \exp(iq\xi)$$  \hspace{1cm} (27)

where $f(\tau)$ is a real field profile, and $q$ is the propagation constant.

The total energy associated with an arbitrary solution, $\psi(t, \xi)$, is $Q = \int_{-\infty}^{\infty} I d\tau$, where the intensity is $I = |\psi|^2 = f^2$. In spatial problems, $Q$ is the power or power flow. In problems related to pulse propagation in optical fibers, where $t$ is regarded as a retarded time, $Q$ is the total pulse energy. For localized solutions (Eq.(27)), $Q$ is finite and it is one of the conserved quantities of Eq.(26).
Similarly, the Hamiltonian is another conserved quantity:

$$H = \int_{-\infty}^{\infty} \left[ \frac{1}{2} f_{\tau}^2 - F(I) \right] d\tau,$$

with $F$ given by $F(I) = \int_{0}^{I} N(I') \, dI'$. The Hamiltonian plays a major role in the dynamics of the infinite-dimensional system. Namely, stationary solutions of equation (15) can be derived from the Hamiltonian using the variational principle $\delta H = 0$.

Now, substituting (27) into (26) and integrating once, we have $f_{\tau}^2 = 2(qI - F)$ or

$$H = qQ - 2K,$$

where $K = \int_{-\infty}^{\infty} F(I) \, d\tau$. This expression can be used to calculate $H$ versus $Q$ curves explicitly. It is easy to show that in the case of a Kerr medium $H(Q) = -\frac{Q^3}{24}$.

### A. Stability

One of the advantages of using $H - Q$ curves is that they can predict the stability of solitons. It is apparent that, if there is more than one branch at a given $Q$, then the lowest branch (i.e. the one with the minimum Hamiltonian) is stable. This conclusion follows directly from the nature of Hamiltonian and does not need a special proof. However, the stability condition can take a more direct form. There is a useful theorem in this regard. For solitons in media with local nonlinearities, we have,

$$\frac{dH}{dq} = -q \frac{dQ}{dq}. \quad (30)$$

Then it follows that

$$\frac{dH}{dQ} = -q. \quad (31)$$

If we start at $q = 0$ and traverse the curve so that $q$ is increasing, then the magnitude of the slope always increases. Furthermore,

$$\frac{d^2 H}{dQ^2} = \frac{-1}{dQ/dq} \quad (32)$$

The denominator on the right-hand-side defines the stability of the lowest-order modes (fundamental solitons) [10,24,26,25,27,28]. Hence, the stability is directly
related to the concavity of the $H$ versus $Q$ curve. Namely, the solitons with $H''(Q) < 0$ are stable while those with $H''(Q) > 0$ are unstable.

Another consequence of (30) is that
\[ \frac{dH}{dq} = 0 \Rightarrow \frac{dQ}{dq} = 0 \text{ or } q = 0. \] (33)

Thus if $Q$ has a stationary point, then so does $H$. For $q > 0$, this produces a cusp on the $H - Q$ diagram. However, we can have $dH/dq = 0$ with $q = 0$ and $dQ/dq \neq 0$. This produces a rounded maximum on the $H$ vs. $Q$ plot and not a cusp.

Clearly, from eq.(30), if we have $q > 0$, then $H$ decreases as $Q$ increases, meaning that $\frac{dH}{dq} < 0$. On the other hand, if $q < 0$ is allowable, then $H$ and $Q$ have the same slope, so that $\frac{dH}{dq} > 0$.

Thus, we can conclude that, for the lowest order modes:
1. Solitons with $H''(Q) < 0$ are stable while those with $H''(Q) > 0$ are unstable.
2. Stability changes only at cusps.

This criterion for stability can be more general than $dQ/dq > 0$, because it involves only conserved quantities which always exist in conservative systems; this is in contrast to $q$, which may not be defined uniquely. This is an important theorem and we illustrate its application in the following example. Moreover, we also consider what happens to unstable solitons if they are excited in the system.

