How big is a small Langmuir probe?

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The area of the sheath around a thin, disk-shaped electrode that is biased below the plasma potential has been computed using a hybrid simulation with cold, collisionless ions and Boltzmann electrons. That is, the “collecting area” of a double-sided, planar Langmuir probe has been determined for the ion saturation current regime. Sheath areas are calculated for probe radii from 10 to 45 electron Debye lengths and for probe biases from −5 to −30 times the electron temperature. The dependence of the sheath area on probe radius and bias is parameterized using simple empirical formulas. © 2000 American Institute of Physics.

I. INTRODUCTION

Langmuir (i.e., electrostatic) probes are widely used to diagnose “low-temperature” and processing plasmas. As has often been noted, the ease of construction and use of such probes is counterbalanced by the difficulty of extracting quantitative information from the measured current–voltage (I–V) characteristic. Theories are most fully developed for highly-symmetric, one-dimensional geometries such as spheres, infinite planes, and infinitely-long cylinders. End and edge effects, which can dominate the probe characteristic, have received only modest attention since they are extremely difficult to analyze.

Thin, disk-shaped Langmuir probes are often used because their electron and ion saturation currents are thought to be weak functions of probe bias and plasma density when the probe radius is large compared to the sheath width (i.e., the probe’s “collecting area” does not change much). In particular, the ion saturation current $I_s$ can be written as

$$I_s = e n_0 \kappa A_s v_B,$$

(1)

where $n_0$ is the plasma density, $\kappa$ is a constant (e.g., $\kappa \approx 0.6$ for a spherical presheath), $A_s$ is the sheath or “effective collecting” area of the probe, and $v_B$ is the Bohm speed (which for cold ions equals the ion acoustic speed $c_s$). The ion saturation current is typically measured at large negative biases to minimize effects due to variations in the plasma potential, as well as to exclude energetic electrons. As a consequence, a probe’s collecting area can significantly exceed its physical area, which effect is (probably) impossible to quantify analytically since the probe geometry is fully two-dimensional.

In this paper, we present a novel method for computing the sheath area $A_s$ around a disk electrode, as described in Sec. II. In Sec. III, we validate our technique for a one-dimensional spherical probe and then compute the dependence of the collecting area on probe bias and radius for small disk probes. To the best of our knowledge, this is the first published report of quantitative results for this widely-used probe geometry. These results should be of interest to all experimentalists using Langmuir probes, as well as to those who wish to calculate the properties of the sheath around other realistic electrode geometries.

II. MODEL

We wish to determine the properties of the collisionless, steady-state sheath surrounding a thin, two-sided conducting disk of radius $r_p$ biased at a negative potential $\phi_p$ (i.e., below the plasma potential) in a stationary, unmagnetized plasma. (We ignore the structure supporting the disk.) Far from the probe the plasma density is $n_0$ and the plasma potential $\phi = 0$. The plasma contains cold, collisionless ions with mass $M_i$, density $n_i$ and fluid velocity $\mathbf{v}_i$, and Boltzmann electrons with a temperature $T_e$. (The assumption of Boltzmann electrons introduces some error for $|\phi_p| \approx 3 T_e$ since a non-negligible fraction of electrons are absorbed by the probe.) That is, the ion temperature is assumed small in comparison to both $T_e$ and $\phi_p$, as is the usual case for “low-temperature” plasmas. For spherical probes, it has been shown that finite-ion-temperature effects reduce the current by $\lesssim 15\%$ for large-radius probes ($\approx 10$ Debye lengths) at moderate biases ($\approx 5 T_e$), and that the discrepancy decreases as the probe size increases.

The steady-state sheath and presheath can be described by the ion fluid equations of continuity and motion together with Poisson’s equation:

$$\nabla \cdot (n_i \mathbf{v}_i) = 0,$$

(2a)

$$\mathbf{v}_i \cdot \nabla \mathbf{v}_i = -e \nabla \phi / M_i,$$

(2b)

$$\epsilon_0 \nabla^2 \phi = -e [n_i - n_0 \exp(e \phi / k T_e)],$$

(2c)

respectively. At a sufficient distance from any finite-sized probe, the presheath must be nearly spherical. Consequently, the plasma source can be moved to infinity, as in the spherical case. Azimuthal symmetry is assumed. In comparison to one-dimensional problems, those in two or three dimensions are significantly more difficult because even for cold ions the direction of the ion velocity can only be found by integrating the ion trajectories.

