A solution space for a system of BPZ equations: rigorous results and applications

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A useful correlation function

With $N \in \mathbb{Z}^+$ and $\phi_{r,s}$ the CFT primary $(r, s)$ Kac operator, we consider the $2N$-point boundary CFT correlation functions

$$F(x) := \left\{ \begin{array}{l} \langle \phi_{1,2}(x_1) \phi_{1,2}(x_2) \cdots \phi_{1,2}(x_{2N}) \rangle, \\
\langle \phi_{2,1}(x_1) \phi_{2,1}(x_2) \cdots \phi_{2,1}(x_{2N}) \rangle, \\
\end{array} \right.$$ 

$x := (x_1, x_2, \ldots, x_{2N}), \quad x_1 < x_2 < \ldots < x_{2N}$.

These functions satisfy a certain elliptic system of $(2N+3)$ PDEs. We wish to answer the following questions about this system:

**Goals:**

1. What is the dimension/content of the solution space?
2. Which solutions have statistical mechanics applications?
3. How to find bulk CFT correlation functions from item 1?
Central charge and SLE$_\kappa$ speed

For convenience, we parameterize the CFT central charge $c \leq 1$ by the SLE$_\kappa$ speed $\kappa > 0$ as shown above, and we write

$$\psi_1 := \begin{cases} 
\phi_{2,1}, & \kappa < 4 \\
\phi_{1,2}, & \kappa \geq 4 
\end{cases}, \quad \theta_1 := \begin{cases} 
h_{2,1}, & \kappa < 4 \\
h_{1,2}, & \kappa \geq 4 
\end{cases} = \frac{6 - \kappa}{2\kappa}.$$
Conformal Ward identities

The correlation function $F(x) = \langle \psi_1(x_1) \psi_1(x_2) \cdots \rangle$ is covariant with respect to Möbius (conformal) maps. Therefore, we have

- **Translations**
  \[
  \sum_j \partial_j F = 0
  \]

- **Dilations (and rotations)**
  \[
  \sum_j [x_j \partial_j + \theta_1] F = 0
  \]

- **Special conformal transformations**
  \[
  \sum_j [x_j^2 \partial_j + 2\theta_1 x_j] F = 0
  \]

The Möbius map must preserve coordinate order on the real axis.

This collection of PDEs is called **conformal Ward identities**.
BPZ equations

The CFT null-state condition says that for each $j \in \{1, 2, \ldots, 2N\}$, $F(x) = \langle \psi_1(x_1) \psi_1(x_2) \cdots \rangle$ also satisfies the BPZ equation

$$\left[ \frac{3 \partial_j^2}{2(2\theta_1 + 1)} + \sum_{k \neq j}^{2N} \left( \frac{\partial_k}{x_k - x_j} - \frac{\theta_1}{(x_k - x_j)^2} \right) \right] F = 0.$$

Taken together with the three conformal Ward identities, $F$ solves an elliptic system of $(2N + 3)$ PDEs. Some questions:

Goals:

1. What is the dimension/content of the solution space?
2. Which solutions have statistical mechanics applications?
3. How to find bulk CFT correlation functions from item 1?
Statement of main results

Let $S_N$ be the vector space (over $\mathbb{R}$) of real-valued (classical) solutions $F$ for the system obeying the power-law bound

$$|F(x)| \leq C \prod_{i<j}^{2N} |x_j - x_i|^\mu_{ij}(p), \quad \mu_{ij}(p) := \begin{cases} -p, & |x_j - x_i| < 1 \\ +p, & |x_j - x_i| \geq 1 \end{cases}$$

for some constants $C, p > 0$ (possibly depending on $F$).

**Theorem:** Suppose that $\kappa \in (0, 8)$. Then

- $\dim S_N = C_N$, with $C_N$ the $N$th Catalan number,

$$C_N := \frac{(2N)!}{N!(N + 1)!}.$$

- $S_N$ has a basis of solutions with known explicit formulas.