**Example: Dual power law nonlinearity.** This nonlinearity is given by $N = I^b + \nu I^{2b}$. When $\nu$ is positive, the refractive index increases monotonously with $I$. We consider $\nu < 0$, when $N(I)$ dependence is not monotonous and we can expect qualitatively new effects. We let $\beta = 2 (1+b) \sqrt{-\frac{\nu q}{2b+1}} \equiv \tanh (B) \ (\beta > 0)$. Thus $0 < \beta < 1$, so that solitons can exist (only) within the range $-\frac{1+b}{4(1+b)\beta} < \nu q < 0$.

Then
\[ Q = \frac{I_m}{b\sqrt{2q}} \int_0^1 \frac{y^{1/2-1} dy}{\sqrt{1-y} \sqrt{1-y \tanh^2 (B/2)}}, \] (34)

where $I_m = \left( \frac{(1+2b)(\text{sech}(B)-1)}{2\nu (1+b)} \right)^{1/2}$, so that
\[ Q = \frac{I_m}{b} \sqrt{\frac{\pi}{2q}} \frac{\Gamma\left(\frac{1}{b}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{b}\right)} F\left(\frac{1}{2}, \frac{1}{b}; \frac{1}{2}; \frac{1}{b}; z\right), \] (35)

where $F$ is the hypergeometric function and $z = \tanh^2 (B/2)$.

The Hamiltonian is given by
\[ H = \sqrt{2S - q Q}. \] (36)

where $S = \frac{I_m \sqrt{\pi}}{2b} \frac{\Gamma\left(\frac{1}{b}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{b}\right)} F\left(-\frac{1}{2}, \frac{1}{b}; \frac{3}{2}; \frac{1}{b}; z\right)$. 


Now we have explicit forms for $Q$ and $H$. In general, when $b < 2$, $Q$ increases and $H$ decreases monotonically with $q$, so that the parametric $H$ versus $Q$ plot decreases monotonically as $q$ increases and is always concave down. Hence, the solitons of the whole family are stable.

![Graph](image)

**FIG. 4.** Hamiltonian versus energy for dual-power-law nonlinearity for the values of the parameters $b = 5/2$ and $\nu = -1$. The dotted arrow shows a transformation which occurs from the unstable branch to the stable one, due to the soliton’s interaction with radiation. The cusp occurs at $q = q_c = 0.0492$ and corresponds to the soliton’s minimum energy of $Q = 2.51$ and maximum Hamiltonian, viz. $H = 0.00783$.

For $b > 2$ however, $Q$ has a minimum and $H$ has a maximum at $q > 0$, thus producing a cusp in the $H$ versus $Q$ plot (see Figure 4). Note that solitons exist only above some threshold energy in this case. The important conclusion from this case is that the upper branch should be unstable, because the Hamiltonian is concave upwards while the lower branch should be stable as it is concave downwards. Numerical simulations show that this is indeed the case.

The exact soliton profile is given by

$$
f(\tau) = \left[ \frac{1}{2q(1+b)} \left[ 1 + \sqrt{1 + \frac{4q\nu}{(1+2b)(1+b)^2} \cosh \left(2b \sqrt{2q} \tau \right)} \right] \right]^{-\frac{1}{2b}}.
$$

**B. Soliton dynamics**

To illustrate further the usefulness of the $H(Q)$ diagrams in predicting dynamics, let us consider a simple example. In Fig.4, the upper unstable branch of solitons corresponds to the range $0 < q < q_c$. The lower stable branch corresponds to the interval $q_c < q < q_{max} = 6/49$. The cusp appears at $q = q_c = 0.0492$. An example of propagation is shown in Fig.5(a). It shows the instability of the upper
branch. Numerical simulations start with the exact solution, Eq.(37), as the initial condition, and take $q = 0.005$, which corresponds to $Q = 2.936$. This soliton is unstable, and due to interaction with radiation, it evolves into a soliton of the stable branch. The initial and the final soliton profiles are shown in Fig.5. The final state, after the radiation waves have dispersed, is a soliton with parameters $q = 0.094$ and $Q = 2.69$. The course of the above transformation is clearly seen in Fig.5. It is represented by the dotted arrow in Fig.4. A physically similar process has been considered in [15] for solitons in birefringent fibers. As a general rule, this analysis shows that the transformation always takes place from an upper right point on the $H(Q)$ diagram to a lower left point on the diagram. Hence the direction of the arrow in Fig.4 must be down and to the left.

