Transitions from electrostatic to electromagnetic whistler wave excitation

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(Received 29 December 2003; accepted 30 January 2004; published online 16 April 2004)

At frequencies below the electron cyclotron and above the lower hybrid frequency in a magnetoplasma, the refractive index is anisotropic and shows resonances at certain angles caused by electron inertia. For a radiating antenna in a plasma, the short wavelengths near this resonance angle may contribute to the radiation pattern of the antenna. A series of experiments is reported, in which waves were excited using a small 1-cm-diam electrostatically coupled antenna into a preformed plasma, densities (~n_e) from 10^{15} to 10^{18} m^{-3} and magnetic fields from 30 to 60 G. Maps of the wave amplitude and phase were made within the plasma by scanning the position of a b-dot probe. At low densities (e.g., n_e<5 \times 10^{16} m^{-3} at 50 G and 3 mTorr), a single amplitude maximum along the group velocity resonance cone angle was measured that decayed as the distance from the antenna increased. The observed radiation pattern in all cases was consistent with that of a point source in an unbounded plasma, and no global eigenmode resonances were found. As the density was increased, the apparent attenuation of the resonance cone waves increased and they appeared to withdraw into the antenna. At high densities (n_e>5 \times 10^{16} m^{-3}) the radiation pattern was characterized by a monotonic increase in wave phase in the axial direction, a central maximum for B_z, and off-axis maxima for B_x and B_y which are consistent with the propagation of m=0 helicon waves, and no evidence of resonance cone structure. This change in the radiation pattern is reproduced numerically in a homogeneous plasma model including an electrostatically coupled antenna with the same geometry as that used in the experiment. © 2004 American Institute of Physics. [DOI: 10.1063/1.1689352]

I. INTRODUCTION

Whistler waves are right-hand-polarized waves that propagate between the electron cyclotron and lower hybrid frequencies (ω_{ce} and ω_{LH}, respectively), typically in “over-dense” laboratory plasmas (ω_{pe}>ω_{ce}, where ω_{pe} is the plasma frequency). Their dispersion is characterized by the Hall term and electron inertia, with the magnetohydrodynamic (MHD) term (ν×B) being comparatively small for frequencies above the lower hybrid frequency. The Hall term dispersion yields the classical ionospheric signature for whistlers; the frequency is inversely proportional to the square of the wavelength and the wave is electromagnetic. Introducing electron inertia produces a resonance at ω_{ce} and also at a propagation angle for the wave group velocity given by sin(θ)=ω/ω_{ce} where θ is called the resonance cone angle. At ω_{ce}, θ is 90° and decreases, or approaches the magnetic field direction, as the frequency decreases until, at ω_{LH}, θ is essentially parallel to the magnetic field B_0. Below ω_{LH}, the refractive index surfaces become closed and the whistler becomes a magnetoacoustic wave it propagates perpendicular to B_0. If the electrostatic approximation is made, then we need only deal with electron inertia and experiments with small punctual antennae in the laboratory have shown the existence of the group velocity resonance cones at low plasma densities, around 10^{15} m^{-3} (Ref. 1). Reflections of these resonance cones from cylindrical boundaries give rise to the electrostatic waves first analyzed by Trivelpiece and Gould. For higher densities, the electromagnetic part of the dispersion becomes more and more important and both the Hall and electron inertia terms need to be addressed. This has been remarked upon by many authors. More recently it has become popular to call the electrostatic waves on the resonance cone Trivelpiece–Gould modes.

Above 0.5ω_{ce} the dispersion is dominated by electron inertia and the waves are often called electron cyclotron waves even though they have an electromagnetic character when propagating along B_0. This latter characteristic is used in electron cyclotron resonance (ECR) plasma sources where the waves are absorbed by Doppler-shifted cyclotron resonance in a magnetic beach at frequencies around 0.9ω_{ce}.

At 0.5ω_{ce} the effect of the Hall term starts to manifest itself by flattening out that part of the refractive index surface close to the axis parallel to B_0. This has the curious effect of channeling the waves along the magnetic field since the group velocity vector is normal to the refractive index surface.

Below 0.5ω_{ce} the refractive index surfaces in an infinite plasma for these waves have two distinct branches: the electromagnetic or helicon branch for low values of k_\perp (where k_\perp is the perpendicular wave number) and the electrostatic or Trivelpiece–Gould (TG) branch for high values of k_\perp.