The cutoff $\kappa < 8$ is natural in $\text{SLE}_\kappa$ (but not so much in CFT?).
Asymptotics of solutions

Why should this theorem be true? To prove it, we consider limits of $F \in S_N$. The CFT fusion rule $\psi_1 \times \psi_1 = \text{i.d.} + \psi_2$ suggests

$$F(x) = \left\langle \psi_1(x_1) \psi_1(x_2) \psi_1(x_3) \psi_1(x_4) \cdots \right\rangle \sim_{x_2 \to x_1} C(x_2 - x_1)^{-2\theta_1} \left\langle \psi_1(x_3) \psi_1(x_4) \cdots \right\rangle + \cdots, \quad C \in \mathbb{R}.$$ 

The $(2N - 2)$-point function on the right side should be in $S_{N-1}$. Thus, we expect the asymptotic behavior (recall $\theta_1 = (6 - \kappa)/2\kappa$)

$$F(x_1, x_2, x_3, x_4, \ldots) \sim_{x_2 \to x_1} (x_2 - x_1)^{1-6/\kappa} G(x_3, x_4, \ldots),$$

and similarly for any other pair $x_{i+1} \to x_i$ of adjacent points. Do all elements of $S_N$ exhibit this asymptotic behavior? Yes!
Lemma: Suppose that $\kappa \in (0, 8)$ and $F \in \mathcal{S}_N$. Then for all $i \in \{1, 2, \ldots, 2N - 1\}$, the limit

$$\lim_{x_{i+1} \to x_i} (x_{i+1} - x_i)^{6/\kappa - 1} F(x) \quad \text{exists, is independent of } x_i, \text{ and is in } \mathcal{S}_{N-1}.$$ 

Question: What happens if we apply another such limit, and then another, and so on?
**A sequence of limits**

As we take successive limits, points of different limits must not collide. For this, we link points pairwise with disjoint arcs in $\mathbb{H}$, and we bring together only the endpoints of an arc that does not nest another arc. After taking the limit, we delete its arc:
A sequence of limits

We take $N - 1$ more limits. Each limit brings together the two endpoints of one non-nesting arc. Eventually, no arcs are left:

Thus, we obtain a sequence of maps sending

$$S_N \rightarrow S_{N-1} \rightarrow \cdots \rightarrow S_1 \rightarrow S_0 = \mathbb{R}.$$  

Their composition is evidently a linear functional $\mathcal{L} : S_N \rightarrow \mathbb{R}$. 

Equivalence classes

Let $[\mathcal{L}]$ be the equivalence class of all functionals with the same diagram as $\mathcal{L}$ (so they differ only in the orders of their limits).

**Lemma:** If $\kappa \in (0, 8)$, then $[\mathcal{L}]F$ is well-defined for all $F \in S_N$.

There are $C_N$ arc connectivity diagrams, and thus $C_N$ such equivalence classes. The Catalan numbers make their appearance!

$N = 2$
$C_2 = 2$

$N = 3$
$C_3 = 5$

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![Diagram 1](image1.png)

Another lemma

With exactly $C_N$ distinct equivalence classes $[L_1], [L_2], \ldots, [L_{C_N}]$, we can obtain an upper bound on $\dim S_N$:

**Lemma:** Suppose that $F \in S_N$. If $\kappa \in (0, 8)$, then the map

$$v : S_N \to \mathbb{R}^{C_N}, \quad v(F)_\alpha := [L_\alpha]F$$

is a linear injection, so $\dim S_N \leq C_N$. (But is this an equality?)
Coulomb gas solutions