![Fig. 5. (a) Evolution of an unstable soliton. The result of this evolution is shown schematically by the arrow in Fig.4. (b) Initial ($\xi = 0$) and final ($\xi > 1400$) soliton profiles. Initially the stationary soliton solution ($q = 0.005$) is unstable, but it evolves into a soliton on the stable branch while emitting small amplitude radiation waves (note ripples in (a)).](image)

The instability eigenvalues of the linearized equations for the upper soliton branch must be complex, as they have real parts which correspond to the deviation from the unstable soliton and imaginary parts which correspond to interactions with radiation. Complex eigenvalues have been proved to exist for Hamiltonian systems in [29–32].

This approach can be generalized to include more complicated Hamiltonian nonlinear systems, including cases with two [14,16] or more coupled NLSEs [33–35], parametric solitons [36,37] and examples of higher-order dimensionality [38]. For example, the curves $H(Q)$ calculated numerically in [18] show clearly that our stability criterion can be applied to a system of coupled NLSEs. The results obtained in [20] also show that this principle can be generalized to the case of $(1+3)$-D solitons. It is quite obvious, then, that $(1+2)$-D cases and spatio-temporal $(1+3)$-D solitons [39,40,43] also could be handled with our approach. This means that, independent of their physical nature, single-soliton solutions of Hamiltonian systems can be well understood and analysed using the concavity of the $H(Q)$ curves.
IV. DISSIPATIVE SYSTEMS

The next level of generalization is to consider solitons in dissipative systems (see Fig.2). The main feature of these systems that they include energy exchange with external sources. These are no longer Hamiltonian and the solitons in these systems are also qualitatively different from those in Hamiltonian systems. In Hamiltonian systems, soliton solutions appear as a result of a balance between diffraction (dispersion) and nonlinearity. Diffraction spreads the beam while nonlinearity focuses it and makes it narrower. The balance between the two results in a stationary solution, which is usually a one-parameter family. In systems with gain and loss, in order to have stationary solutions, gain and loss must be also balanced. This additional balance results in solutions which are fixed. The shape, amplitude and the width are all fixed and depend on parameters of the equation. This situation is presented qualitatively in Fig.6. However, the common feature is that solitons, when they exist, can again be considered as ‘modes’ of dissipative systems.

**FIG. 6.** Qualitative difference between the soliton solutions in Hamiltonian and dissipative systems. In Hamiltonian systems, soliton solutions are the result of a single balance, and comprise one- or few-parameter families, whereas, in dissipative systems, the soliton solutions are the result of a double balance and, in general, are isolated. There can be exceptions to this rule [87–89], but, usually, the solutions are fixed (i.e. isolated from each other). On the other hand, it is quite possible for several isolated soliton solutions to exist for the same equation parameters. This is valid for (1+1)-dimensional as well as for (2+1)-dimensional cases. In the latter case, the terms “localized structures” [70], “bullets” [71,72] or “patterns” [90] are also used along with the term “solitons” [73].