The electromagnetic branch has been used in laboratory
and industrial plasma sources to couple radio frequency power by means of an external antenna. These so-called helicon sources are very efficient producers of high-density plasmas at low filling pressures. A great deal of interest has been generated over the last three decades in the propagation and damping of helicon waves in bounded plasmas for this reason. In recent years it has been argued that this efficiency may be explained by power coupling to TG modes, which are themselves heavily damped due to their small wavelength and phase velocity which is lower than the electron thermal velocity. TG modes can be launched directly by an external antenna and propagate along the periphery of the plasma (for this reason they are also described as surface waves) and are also thought to arise from helicon modes by a mode coupling process in regions of high-density gradient near the periphery. In this picture, the mode coupling results from the inability of either the helicon or TG wave alone to satisfy the radial boundary conditions; hence the eigenfunctions formed are comprised of a combination of the two wave types. To some extent, the classification is one of semantics as the dispersion of the actual wave in the plasma involves both terms. 

The experiments described in this paper were designed to investigate the conditions under which either the electrostatic nature or the electromagnetic nature of the wave was dominant. Test waves were launched from a small 1-cm-diam disk antenna into the center of a preformed, weakly magnetized cylindrical plasma. Section II gives a short overview of the theory for whistler wave excitation that is relevant to the experiment. In Sec. III, we describe the experimental method whereby the wave fields are measured on a two-dimensional grid over a series of plasma pulses. Section IV reports the results from density scan experiments carried out under various conditions that show a dramatic transition in the radiated field structure. In Sec. V we reproduce the experimental results by means of a simple model of the antenna in an infinite homogeneous plasma, and Sec. VI presents our conclusions.

II. THEORETICAL BACKGROUND: WHISTLER WAVE EXCITATION FROM A POINT SOURCE

The purpose of this section is to illustrate the form that propagating whistler wave fronts should take in real space when excited from an ideal point source antenna, using the geometrical method of Stix. This is a necessary background for understanding the results presented in this paper. The general equation for plasma waves in a homogeneous cold plasma is

$$k \times k \times E + \frac{\omega^2}{c^2} \epsilon \cdot E = \omega^2 \epsilon - A \cdot E = 0,$$  

(1)

where $\omega$ is the wave angular frequency, $c$ is the speed of light, $\epsilon$ is the cold plasma dielectric tensor, and $E(\omega, k)$ is the wave electric field vector in Fourier space.

In the regime of frequencies where $\omega_L \ll \omega < \omega_{ce}$, ion motions and displacement currents can be neglected in the relevant terms of the dielectric tensor, although electron inertia is retained. The general solution to Eq. (1) is given by det$(A) = 0$. With these restrictions, one arrives at the well-known dispersion relation for whistler waves in a homogeneous plasma:

$$\frac{k^2 c^2}{\omega^2} = \frac{\omega_{pe}^2}{\omega^2 \left( \omega_{ce} \cos(\theta) - \omega \right)},$$  

(2)

where $\theta$ is the angle with respect to the dc magnetic field $B_0$ of the wave vector $k$ [which has magnitude $k$ and Cartesian components with respect to the magnetic field $k_1 = k \cos(\theta)$ and $k_2 = k \sin(\theta)$].

Figure 1(a) shows the dispersion relation for whistlers with $\omega / \omega_{ce} < 0.5$. In this case $k$ asymptomatically approaches a specific angle as $k_1$ becomes large, known as the phase velocity resonance cone angle given by $\cos(\theta_{RC}) = \omega / \omega_{ce}$. Beyond this angle the refractive index becomes imaginary and field oscillations are evanescent. Fisher and Gould show from the electrostatic wave equation $k \cdot e \cdot k = 0$ that, for $\omega \gg \omega_L$ and arbitrary density, the dispersion relation is given by

FIG. 1. The variation of (a) $k$, (b) $\nu_L$, and (c) surface of constant phase (wave front) with $\theta$ for $\omega / \omega_{ce} = 0.1$ with $\omega_{pe} / (\omega_{pe} - \omega) = 100$. The phase and group velocity resonance cone angles are indicated by dashed lines in (a) and (c), respectively.
\[
\sin^2(\theta_{RC,\phi}) = \left(1 - \frac{\omega^2}{\omega_{pp}^2}\right) \left(1 - \frac{\omega^2}{\omega_{ce}^2}\right).
\]

This reduces to \(\cos(\theta_{RC,\phi}) = \omega/\omega_{ce}\) when \(\omega_{pe} \gg \omega\), indicating that the whistler wave becomes electrostatic as \(\theta\) approaches \(\theta_{RC,\phi}\).