For $\beta \in \{1, 2, \ldots, C_N\}$, we define the **Coulomb gas function** as (Feigin, Fuchs, Dotsenko, Fateev, Dubédat, Kytölä, Peltola)

$$
\mathcal{F}_\beta(\kappa \mid x_1, x_2, \ldots, x_{2N}) := \left[ \frac{\nu(\kappa)\Gamma(2 - \frac{8}{\kappa})}{\Gamma(1 - \frac{4}{\kappa})^2} \right]^N \\
\times \left( \prod_{j<k}^{2N} (x_k - x_j)^{2/\kappa} \right) \int_{\Gamma_N} du_N \int_{\Gamma_{N-1}} du_{N-1} \cdots \int_{\Gamma_2} du_2 \\
\times \int_{\Gamma_1} du_1 \left( \prod_{l=1}^{2N} \prod_{m=1}^N (x_l - u_m)^{-4/\kappa} \right) \left( \prod_{p<q}^N (u_p - u_q)^{8/\kappa} \right),
$$

and a **Coulomb gas solution** as a linear combination of $\{\mathcal{F}_\beta\}$.

- For all $\beta \in \{1, 2, \ldots, C_N\}$, we have $\mathcal{F}_\beta \in \mathcal{S}_N$.
- $\nu(\kappa) := -2 \cos(4\pi/\kappa)$ is the $O(\nu)$ model loop-gas fugacity.
- We phase the integrand’s power functions so $\mathcal{F}_\beta(\boldsymbol{x})$ is real.
Coulomb gas solutions

- The integration contours of $\mathcal{F}_\beta$ pair the $x_j$ in $\beta$th connectivity:

  \[ N = 2 \]
  \[ C_2 = 2 \]

  $\mathcal{F}_1$  $\mathcal{F}_2$

  \[ N = 3 \]
  \[ C_3 = 5 \]

  $\mathcal{F}_1$  $\mathcal{F}_2$  $\mathcal{F}_3$  $\mathcal{F}_4$  $\mathcal{F}_5$

- If $\kappa \leq 4$ so these integrals diverge, then we analytically continue them by replacing all contours with Pochhammer contours:

  \[ \frac{1}{4 \sin^2(4\pi/\kappa)} \]
Rank of $\mathcal{B}_N$

To prove $\dim S_N = C_N$, we show that $\mathcal{B}_N := \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{C_N}\} \subset S_N$ is linearly independent. We do this indirectly:

$$v(\mathcal{B}_N) := \{v(\mathcal{F}_1), v(\mathcal{F}_2), \ldots, v(\mathcal{F}_{C_N})\} \subset \mathbb{R}^{C_N}$$

is linearly independent iff $\mathcal{B}_N$ is linearly independent because $v$ is injective. We find $v(\mathcal{F}_\beta)_\alpha := [\mathcal{L}_\alpha]\mathcal{F}_\beta$ is given by the diagram

$$[\mathcal{L}_\alpha]\mathcal{F}_\beta = \nu^{l_{\alpha,\beta}} = \text{number of loops}$$

(two in this example)

Again, we have the “$O(\nu)$ loop fugacity” $\nu(\kappa) := -2 \cos(4\pi/\kappa)$. 
The meander matrix

Meander matrix: The $C_N \times C_N$ Gram matrix $M_N(\nu)$ of the “bilinear form” $[\mathcal{L}_\alpha] \mathcal{F}_\beta$. The $(\alpha, \beta)$th entry of $M_N(\nu)$ is $\nu^{l_{\alpha,\beta}}$.

Example: The $N = 2$ meander matrix $M_2$:

$$
\begin{pmatrix}
\nu^2 & \nu \\
\nu & \nu^2
\end{pmatrix}
$$
The main theorem

The determinant of the meander matrix is usually nonzero (and if it vanishes, then there is a “fix”). This implies the following:

**Theorem:** Suppose that $\kappa \in (0, 8)$. Then

1. CG functions form a basis for $S_N$ if and only if $\kappa$ is not among the finite set of zeros of the meander determinant.
2. $\dim S_N = C_N$, with $C_N$ the $N$th Catalan number.
3. Each element of $S_N$ is either a Coulomb gas solution or a limit as $\kappa \to \kappa$ of a Coulomb gas solution.
4. The map $v : S_N \to \mathbb{R}^{C_N}$ with components $v(F)_{\alpha} := [L_{\alpha}]F$ is a vector-space isomorphism.
5. The set $B_N^* := \{[L_1], [L_2], \ldots, [L_{C_N}]\}$ is a basis for $S_N^*$.