We will concentrate here on equation (1), the complex Ginzburg-Landau equation (CGLE). Many non-equilibrium phenomena, such as convection instabilities [80], binary fluid convection [81] and phase transitions [82], can be described by this equation. In optics, this equation (or a generalization of it) describes the essential features of processes in lasers [83,91,84–86], optical parametric oscillators [92], spatial soliton lasers [93,70], Fabry-Perot cavities filled with nonlinear material and driven by an external field [71–73], free-electron laser oscillators [74] and all-optical transmission lines [75]. Planar soliton systems with gain [76,77] or
with light-guiding-light phenomena [110] are also particular examples which can be described by the CGLE. This equation is essentially the nonlinear Schrödinger equation (NLSE) with gain and loss, where both gain and loss are frequency- and intensity-dependent.

![Diagram of solitons](image)

**FIG. 7.** Qualitative description of solitons in dissipative systems. The soliton has areas of consumption as well as dissipation of energy which can be both frequency (spatial or temporal) and intensity dependent. Arrows show the energy flow across the soliton. The soliton is a result of a complicated dynamical processes of energy exchange with the environment and between its own parts.

Another simple qualitative picture is presented in Fig.7. In order to be stationary, solitons in dissipative systems need to have regions where they extract energy from an external source, as well as regions where energy is dissipated to the environment. A stationary soliton is the result of a dynamical process of continuous energy exchange with the environment and its redistribution between various parts of the soliton. Hence the soliton by itself is an object which is far from equilibrium. In this sense, it is more like a living thing than an object of the inanimate world. It is like a species in biology which is fixed (or isolated) in its properties.

Equation (1) has been written in such a way that if the right-hand side of it is set to zero, we obtain the standard NLSE. For spatial solitons in wide-aperture laser cavities, the form of the coefficients in this equation can be different [93]. The equation can also have additional terms related to finite aperture [93] and other forms of local [70] or nonlocal [94,95] nonlinearity. The theory of phase-sensitive amplification (or 'parametrically amplified' optical systems) [96] also uses a different form of the CGLE which is called a "parametric Ginzburg-Landau equation". In this paper, we will retain the form presented above as the basic one, as it gives the main properties of solitons in dissipative systems. We will also concentrate on the (1+1) dimensional case, as it is the fundamental one which allows us to understand some of the features of solitons in (2+1) dimensional cases.

Equation (1) is nonintegrable, and only particular exact solutions can be obtained. In general, initial value problems with arbitrary initial conditions can only be solved numerically. The cubic CGLE, obtained by setting \( \mu = \nu = 0 \) in (1), has been studied extensively [97–101,109]. Exact solutions to this equation can be obtained using a special ansatz [98], Hirota bilinear method [101] or reduction to systems of linear PDEs [102]. However, it was realized many years ago that the
soliton-like solutions of this equation are unstable.

The case of the quintic CGLE has been considered in a number of publications using numerical simulations, perturbative analysis and analytic solutions. Originally, this equation was used mainly as a model for binary fluid convection [103–105]. The existence of soliton-like solutions of the quintic CGLE in the case $\epsilon > 0$ has been demonstrated numerically [104,105]. A qualitative analysis of the transformation of the regions of existence of the soliton-like solutions, when the coefficients on the right-hand-side change from zero to infinity, has been made in [106]. An analytic approach, based on the reduction of (1) to a three-variable dynamical system, which allows us to obtain exact solutions for the quintic equation, has been developed in [88,107]. The most comprehensive mathematical treatment of the exact solutions of the quintic CGLE, using Painlevé analysis and symbolic computations, is given in [108]. The general approach used in that work is the reduction of the differential equation to a purely algebraic problem. The solutions include solitons, sinks, fronts and sources. The great diversity of possible types of solutions requires a careful analysis of each class of solutions separately. In the brief review which follows, we will concentrate solely on soliton-like solutions.