An important result shown in Stix\textsuperscript{11} is that the direction of \(\mathbf{v}_p\) is always perpendicular to the \(k\) surface shown in Fig. 1(a). Hence the phase velocity resonance cone angle also implies the so-called group velocity resonance cone angle, given by

\[
\sin^2(\theta_{RC,z}) = \sin^2\left(\frac{\pi}{2} - \theta_{RC,\phi}\right) = 1 - \left(1 - \frac{\omega^2}{\omega_{pp}^2}\right) \left(1 - \frac{\omega^2}{\omega_{ce}^2}\right),
\]

beyond which the wave energy cannot propagate.

The \(k\) surface in Fig. 1(a) has two minima occurring off axis and a local maximum on axis, allowing two values of \(k_z\) for a given value of \(k_\phi\) (within a specific range of \(k_\phi\)). As mentioned in the Introduction, the solution with higher \(k_z\) corresponds to the TG wave and the solution with smaller \(k_z\) corresponds to the helicon wave. It can be shown that the minimum in the dispersion curve separating the helicon and TG waves occurs at the so-called Gendrin angle \(\theta^*\) with respect to the magnetic field given by

\[
\cos(\theta^*) = \frac{2\omega}{\omega_{ce}}.
\]

The shape taken by the actual wave fronts that would be emitted by an ideal point source may be inferred from the \(k\) surface. Following the method of Stix\textsuperscript{11}, we consider the phase velocity surface, given by plotting \(\omega/\kappa\) as a function of \(\theta\), as shown in Fig. 1(b) for the case in Fig. 1(a). This plot represents the locus of points marked by the tip of the phase velocity vector \(\mathbf{v}_\phi\) as a function of \(\theta\) and represents the distance a planar wave travels in time \(\delta t\), having started from the origin. An ideal point source may be considered as the constructive interference of an infinite number of planar waves of all angles intersecting at the origin. The surface of constant phase after the time \(\delta t\) may then be built up from Fig. 1(b) using Huygen’s principle by considering the points of intersection of each planar wave front with its neighboring planar waves which have slightly different \(\theta\). This geometrical method is analogous to the stationary phase approximation by which the surface of constant phase may be constructed analytically. The surface of constant phase is shown in Fig. 1(c). This curve asymptotes along a particular angle, the group velocity resonance cone angle \(\theta_{RC,z} = \pi/2 - \theta_{RC,\phi} = \sin^{-1}(\omega/\omega_{ce})\), as expected from the above discussion of the wave group velocity. It is interesting to note the self-intersection of this surface on the \(z\) axis (parallel to the magnetic field) and the existence of two cusp points. These points correspond to the inflection points in the helicon part of the \(k\) surface of Fig. 1(a), at which point the angle of \(\mathbf{v}_p\) with respect to the magnetic field reaches a local maximum value. Keeping in mind that the group velocity direction is perpendicular to the \(k\) surface, it is clear that the group velocity is directed along the magnetic field at the boundary between the helicon and TG branches at \(\theta^*\), and this corresponds to the point of self-intersection on the \(z\) axis in the surface of constant phase.

Note that in an experiment with a well-defined excitation frequency, an infinite train of these wave fronts are launched by the antenna, and the radiation pattern observed is also the result of their interference.

III. EXPERIMENTAL SETUP

The large volume helicon source WOMBAT (waves on magnetized beams and turbulence) consists of a glass source tube 50 cm in length and 18 cm in diameter attached at one end to a stainless steel diffusion chamber 200 cm in length and 90 cm in diameter. The other end of the source is terminated with a grounded stainless steel plate through which various probes can be inserted. A weak axial dc magnetic field between 20 and 60 G in this experiment is maintained by a set of external solenoids surrounding the source and internal solenoids in the diffusion chamber. In all experiments the magnetic field in the diffusion chamber is set to half the value in the source to ensure that the field decreases monotonically from the end of the source to the diffusion chamber. A turbo molecular pump located at the end of the diffusion chamber opposite the source maintains a base pressure of \(5 \times 10^{-6}\) Torr in the vessel. An argon gas operating pressure of between 1 and 5 mTorr (measured by a capacitance manometer in the diffusion chamber) is set by adjusting the input flow rate.

Plasma is produced in this experiment by coupling up to 2 kW rf power at 13.56 MHz via a \(\pi\) matching network to a double half-turn (DHT) external antenna (that is, the arms of the antenna are outside the 18-cm-diam source tube) situated 7 cm from the source endplate. The characteristics of the plasma produced by this antenna have been reported previously.\textsuperscript{12} The power to the DHT antenna is pulsed with a duration of 20 ms and 10% duty cycle. The time variation of the plasma parameters, such as plasma density and wave magnetic field measurements at a given location, were found to be very reproducible from pulse to pulse. This enabled the compilation of spatiotemporal information from local measurements over a series of repeated pulses in which the position of the measuring diagnostic was scanned. During each series of plasma pulses, plasma waves at a frequency of 29.5 MHz were excited by means of a small electrostatically coupled antenna. This antenna consists of a 1-cm-diam flat nickel disk inserted into the source from the endplate along the axis. The rf generator for this antenna outputs 200 W into a parallel combination of a 50-\(\Omega\) dummy load and the antenna, the latter via a dc blocking capacitor.