(Item 5 follows almost immediately from the bijectivity of $v$.)
Some physical applications

Next, we use the above methods and results to find particular elements of $S_N$ with these physical applications.

1. The pure partition functions of multiple $\text{SLE}_\kappa$.
2. Crossing-event probabilities, generalizing Cardy’s formula.
Application: multiple $\text{SLE}_\kappa$

Critical $Q$-state FK-Potts model (i.e., percolation with $Q$ colors) on a regular lattice in a $2N$-gon $\mathcal{P}$ with every other side fixed:

Boundary clusters touch fixed sides. Random boundary arcs trace the boundary clusters and join the vertices of $\mathcal{P}$ pairwise.
Multiple Schramm-Loewner evolution

After conformally sending $\mathcal{P}$ onto $\mathbb{H}$ and shrinking the lattice spacing, the boundary arcs fluctuate to the law of multiple $\text{SLE}_\kappa$.

$$Q = 4 \cos^2 \left( \frac{4\pi}{\kappa} \right), \quad \kappa > 4, \quad Q \in \{1, 2, 3, 4\}.$$ 

This is a stochastic process that grows $N$ random curves into $\mathbb{H}$ from $2N$ marked points in $\mathbb{R}$. (It is a generalization of $\text{SLE}_\kappa$.)

The stochastic DEs of multiple $\text{SLE}_\kappa$ require a (deterministic, positive-valued) $\text{SLE}_\kappa$ partition function $F \in S_N$ for input.
Pure SLE$_\kappa$ partition functions

We expect that there are $C_N$ positive-valued elements of $S_N$, $Z_1$, $Z_2$, ..., $Z_{CN}$, called \textbf{pure SLE$_\kappa$ partition functions}, such that

$$F = Z_\beta \iff \begin{cases} \text{multiple-SLE$_\kappa$ curves a.s. close pairwise into arcs} \\ \text{joining polygon vertices in the } \beta \text{th connectivity.} \end{cases}$$

Non-rigorous physical arguments reveal the asymptotic behavior of each $Z_\beta(x)$ as $x_{i+1} \to x_i$. This leads to the following guess...
Connectivity weights

\[ [\mathcal{L}_\alpha] = \langle \quad \rangle \rightarrow \Pi_\alpha = \langle \quad \rangle \]

**Definition:** For all \( \kappa \in (0, 8) \), let the \( \alpha \)th connectivity weight \( \Pi_\alpha \) be the element of \( S_N \) dual to \( [\mathcal{L}_\alpha] \in \mathcal{B}_N^* \), that is,

\[ [\mathcal{L}_\alpha] \Pi_\beta = \delta_{\alpha, \beta}, \]

and \( \mathcal{B}_N := \{ \Pi_1, \Pi_2, \ldots, \Pi_{CN} \} \) be the basis for \( S_N \) dual to \( \mathcal{B}_N^* \). We find explicit formulas for all of the \( \Pi_\beta \) by isolating them from

\[ \mathcal{F}_\alpha = \sum_{\beta=1}^{CN} \nu^{l_{\alpha, \beta}} \Pi_\beta, \quad 1 \leq \alpha, \beta \leq CN. \]

**Conjecture:** \( \Pi_\beta \) is the \( \beta \)th pure SLE\( _\kappa \) partition function \( Z_\beta \).
Application: crossing probabilities

Critical $Q$-state FK-Potts model (i.e., percolation with $Q$ colors) on a regular lattice in a $2N$-gon $\mathcal{P}$ with every other side fixed:

What is the probability of a particular topological configuration of crossing paths (red) between the various fixed sides of $\mathcal{P}$?