A. Balance equations and perturbation theory

The CGLE has no known conserved quantities. Instead, the energy associated with solutions $\psi$ is $Q = \int_{-\infty}^{\infty} |\psi|^2 d\tau$, and its rate of change with respect to $\xi$ is [12]:

$$\frac{d}{d\xi} Q = F[\psi],$$  \hspace{1cm} (38)

where the real functional $F[\psi]$ is given by

$$F[\psi] = 2 \int_{-\infty}^{\infty} \left[ \delta |\psi|^2 + \epsilon |\psi|^4 + \mu |\psi|^6 - \beta |\psi_\tau|^2 \right] d\tau.$$  

Similarly [12], the momentum is $M = Im \left( \int_{-\infty}^{\infty} \psi^*_\tau \psi d\tau \right)$, and its rate of change is defined by

$$\frac{d}{d\xi} M = J[\psi],$$  \hspace{1cm} (39)

where $J[\psi] = 2 Im \int_{-\infty}^{\infty} \left[ (\delta + \epsilon |\psi|^2 + \mu |\psi|^4)\psi + \beta \psi_\tau \right] \psi^*_\tau \ d\tau$. By definition, this functional is the force acting on a soliton along the $\tau$-axis. There are only two rate equations, viz. (38) and (39), which can be derived for the CGLE. They can be used for solving various problems related to CGLE solitons.

If the coefficients $\delta$, $\beta$, $\epsilon$, $\mu$ and $\nu$ on the right-hand side are all small, then soliton-like solutions of (1) can be studied by applying perturbative theory to the soliton solutions of the NLSE. Let us consider the right-hand side of equation (1), with $D = +1$, as a small perturbation and write the solution as a soliton of the NLSE, viz.
\[ \psi(\tau, \xi) = \frac{\eta}{\cosh[\eta(\tau + \Omega \xi)]} \exp[-i\Omega \tau + i(\eta^2 - \Omega^2)\xi/2]. \] (40)

In the presence of the perturbation, the parameters of the soliton, viz. the amplitude \( \eta \) and frequency (or velocity) \( \Omega \), change adiabatically. The equations for them can be obtained from the balance equations for the energy and momentum. Using (40) and (38), we have the equations for the evolution of \( \eta(\xi) \) and \( \Omega(\xi) \):

\[ \frac{d\eta}{d\xi} = 2\eta \left[ \delta - \beta \Omega^2 + \frac{1}{3}(2\epsilon - \beta)\eta^2 + \frac{8}{15}\mu \eta^4 \right], \quad \frac{d\Omega}{d\xi} = -\frac{4}{3} \beta \Omega \eta^2. \] (41)

The dynamical system of equations, (41), has two real dependent variables and the solutions can be presented on the plane. An example is given in Fig. 8. It has a line of singular points at \( \eta = 0 \), and, depending on the equation parameters, may have one or two singular points on the semi-axis \( \Omega = 0, \eta > 0 \). The values of \( \eta^2 \) for singular points are defined by finding the roots of the biquadratic polynomial in the square brackets in (41). When the roots are negative (hence, \( \eta \) is imaginary), there are no singular points and hence no soliton solution. If both roots of the quadratic polynomial (in \( \eta^2 \)) are positive (so that both \( \eta \) are real), then there are two fixed points and two corresponding soliton solutions. Both roots are positive when either \( \beta < 2\epsilon, \mu < 0 \) and \( \delta < 0 \) or \( \beta > 2\epsilon, \mu > 0 \) and \( \delta > 0 \).

![Phase portrait](image)

**FIG. 8.** The phase portrait of the dynamical system (41) for \( \delta = -0.03, \beta = 0.1, \epsilon = 0.2 \) and \( \mu = -0.11 \). The upper fixed point is a sink which defines the parameters of a stable approximate soliton-like solution of the quintic CGLE. Any soliton-like initial condition in close proximity to a fixed point will converge to a stable stationary solution. The points on the line \( \eta = 0 \) are stable when \( \delta < 0 \) and \( \beta > 0 \). This condition is needed for the background state \( \psi = 0 \) to be stable.