The principal \textit{in situ} diagnostics used in this experiment are a three-axis, \(b\)-dot probe for rf magnetic field measurements and a Langmuir probe used in ion saturation for density measurements and at the floating potential for rf measurements.

The \(b\)-dot probe consists of three windings of copper wire (each with five turns and a cross sectional area of 1
cm²) arranged in orthogonal directions, mounted on the end of a 4 cm length of soap stone that is in turn mounted on a hollow stainless steel shaft with a right angle bend 4 cm from the probe end, producing a radial arm of length 8 cm. A glass sleeve fits over the probe windings and the end of the shaft to prevent contact with the plasma. The shaft has a length of 60 cm and is inserted through a port in the endplate at a radius of 8 cm in order to allow radial scans from the center to the edge of the source by rotating the shaft about its axis thereby moving the radial arm of the probe along an arc that passes through the axis of the source as shown in Fig. 2 and axial scans to be made by further inserting the shaft. A twisted pair for each probe runs along the inside of the shaft to three hybrid combiners which remove electrostatic (common mode) pickup, leaving only the inductively coupled, differential signal from the wave magnetic field components at the position of the probes. These signals are input to an analog rf amplitude and phase detector, with the phase reference given by the rf generator signal for the electrostatic antenna. The $A \sin(\phi)$ and $A \cos(\phi)$ outputs from the phase detector (where $A$ is the amplitude of the rf signal and $\phi$ is its phase with respect to the reference) are digitized and recorded with 16-bit precision and 14-kHz sampling frequency.

In the experiment, the orientations of two of the $b$-dot probes change as the radial arm is moved along its arc, so the output signals from the probes must be calibrated in order to correctly deduce field components in laboratory coordinates. This was done by measuring the probe outputs as the probe array was rotated inside a test magnetic field provided by a Helmholtz coil.

The Langmuir probe consists of a tungsten wire ($r = 0.75 \text{ mm radius, } l = 1.0 \text{ cm exposed length}$) inserted into a ceramic tube (with an exposed length of 4 cm) that is in turn inserted into a stainless steel shaft with a right angle bend 4 cm from the end. The axial length of the shaft is about 60 cm and is inserted through the endplate through another port at a radius 8 cm, allowing axial and radial scans to be made throughout the source (along a different arc to the $b$-dot probe) as shown in Fig. 2. When used in ion saturation (with bias voltage $V_{\text{bias}} = -90 \text{ V}$), the plasma density $n_e$ is estimated according to

$$I_{\text{sat}} = 0.6 A_s n_e e v_B,$$

where $A_s$ is the surface area of the sheath surrounding the probe, $e$ is the charge of an electron, and $v_B = (kT_e / m_i)^{1/2}$ is the Bohm velocity (where $kT_e / e$ is the electron temperature in eV and $m_i$ is the ion mass). The voltage across a 100-Ω sense resistor from the Langmuir probe was digitized to give $I_{\text{sat}}$ as a function of time during experiments. Previous measurements in WOMBAT under similar conditions indicate an electron temperature of 3 eV, yielding a Bohm velocity of $2.8 \times 10^3 \text{ m/s}$.

At high densities, the sheath thickness is small compared to the probe radius, and $A_s$ can be replaced by the probe surface area. As $n_e$ is decreased, the sheath thickness $r_s$ increases according to the Child–Langmuir law:

$$r_s = x \Lambda_D e,$$

where
This causes \( n_e \) to be overestimated by Eq. (6) unless \( r \gg r_s \). A simplistic attempt to account for the sheath thickness at low densities is made in the following way. Assuming the sheath around the probe takes the shape of a cylinder with a hemispherical end, the sheath surface area can be estimated by \( A_s = 2 \pi(r+r_s)(r+r_s+1) \), and Eq. (6) can be rewritten in terms of \( \lambda_{De} \):

\[
\lambda_{De}^{2}(x^2 - a I_{e\phi}) + \lambda_{De} x (2r+1) + r(r+1) = 0, \tag{8}
\]

where \( a = e/(1.2 m_{e} \nu_{0}^{4} \pi \varepsilon_{0}) \). Solving this equation then yields an estimate of the plasma density through the Debye length scaling shown in Eq. (7).