Equations (2a)–(2c) are nondimensionalized using the variables

$\epsilon_0 = \frac{e^2}{M_i}$

and

$\phi_p = \frac{e \phi_p}{k T_e}$,
where the electron Debye length and ion sound speed (or Bohm speed) are
\[ \lambda_e = \sqrt{\epsilon_0 k T_e / e^2 n_0}, \quad c_s = \sqrt{k T_e / M_i}. \]
The governing equations then become
\[ \nabla \cdot (n u) = 0, \quad (u \cdot \nabla) u = \nabla \eta, \quad \nabla^2 \eta = n - \exp(-\eta). \]
These equations do not contain any specifiable parameters. Consequently, solutions are characterized by only two parameters: the probe radius and the probe bias,
\[ \rho_p = r_p / \lambda_e, \quad \eta_p = -e \phi_p / k T_e, \]
respectively. (A cylindrical probe has three parameters, radius, bias, and length.)

Boundary conditions must be specified on the probe and in the plasma. On the probe, the potential \( \phi_p < 0 \), or the dimensionless potential \( \eta_p > 0 \), is given. In the plasma, there are two ways of treating the "boundary." The first is to assume a well-defined plasma-sheath boundary (e.g., the Child–Langmuir Law). In this case, the ion density and velocity as well as the potential and electric field must be specified on the boundary. In two dimensions, this approach is complicated by the need to determine a boundary surface consistent with the chosen boundary conditions (i.e., this is an "eigen-boundary" problem). The second approach is to specify an ion current far from the probe, so that a presheath and sheath form naturally. For example, for a spherical probe, the current is specified on a spherical surface at a radius \( r > r_p \). This method is preferable on physical grounds as it is not necessary to make the somewhat arbitrary division between plasma and sheath.

We use a novel computational approach to sidestep the problem of determining the plasma boundary as well as to calculate the ion density and potential in the sheath. We solve the time-dependent problem of sheath expansion away from a pulsed electrode using a self-consistent plasma simulation. The bias on the probe is of the form
\[ \phi = \phi_p [1 - \exp(-t/t_r)], \]
where \( t_r \) is a rise time. For one-dimensional spherical electrodes, when the speed of the expanding sheath edge falls below \( c_s \), a "presheath" separates from the sheath and moves into the plasma at the sound speed. The sheath then becomes stationary while the presheath expands into the ambient plasma feeding ions to the sheath. As we demonstrate below, the structure of the stationary sheath with an expanding presheath approaches that predicted for a stationary sheath with a stationary presheath having a finite ion current.

### III. RESULTS AND DISCUSSION

#### A. One-dimensional sphere

To validate our method, we have applied it to a one-dimensional spherical probe, for which the time-independent result can readily be calculated for cold ions. Rather than solve the ion fluid equations directly, we use the particle-in-cell (PIC) technique to solve the ion Vlasov equation. Physical parameters are: probe radii \( r_p = 10 \) and \( 45 \lambda_e \) for probe biases \( \phi_p = -5 \) to \(-30 k T_e / e\). Simulation parameters are: \( t_r = \omega_{pi}^{-1} \), grid spacing \( \Delta x = 1/2 \lambda_e \), time step \( \Delta t = 1/32 \omega_{pi}^{-1} \), and initially 64 cold ion particles are placed in each simulation cell.

We consider two possible definitions for the sheath edge, \( u = 1 \) and \( \eta = 1/2 \), as shown in Fig. 1, where we plot the sheath area normalized by the spherical probe area \( A_p = 4 \pi r_p^2 \) as a function of time. The condition \( u = 1 \) is the Bohm criterion for (static) sheath formation, while the condition \( \eta = 1/2 \) is the predicted value of the potential at the edge of a spherical presheath. The sheath edge defined by \( \eta = 1/2 \) displays a large transient overshoot, as observed in experiment. The sheath edge defined by \( u = 1 \) also overshoots, but to a lesser extent. The two curves eventually merge, indicating that a stationary sheath has been established. However, even before a static sheath is achieved, the \( u = 1 \) curve is quite close to its asymptotic value. Consequently, in what follows, we use the condition \( u = 1 \) to estimate the steady-state position of the sheath edge.