The stability of at least one these fixed points requires \( \beta > 0 \). Moreover, the stability of the background requires \( \delta < 0 \). In the latter case we necessarily have \( \beta < 2\epsilon, \mu < 0 \) and the upper fixed point is a sink (as shown in Fig. 8) which defines the parameters of a stable approximate soliton solution of the quintic CGLE. The background \( \psi = 0 \) is also stable, so that the whole solution (soliton plus background) is stable. Finally, when only one of the roots is positive, there is a
singular point in the upper half-plane and there is a corresponding soliton solution. However either the background or the soliton itself is unstable, so that the total solution is unstable. The term with $\nu$ in the CGLE does not influence the location of the sink. It only introduces an additional phase term, $\exp(8i\nu \eta^4 \xi/15)$, into the solution of Eq.(40).

In the case of cubic CGLE, $\mu = 0$ and $\nu = 0$. The stationary point is then

$$\eta = \sqrt{3\delta/(\beta - 2\epsilon)}, \quad \Omega = 0. \quad (42)$$

It is stable provided that $\delta > 0$, $\beta > 0$ and $\epsilon < \beta/2$. Clearly, in this case the soliton and the background cannot be stable simultaneously. Hence, this approach shows that to have both the soliton and the background stable, we need to have quintic terms in the CGLE (see also [78]).

This simple approach shows that, in general, the CGLE has stationary soliton-like solutions, and that for the same set of equation parameters there may be two of them simultaneously (one stable and one unstable). Moreover, this approach shows that soliton parameters are fixed, as depicted in Fig.6. This occurs because the dissipative terms in (1) break the scale invariance associated with the conservative system.

Despite its simplicity and advantages in giving stability and other properties of solitons, the perturbative analysis has some serious limitations. Firstly, it can be applied only if the coefficients on the right-hand side of (1) are small, and this is not always the case in practice. Correspondingly, it describes the convergence correctly only for initial conditions which are close to the stationary solution. Secondly, the standard perturbative analysis cannot be applied to the case $D < 1$, when the NLSE itself does not have bright soliton solutions. However, (1) has stable soliton solutions for this case as well.

Exact analytical solutions can be found only for certain combinations of the values of the parameters [12,79]. In general we need to use some numerical technique to find stationary solutions. One way to do it is by reducing Eqs. (15) to a set of ODEs. We do that by seeking solutions in the form:

$$\psi(t, \xi) = \psi_0(\tau) \exp(-i\omega \xi) = a(\tau) \exp[i\phi(\tau) - i\omega \xi], \quad (43)$$

where $a$ and $\phi$ are real functions of $\tau = t - \nu \xi$, $\nu$ is the pulse velocity and $\omega$ is the nonlinear shift of the propagation constant. Substituting (43) into (15), we obtain an equation for two coupled functions, $a$ and $\phi$. Separating real and imaginary parts, we get the following set of two ODEs:

$$\left[ \omega - \frac{1}{2} D\phi'^2 + \beta \phi'' + \nu \phi' \right] a + 2\beta \phi' a' + \frac{1}{2} Da'' + a^3 + \nu a^5 = 0,$$

$$(-\delta + \beta \phi'^2 + \frac{1}{2} D\phi'') a + (D\phi' - \nu) a' - \beta a'' - \epsilon a^3 - \mu a^5 = 0, \quad (44)$$

where each prime denotes a derivative with respect to $\tau$.

It can be transformed into:

$$\omega a + \nu \frac{M}{a} - \frac{DM^2}{2a^3} + \frac{\beta M'}{a} + \frac{D}{2} a'' + a^3 + \nu a^5 = 0,$$