### IV. EXPERIMENTAL RESULTS

In this series of experiments, the rf power coupled to the DHT antenna was linearly ramped within a 20-ms pulse to produce a plasma in which the density was scanned as a function of time and a constant frequency (29.5 MHz) signal was applied to the test wave antenna. In each pulse the local wave magnetic field was measured as a function of time at a single location and the position of the measurement was incremented from pulse to pulse radially and axially on a two-dimensional (2D) grid. These series of measurements were carried out at source magnetic fields between 20 and 60 G, neutral filling pressures between 0.5 and 5 mTorr, and plasma densities reaching a maximum value of \( 5 \times 10^{19} \text{ m}^{-3} \) during the rf power ramp. The time traces of forward and reflected rf power to the DHT antenna and the plasma density on axis at \( z = 10 \text{ cm} \) for a typical discharge with \( B_0 = 40 \text{ G} \) are shown in Fig. 3. This figure shows that the density does not ramp linearly with the input power, but rather begins to rapidly increase once the power has exceeded a threshold value. This point marks the well-known transition from capacitive to inductive coupling of rf power to the plasma, which is a well-known characteristic of helicon sources. At the low magnetic fields used in this experiment, this transition is smooth and repeatable, and does not produce a sudden jump in density (as is the case at magnetic fields above 60 G in WOMBAT).

#### A. Electrostatic whistler characteristics

Figure 4 shows a 2D map of the amplitude of \( B_z \) at a time slice during the power ramp corresponding to a source density of \( 1.8 \times 10^{19} \text{ m}^{-3} \). The magnetic field was set to 50 G and the pressure to 3 mTorr in this example. The field clearly peaks along a specific angle extending from the antenna position, which is located on axis at \( z = 1.0 \text{ cm} \). The position of the peak along a series of arcs shown in the figure was determined by interpolation (circles). A least-squares fit to these points was used to obtain the angle of the peak with respect to the axial magnetic field (\( \theta_{BC exp} \)) for comparison with \( \theta_{BC exp} \) from Eq. (4), which is also shown in the figure. This procedure was carried out for each time slice during the power ramp to produce Fig. 5 for the case where \( B_0 = 50 \text{ G} \). The domain of the data points shown corresponds to the extent in density in which the resonance cone maximum was visible on the 2D maps of \( B_z \) amplitude for each time slice. This figure shows that at low densities, \( \theta_{BC exp} \) closely follows the \( \theta_{BC exp} \) curve as \( n_e \) is increased, until \( n_e = 1.3 \times 10^{18} \text{ m}^{-3} \), whereupon the measured resonance cone angle begins to increase from the expected value. As the density is further increased, a transition in the characteristics of the field pattern takes place, which will be described in the next section.

![FIG. 3. (a) Forward and reflected rf power vs time input to the DHT antenna, for a typical plasma discharge. (b) Corresponding electron density vs time, measured at \( r = 0, z = 10 \text{ cm} \).](image)

![FIG. 4. 2D map of \( B_z \) amplitude, as a function of radius and axial position. This is a single time slice compiled from a series of density ramp discharges. The plasma conditions are \( B_0 = 50 \text{ G} \). \( n_e \) as indicated, pressure = 3 mTorr. Circles indicate the position of the peak in amplitude along a series of concentric arc constructions as shown (dotted curves). The solid line is a linear fit to these points, and the dashed line is the group velocity resonance cone angle for these conditions.](image)
Figure 6 shows the variation in \( \sin(\theta_{RC_{expt}}) \) with the square root of the left-hand side of Eq. (4) for various magnetic fields from 30 to 60 G. The data shown in Fig. 5 were replotted for 50 G in this figure. The agreement for these data set with the expected variation of \( \theta_{RC} \) with \( n_e \) at low density and subsequent departure from the theory are indicated by the data following the diagonal line from right to left, before abruptly departing from the diagonal. The cases for 40 and 60 G show similar trends albeit with lower signal to noise; however, the data appear to be offset vertically by about 3° in these cases. This discrepancy is most likely the result of a small misalignment in the magnetic field direction that went unnoticed during the course of the measurements. Such a misalignment could be caused by an accidental adjustment in position of the source solenoids that occurred after the 30 and 50 G experiments and before the 40 and 60 G experiments (these two experiments were carried out on a later date). While a systematic error such as this is regrettable, it nevertheless highlights the level of precision in the technique used to determine \( \theta_{RC_{expt}} \) from the magnetic field measurements.