For critical percolation ($Q = 1$) in the rectangle ($N = 2$), the answer is given by the (famous) Cardy-Smirnov formula.
Crossing events in $2N$-gons

- Sides of $\mathcal{P}$ alternate from fixed to free. This is a **fixed/free side-alternating boundary condition (FFBC)**.

- Boundary clusters (filled gray) touching the fixed sides join those sides in one of $C_N$ crossing configurations.

- Boundary arcs (colored red) trace the boundary clusters’ perimeters and fluctuate to the law of multiple $\text{SLE}_\kappa$.

- Probability of $\alpha$th crossing configuration $\mathcal{E}_\alpha \sim$ Probability of multiple $\text{SLE}_\kappa$ curves closing in $\alpha$th connectivity $\sim \Pi_\alpha$. 
Independent vs. mutual wiring FFBC events

There may be many different FFBC events. They may arise in, for example, the Potts model, from coloring the fixed sides of $\mathcal{P}$.

- mutually wired: constrain all lattice sites in a pair of fixed sides to be in the same state,
- independently wired: do not impose the above constraint on the pair of fixed sides.

Let $\mathcal{E}_\beta$ be the FFBC where any two fixed sides that are joined together in the $\beta$th connectivity are mutually wired together.
The crossing probability formula

Prediction (supported by computer simulation) for probability of \( \alpha \)th crossing event \( \mathcal{E}_\alpha \) conditioned on \( \beta \)th FFBC event \( \mathcal{E}_\beta \):

\[
\mathbb{P}(\mathcal{E}_\alpha \mid \mathcal{E}_\beta) = \frac{\text{partition function for crossing event } \mathcal{E}_\alpha}{\text{partition function for FFBC event } \mathcal{E}_\beta} 
\]

\[\Rightarrow \quad \mathbb{P}(\mathcal{E}_\alpha \mid \mathcal{E}_\beta) = Q^{l_{\alpha,\beta}/2} \frac{\Pi_{\alpha}(\kappa \mid x_1, x_2, \ldots, x_{2N})}{\mathcal{F}_\beta(\kappa \mid x_1, x_2, \ldots, x_{2N})}.\]

\( x_j \) = image of the \( j \)th vertex under a conformal map \( \mathcal{P} \) onto \( \mathbb{H} \).
Now we consider the bulk CFT correlation function, with $2N$ holomorphic coordinates and $2N$ antiholomorphic coordinates,

$$F(z, \bar{z}) = \langle \psi_1(z_1, \bar{z}_1) \psi_1(z_2, \bar{z}_2) \cdots \psi_1(z_{2N}, \bar{z}_{2N}) \rangle \in S_N \otimes \bar{S}_N.$$ 

A generic element of $S_N \otimes \bar{S}_N$ is a multivalued function of the domain $\mathbb{C}^{2N} \times \bar{\mathbb{C}}^{2N}$, where $\bar{\mathbb{C}}^{2N}$ is an independent copy of $\mathbb{C}^{2N}$.

$$\mathbb{C}_\times^{2N} := \{ z = (z_1, z_2, \ldots, z_{2N}) \mid z_j \neq z_k \text{ if } j \neq k \}.$$ 

More precisely, a generic element of $S_N \otimes \bar{S}_N$ is a function of $\mathcal{U} \times \bar{\mathcal{U}}$, where $\mathcal{U}$ denotes the universal cover of $\mathbb{C}_\times^{2N}$.