$$-\delta a - \nu a' + \frac{\beta M^2}{a^2} + \frac{DM'}{2a} - \beta a'' - \epsilon a^3 - \mu a^5 = 0, \quad (45)$$
where \( M = a^2 \phi' \).
Separating derivatives, we obtain:
\[
M' = \frac{2(D\delta - 2\beta\omega)}{1+4\beta^2} a^2 + \frac{2(D\xi - 2\beta)}{1+4\beta^2} a^4 + \frac{2(D\rho - 2\beta
u)}{1+4\beta^2} a^6 - \frac{4\beta v}{1+4\beta^2} M + \frac{2Dv}{1+4\beta^2} a y ,
\]
\[
y' = M^2 \frac{2(D\omega + 2\beta \delta)}{1+4\beta^2} a - \frac{2(D + 2\beta \epsilon)}{1+4\beta^2} a^3 - \frac{2(D\nu + 2\beta \mu)}{1+4\beta^2} a^5 - \frac{4\beta v}{1+4\beta^2} y - \frac{2Dv}{1+4\beta^2} M a ,
\]
\[
a = y .
\]
This set contains all stationary and uniformly translating solutions. The parameters \( v \) and \( \omega \) are the eigenvalues of (46). In the \((M, a)\) plane, the solutions corresponding to pulses are closed loops starting and ending at the origin. The latter happens only at certain values of \( v \) and \( \omega \). If \( v \) and \( \omega \) differ from these fixed values, the trajectory cannot comprise a closed loop. By properly adjusting the eigenvalue \( \omega \), it is possible to find the soliton solution with a "shooting" method.

B. Multiplicity of solutions

Numerical simulations show that a multitude of soliton solutions of the CGLE exist. They have a variety of shapes and stability properties. They can even be partly stable and partly unstable. The shapes are influenced greatly by the singular points of equations (46). Trajectories on the phase portrait are deflected from their smooth motion near the singular points. As a result, the trajectory can have additional loops and the soliton shape can become multi-peaked. Even when the soliton shape is smooth (bell-shaped), the singular points can influence the soliton evolution in \( \xi \).

Singular points have a certain physical meaning. They define a continuous wave solution with the same propagation constant as the soliton. Thus, there is close relation between solitons and CW solutions. When the CW solution is stable itself, it might lead to the existence of "front" solutions. If it is not stable, it still has a role is in influencing the soliton shape.

The vast majority of soliton solutions is unstable. In fact, in some regions of the parameter space, all of them can be unstable. However, even in this case, unstable solutions can emerge from an arbitrary pulse and exist in this form for some distance. Surprisingly, some solutions disappear because of explosions [111] and reappear repeatedly after this visibly chaotic stage of evolution. The phenomenon of explosions never occur if solitons are stable for those values of the parameters.

Our investigation shows once again that dissipative solitons are qualitatively different from Hamiltonian solitons.

1. Solitons in dissipative systems are fixed solutions in contrast to Hamiltonian solitons which are one- or two-parameter families.

2. There is a multiplicity of soliton solutions in dissipative systems for a single equation (CGLE). In the case of Hamiltonian systems, one equation usually has one family of solitons. The number of soliton families increases with the number of coupled equations [12].
3. This multiplicity of solitons in dissipative systems appears in the form of two subclasses: high amplitude solitons and low amplitude solitons. In the particular case we are considering, this is a consequence of the quintic-cubic nonlinearity in the problem. However, the quintic-cubic nonlinearity is the nonlinearity of minimum complexity which admits stable solitons in the system [78]. In the case of the cubic CGLE, all soliton solutions are unstable (with the exception of arbitrary amplitude solitons [12]).

4. In dissipative systems, several solitons which belong to different branches can be stable simultaneously. For the cubic-quintic nonlinearity, the stable solitons belong to the subclass of high amplitude solitons. In contrast, in the case of Hamiltonian systems, only the lowest branch of solitons (fundamental mode) is usually stable [12]. In other words, the branch of solitons with the lowest Hamiltonian is stable.

5. In dissipative systems, the singular points of (46) partition the soliton into pieces with different stability properties. This can happen even to a “ground state” soliton with the perfect “bell-shape” profile. Such partitions can lead to some complicated behavior of dissipative solitons. The most striking example of complicated dynamics is “soliton explosions” [111]. In contrast, the “ground-state” solitons of Hamiltonian systems must be considered as single particles and are stable (or unstable) as a whole.

References