**B. ES to EM transition**

Figure 7 shows a series of 2D maps of the \( B_z \) amplitude (top row) and phase with respect to the antenna (bottom row) as the density is increased between \( 2 \times 10^{15} \) and \( 2 \times 10^{17} \) m\(^{-3}\), for \( B_0 = 50 \) G. The antenna is located in the lower left-hand corner of each frame (centered at \( r = 0 \)). The top row of the figure shows that as the density increases a clear transition in the structure of the wave field occurs, with the amplitude maximum along \( \theta_{RC} \) becoming increasingly
localized close to the antenna, before disappearing completely to be replaced by an amplitude maximum extending from the antenna along the axis of the source. These figures also show the apparent widening of the resonance cone angle with increasing density before the transition, as noted in the last section. The frames in the bottom row show an abrupt change in phase as the resonance cone is crossed, which is the expected result of crossing the resonance with a finite sampling grid. As the density is increased the phase begins to ramp in the axial direction at an increased rate, indicating an increase in the parallel wave number $k_i$, which is consistent with the dispersion expected for electromagnetic helicon waves. This will be examined more closely in the next section.

The transition from electrostatic (ES) to electromagnetic (EM) waves appears to be characterized by a threshold density $n_e^*$. The following method was used to estimate $n_e^*$ for each magnetic field setting. For each time slice, the amplitude of the wave field along each of the arcs shown in Fig. 4 was parametrized by the angle $\theta$ with respect to the magnetic field, and an average over the set of arcs was taken to give a single curve representing the wave amplitude as a function of $\theta$. These curves were each normalized by their maximum values and were used to plot a contour map with $\theta$ on the y axis and $n_e$, for each time slice on the x axis, as shown in Fig. 8. The angle at which the peak occurs ($\theta_{\text{peak}}$) in this contour map corresponds roughly to $\theta_{\text{BCG}}$ at low densities, and at high densities the peak occurs close to $\theta=0$. In between these limits exists a transition region in which $\theta_{\text{peak}}$ varies rapidly as a function of $n_e$. The value of $n_e^*$ was estimated by the point at which $d\theta_{\text{peak}}/dn_e$ reaches its maximum negative value. Figures 9(a) and 9(b) show the variation of $n_e^*$ with magnetic field (at a pressure of 3.0 mTorr) and neutral filling pressure (at a magnetic field of 50 G), respectively, according to this method. This figure shows that the transition density rises at an increasing rate as either $B_0$ or pressure are decreased. An explanation for these trends will be offered in Sec. V.

C. Electromagnetic whistler characteristics

This section describes the structure and dispersion of waves launched from the antenna at densities above $n_e^*$. Figure 10 shows 2D maps of the amplitude and phase for wave components $B_r$ and $B_\phi$, taken at $n_e = 2 \times 10^{17} \text{ m}^{-3}$ and $B_0 = 50 \text{ G}$. These components have a single, clear peak off
axis, while the corresponding $B_z$ component shown in Fig. 7 has a single peak on axis, which indicates that the waves have axisymmetric structure (azimuthal mode number $m = 0$). This is the expected result given the axis symmetry of the antenna. In each case the amplitude maximum extends down the source axis, indicating a group velocity direction parallel to the magnetic field.

Figure 11 illustrates the procedure carried out to measure the dispersion of these waves. The most straightforward direction to analyze is parallel to the magnetic field, as close as possible to the source axis; therefore, the axial variation of the real and imaginary parts of the $B_z$ component along the innermost radial grid point is considered. An example is shown in Fig. 11(a) for $n_e = 5 \times 10^{17} \text{ m}^{-3}$, $B_0 = 50 \text{ G}$. These signals consist of 23 measurements made every 2 cm. This figure shows that the signal amplitude drops to a small value before the limit in measurements is reached. The fast Fourier transform (FFT) of the signal is therefore taken with zero padding up to 128 points, the amplitude of which is shown in Fig. 11(b). The main peak in this signal is considered to indicate the dominant $k_\parallel$ for the wave field shown. The FFT amplitude for each time slice during the density ramp is shown as a contour plot in Fig. 11(c), with the peak $k_\parallel$ marked by crosses in the figure, and shows a square root dependence with $n_e$.

Naively, one would expect that $k_\parallel$ for wave propagation parallel to the magnetic field would be given from the dispersion relation by setting $k_\perp$ to zero. For whistlers, Eq. (2) shows that $k_\parallel$ is given by $\omega_{pe}/(c\omega_{ce}/\omega - 1)^{1/2}$ in this case. Figure 12(a) shows the measurements of $k_\parallel$ obtained by the procedure above versus this function, for magnetic fields between 30 and 60 G. These data sets follow straight lines close to that expected from this dispersion relation; however, the data sets also show a slight vertical offset that increases as $B_0$ is reduced. In Sec. II it was noted that the wave group velocity is also parallel to $B_0$ at the Gendrin angle $\theta_0$, given by Eq. (5). Inserting this value for $\theta$ into Eq. 2 implies that $k_\parallel$ is given by $2\omega_{pe}/(c\omega_{ce})$ in this case. Figure 12(b) shows the same $k_\parallel$ measurements as (a) plotted against this function. The data more closely fall on the diagonal of the figure, and the systematic dependence on $B_0$ is removed. This indicates that the waves launched by the antenna at densities above $n_e^*$ appear to be a special class of whistler known as a Gendrin mode, rather than a $k_\perp = 0$ helicon mode.