**Question:** By definition, the function $F$ is single-valued on $\mathbb{C}_\times^{2N}$. Is there such a function in $S_N \otimes \bar{S}_N$? Is it unique? A formula?
The braid group action

We identify a path $\gamma : [0, 1] \to \mathcal{U}$ whose endpoints project to the same point $z \in \mathbb{C}_x^{2N}$ with an element $\sigma$ of the braid group $\text{Br}_{2N}$:

$$F \in \mathcal{S}_N \otimes \bar{\mathcal{S}}_N$$ is single-valued on $\mathbb{C}_x^{2N} \times \bar{\mathbb{C}}_x^{2N}$ iff invariant under

$$F(\gamma(0), \bar{\gamma}(0)) \xrightarrow{\sigma} F(\gamma(1), \bar{\gamma}(1)), \quad \gamma(0) = z \in \mathbb{C}_x^{2N}.$$

This defines a $\text{Br}_{2N}$-representation: $\rho \otimes \bar{\rho} : \text{Br}_{2N} \to \text{GL} \mathcal{S}_N \otimes \bar{\mathcal{S}}_N$. 

\[ \sigma_i = \text{ith braid group generator} \]
Holomorphic vs. antiholomorphic

This representation $\rho \otimes \bar{\rho}$ comprises two parts:

1. Holomorphic: $\rho : \text{Br}_{2N} \to \text{GL } \mathcal{S}_N$, $F(\gamma(0)) \xrightarrow{\sigma} F(\gamma(1))$,
2. Antiholomo.: $\bar{\rho} : \text{Br}_{2N} \to \text{GL } \bar{\mathcal{S}}_N$, $F(\bar{\gamma}(0)) \xrightarrow{\sigma} F(\bar{\gamma}(1))$.

They are related by $\bar{\rho}(\sigma_i) = \rho(\sigma_i^{-1})$ for all $\text{Br}_{2N}$-generators $\sigma_i$. 
An explicit correlation function formula

Using the relation of the previous slide, we can construct an isomorphism of representations

\[ T : (\rho \otimes \bar{\rho}, \mathcal{S}_N \otimes \bar{\mathcal{S}}_N) \rightarrow (\rho_{\text{End}\mathcal{S}_N}, \text{End}\mathcal{S}_N). \]

Also, \((\rho, \mathcal{S}_N)\) is irreducible, so Schur’s lemma says that both sides have a unique trivial submodule \(M\). That is, if \(F \in M\), then

\[ F(\gamma(0), \bar{\gamma}(0)) = F(\gamma(1), \bar{\gamma}(1)), \quad \text{for all } \sigma \in \text{Br}_{2N} \text{ and } \gamma \sim \sigma. \]

This function is (up to a constant) the correlation function we seek. By prudent guessing, we can find its explicit formula:

\[
\langle \psi_1(z_1, \bar{z}_1) \psi_1(z_2, \bar{z}_2) \cdots \psi_1(z_{2N}, \bar{z}_{2N}) \rangle \\
\propto \sum_{\beta=1}^{C_N} \mathcal{F}_\beta(\kappa \mid z) \Pi_\beta(\kappa \mid \bar{z}) = \sum_{\beta=1}^{C_N} \Pi_\beta(\kappa \mid z) \mathcal{F}_\beta(\kappa \mid \bar{z}).
\]
Summary

In this talk, we have

1. defined $S_N =$ space of power-law-bounded functions solving the PDEs of the CFT correlation function $\langle \psi_1 \psi_1 \cdots \psi_1 \rangle$,

2. determined the dimension of $S_N$ ($\dim S_N = C_N$) and a basis of explicit functions (Coulomb gas functions) for it,

3. used the tools developed in the proof of item 2 to predict explicit formulas for all $\text{SLE}_\kappa$ pure partition functions,

4. used the $\text{SLE}_\kappa$ pure partition functions to predict a formula for Potts model crossing probabilities in polygons,

5. used item 2 results to find a unique single-valued formula for the bulk $2N$-point correlation function $\langle \psi_1 \psi_1 \cdots \psi_1 \rangle$. 
The end

Thank you.