In Fig. 2, we compare values of the sheath area computed using our computational technique to solutions of the time-independent, cold-ion sheath equation for a spherical probe. Results at times \( t_\omega_{pi} = 50 \) and 100 indicate that the sheath is indeed attaining the same structure as a time-independent sheath with a stationary presheath. At \( t_\omega_{pi} = 50 \), the sheath area calculated from the simulation overestimates the stationary sheath area by not more than 10%.
B. Two-dimensional disk

To compute the sheath structure around the disk probe, we use a two-dimensional (2D) code with PIC ions. The ion velocities lie in the \( r-z \) plane (i.e., the code is ‘‘2D2v’’), so that orbits with a nonzero azimuthal angular momentum are not considered. However, because of the assumed azimuthal symmetry, no forces act in the azimuthal direction. The sheath edge is defined as the surface on which the average ion speed equals the ion acoustic speed \( \left( u = u_1 \right) \), and the sheath area is normalized by the double-sided probe area, \( A_p \). Simulation parameters are: \( t_{\pi} = \omega^{-1} \), grid spacing \( \Delta r = \Delta z = \lambda_{e0} \), time step \( \Delta t = 1/32 \omega^{-1} \), and 100 cold ion particles are initially loaded into each cell. Results are given at \( t_{\pi} = 50 \) because of the computational burden of the 2D code. As indicated in Fig. 2, this may lead us to overestimate the sheath area by less than 10%. The size of the computational grid is chosen such that the presheath does not reach the simulation-box boundary during the time simulated—the grid is about \( 100 \times 100 \), so that there are approximately one million ion particles.

The expansion of the sheath edge with increasing probe bias is indicated in Fig. 3 for probe radii \( r_p/\lambda_{e0} = 15 \) and 30. [Note that a change in \( r_p/\lambda_{e0} \) may be due to a change in either the probe radius or the Debye length (i.e., \( n_0 \) or \( T_e \).)] The sheath edges display the characteristic oblate ellipsoid-like shape observed experimentally.\(^{13-15} \) The sheath is seen to be nearest to the probe at the edge of the disk, while for the same probe bias the sheath is much closer to the smaller probe. As the probe bias increases, the sheath shape changes and the sheath edges become more spherical. Note that even for small biases there is no region near the axis where the sheath edge is flat (i.e., well approximated by one-dimensional planar theory).

The potential dependence of the normalized sheath area

\[ A_s = \frac{A_s}{A_p} = 1 + a r_p^b \]

FIG. 2. Normalized sheath area vs probe bias for spherical probes of radii \( r_p = 10 \) and \( 45 \lambda_{e0} \). Time-independent, cold-ion sheath theory is compared with results from our time-dependent code at \( t_{\pi} = 50 \) and 100 (assuming the sheath edge is defined by \( u = 1 \)). The sheath area we calculate is in good agreement with time-independent theory.

FIG. 3. Expansion of the steady-state sheath with probe bias for disk probes of radii (a) \( r_p = 15 \lambda_{e0} \) and (b) \( r_p = 30 \lambda_{e0} \) for biases \( \eta_p = -5, -10, -15, -20, -25 \), and \(-30 \). The average ion velocity equals the Bohm speed. The sheath edges show a characteristic ellipsoid-like shape.

FIG. 4. Potential dependence of the normalized sheath area for disks of radii \( r_p = 10, 15, 20, 30, \) and \( 45 \lambda_{e0} \) together with fitted curves [Eq. (8), solid lines]. Results are shown on linear axes in (a) and logarithmic axes in (b), clearly showing a power-law dependence. The small cross in (a) is the result of a more extensive computation.
Here considered. We have quantified the radius-dependence of to the sheath area data in Fig. 4 to the sheath area data in Fig. 4(a) for \( r_p / \lambda_{e0} = 10, 15, 20, 30, \) and 45, covering a range of \( \approx 20 \) in plasma density (for a constant \( T_e \)). The curves display a decreasing curvature with increasing probe bias that is familiar to experimentalists, while smaller probes show larger “edge” effects. However, even for the largest probe the area of the sheath exceeds that of the disk by up to a factor of 2. As shown in Fig. 4(b), the computed sheath area \( A_s / A_p - 1 \) exhibits an approximate power-law dependence on \( \eta_p \) (or \( \rho_p \)) when \( \rho_p \) (\( \eta_p \)) is fixed,\(^3\) where the exponent has a weak residual dependence on \( \rho_p \) (\( \eta_p \)). For concreteness, we have fitted the function