V. MODELING WAVE EXCITATION FROM THE ANTENNA

This section presents a calculation of the waves launched into a uniform plasma from the disk antenna used in the experiment. This model computes solutions for $E(t,x)$ by inverting the Fourier transform for the inhomogeneous case...
of the wave equation [Eq. (1)], in which the source term is given by the following expression for an oscillating current density representing the disk:

$$ j(t,x) = j_0 \exp(-i\omega t)H(z,d) \int_0^b dr \exp \left[ \frac{r-r^*}{\delta} \right]. $$

(9)

This current flows uniquely in the axial direction. The function $H(z,d)$ is equal to unity for $-d < z < d$ where $d$ is the location of the disk and is zero for $|z| > d$. The radial convolution integral has an approximately uniform value for $0 < r < b$, where $b$ is the radius of the disk. The parameter $\delta$ allows for radial smoothing near the boundary at $r = b$. Such a current represents a pair of radially smeared charged disks located at $z = \pm d$, as can be seen by taking its divergence. The current also describes the case of a charged disk at $z = +d$ with an image located in the wall at $z = -d$. Correctly accounting for the behavior of the source near a conducting plane where the plasma density goes to zero is problematic for a homogeneous plasma model. In this paper we allow the image to be multiplied by a factor $<1$ to investigate the effect on the fields.

The above equation may be rewritten in terms of the error function

$$ j(t,x) = -\sqrt{\frac{\pi}{4}} \delta j_0 \exp(-i\omega t)H(z,d) \times \left( \text{erf} \left[ \frac{r-b}{\delta} \right] - \text{erf} \left[ \frac{r}{\delta} \right] \right) = H(z,d)G(r). $$

(10)

The first step to solving Eq. (1) is to evaluate the Fourier transform of the current in Eq. (9) using the formula for the Fourier transform in cylindrical coordinates:

$$ J_\psi(\omega,k_z,k_z^*,\psi) = \int_0^\infty r dr \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz j(t,r,\theta,z) \times \exp(-i[k_z r \cos(\theta - \psi) + k_z z]), $$

(11)

where $\psi$ is the equatorial angle in $k$ space. Using Eq. (10) leads to the following result:

$$ J_\psi(\omega,k_z,k_z^*,\psi) = \frac{\exp(ik_z d) - R \exp(-ik_z d)}{ik_z} \times \int_0^\infty r dr G(r)J_0(k_z r), $$

(12)

where $R$ is the parameter we use to vary the weight of the image. The components of $E(\omega,k)$ are obtained by replacing the zero on the right-hand side of Eq. (1) by $i\omega\mu_0 J_x$ using Eq. (12), rewriting the left-hand side of the result in terms of $k_z$ and $k_z$ and then inverting the resulting $3 \times 3$ matrix.

The Fourier transform of the magnetic field $B(\omega,k)$ can be obtained from $E(\omega,k)$ through Faraday’s law, $k \times E(\omega,k) = \omega B(\omega,k)$. The inverse Fourier transform is taken to obtain a desired field component in configuration space. The $z$ integral in this procedure is calculated analytically using the residue theorem, the radial integral is done numerically, and cylindrical symmetry is assumed. The disk antenna is positioned at $z = d = +1$ cm, and the reflection of wave fields from the endplate ($z = 0$) is modeled by including an image of the antenna at $z = -1$ cm, with the amplitude of the image relative to the source specified by the reflection coefficient $R$ as described above. The plasma is described in the model by the parameters $n_e$, $B_0$, and the effective collision frequency $\nu$. This last term is the only source of wave damping included in the model.