\[
A_s / A_p = 1 + a \eta_p^b
\]

to the sheath area data in Fig. 4(a), where \( a \) and \( b \) are fitting parameters that depend on the probe radius. In each case, the fitted curve and the computed areas are in excellent agreement. The dependence of \( a \) and \( b \) on \( \rho_p \) is shown in Fig. 5. Here \( a \) depends strongly on \( \rho_p \), while \( b \) lies between 0.6 and 0.7 and is nearly independent of \( \rho_p \) for the values of \( \rho_p \) here considered. We have quantified the radius-dependence of \( a \) and \( b \) by fitting each with a power-law curve (Fig. 5) giving

\[
a = 2.28 \rho_p^{-0.749}, \quad b = 0.806 \rho_p^{-0.03692}.
\]

Together, Eqs. (8) and (9) describe the computed dependence of the sheath area on probe radius and bias over the entire range of probe radii considered. In combination with Eq. (1), this allows a quantitative determination of the plasma density from the bias-dependence of the ion saturation current to a disk probe given the electron temperature and plasma potential. Calculating the plasma density will be an iterative process since the dimensionless probe radius depends on the Debye length, which in turn depends on the plasma density.

The ion saturation current measured at a fixed probe bias is often used to estimate the relative variation in plasma den-

![Fig. 5. Dependence of the fitting coefficients \( a \) and \( b \) on probe radius together with fitted power-law curves [Eqs. (9)]. This completely parameterizes the collecting area of a small disk-shaped Langmuir probe as a continuous function of probe radius and bias in the limit of zero ion temperature.](image)

sity, for example, with position, neutral pressure or discharge power. That is, \( I_s \propto n_0 \) is assumed, though it is more correct to say \( I_s \propto A_s n_0 \). One implication of this work is that variations in the ion saturation current (for a fixed \( \eta_p \)) can significantly underestimate the true variation in \( n_0 \), particularly for smaller probes, because the collecting area increases as \( n_0 \) decreases [i.e., the (dimensionless) probe “shrinks” as the plasma density decreases]. A linear extrapolation of the ion-saturation current to zero bias can reduce the error somewhat. However, such a procedure is ill-defined and will lead to a systematic overestimation of \( n_0 \) because of the variation of the curvature of the characteristic with probe radius and bias.

There are a number of uncertainties in our theory. First, there is the systematic overestimation of the sheath area discussed above. This is illustrated by the small cross in Fig. 4(a), which is the result of a more extensive calculation to \( t \omega_p = 50 \). This point is only 3.8% below that calculated for \( t \omega_p = 50 \). Second, finite-ion-temperature effects have been neglected. As discussed above, we estimate this results in an overestimate of the sheath area of not more than 15%. Finally, we find that the value of \( \kappa \) in Eq. (1), which represents the ratio of the ion density at the sheath edge to that in the ambient plasma, depends on the curvature of the sheath edge.

We observe \( \kappa \approx 0.5 \) where the sheath edge is flatter (the value for a planar sheath) with \( \kappa \) increasing to \( \approx 0.6 \) where the sheath edge has high curvature. Consequently, using \( \kappa \approx 0.55 \) should not be too bad in any case.

**IV. SUMMARY**

We have presented a new technique for computing the stationary sheath structure around two-dimensional electrodes. This technique was used to calculate the dependence of the sheath area on probe bias and radius for a thin, disk-shaped Langmuir probe assuming cold ions. We believe that this is the first published report of such a calculation. The technique described here can be easily extended to related problems. The effects of ion-neutral collisionality can be studied by including Monte Carlo collisions (i.e., “PIC/MCC”). Finite-ion-temperature effects, which will be relatively more important for smaller probes, can be treated by initializing the ion velocities from a Maxwellian as well as by including the azimuthal velocity component. Finally, other probe geometries (including three-dimensional problems) can be easily investigated.

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