The model output, which consists of an electric or magnetic field component discretely sampled on a 2D $(r,z)$ grid, is analyzed with the same methods that were used on the experimental data. Scans in $n_e$ and $B_0$ were carried out over similar ranges used in the experiment, and it was found that the ES and EM wave characteristics at low and high densities are reproduced by the model. Examples of the amplitude and phase of $B_z$ for $B_0 = 50$ G, $n_e = 10^{16}$ and $10^{17}$ m$^{-3}$ are shown in Figs. 13(a) and 13(b), respectively, and show a close resemblance to the measurements shown in Figs. 7(a) and 7(f). Values of $\theta_{\text{BC}}$ and $k_z$ were obtained for $n_e < 10^{16}$ m$^{-3}$ and $n_e > 10^{17}$ m$^{-3}$, respectively, and are shown in Figs. 14(a) and 14(b). In Fig. 14(a), $\sin(\theta_{\text{BC}})$ is plotted against the square root of the right-hand side of Eq. (4) and shows close agreement with the electrostatic theory. In Fig. 14(b), $k_z$ is plotted against 2$\omega \omega_{\text{pe}}/(c \omega_{\text{ce}})$ and lies along the diagonal independent of $B_0$, hence the model predicts the excitation of a Gendrin mode.

The transition density $n_e^*$ is computed by calculating the radial variation of $B_z$ at a single axial position (e.g., $z = 20$ cm) as $n_e$ is scanned and monitoring the radial variation of the peak in $|B_z| (r_{\text{peak}})$ with $n_e$. The transition is consid-
ered to occur at the point where \( dr_{peak}/dn_e \) reaches its maximum negative value, as shown in Fig. 15 for the case where \( B_0 = 50 \, \text{G} \) and \( R = 0 \). This calculation is performed for parameter scans in \( B_0 \) between 25 and 70 G (with \( \nu_{\text{coll}} = 24 \, \text{MHz} \)) and \( \nu_{\text{coll}} \) between 0 and 50 MHz (with \( B_0 = 50 \, \text{G} \)). The results are shown in Figs. 16(a) and 16(b), respectively. In each scan, the model is run for cases with \( R \) set to 0 and 1. In order to compare the experimental data from Fig. 9 with the model, an assumption has to be made about the collision rate constant \( K \) in order to determine the effective collision frequency from the neutral filling pressure. According to Lieberman and Lichtenberg, for an electron temperature of 3 eV and elastic electron–neutral collisions in argon, \( K \) should be \( 10^{-13} \, \text{m}^3/\text{s} \). This parameter was adjusted to \( 2.5 \times 10^{-13} \, \text{m}^3/\text{s} \) to estimate \( n_{\text{coll}} \) from the measured neutral pressure.

Having compared results from the model and experiment and found reasonable agreement, what does the model tell us about the origin of the observed transition in wave charac-

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**FIG. 14.** (a) Variation of \( \sin(\theta_{\text{LC}}) \) obtained from the model with the square root of the right-hand side of Eq. (4), for \( B_0 \) between 30 and 60 G and densities between \( 10^{14} \) and \( 10^{16} \, \text{m}^{-3} \). (b) The variation of \( k_\perp \) with \( 2\omega_{pe}/(c\omega_{ce}) \) for magnetic fields between 30 and 60 G and densities up to \( 10^{18} \, \text{m}^{-3} \).

**FIG. 15.** 2D map showing the radial variation of \( |B_z| \) at \( z = 20 \, \text{cm} \), computed as a function of \( n_e \) from the model with \( B_0 = 50 \, \text{G} \) and \( R = 0 \). The radial variation of the peak value is shown as a solid line in the figure, and the transition density is shown as a vertical dashed line.

**FIG. 16.** Variation of model \( n_e^* \) with (a) \( B_0 \) (at \( \nu_{\text{coll}} = 24 \, \text{MHz} \)) and (b) \( \nu_{\text{coll}} \) (at \( B_0 = 50 \, \text{G} \)) for reflection coefficients \( R = 0 \) and 1 (dotted and solid lines, respectively). The experimental data from Fig. 9 are replotted for comparison in this figure, using an effective collision rate constant of \( 2.5 \times 10^{-13} \, \text{m}^3/\text{s} \) to estimate \( \nu_{\text{coll}} \) from the measured neutral pressure.
characteristics? Perhaps more is said by noting what phenomena are excluded by the simplicity of the model. For example, since there are no plasma inhomogeneities or radial boundary conditions in the model, it is incapable of producing any type of mode coupling or mode conversion; hence these phenomena are ruled out as possible causes of the transition. The only effect of collision frequency in the model is in the damping rate of the waves excited by the antenna. Therefore, the fact that \( n_e^* \) converges to a constant value as \( \nu_{\text{coll}} \) approaches zero implies that wave damping cannot be the underlying cause of the transition, because in this case no transition would be possible without wave damping and \( n_e^* \) would become infinite (or zero) as \( \nu_{\text{coll}} \) approaches zero. That said, the model and experiment both indicate that \( \nu_{\text{coll}} \) has a significant effect on the value of the transition density. The simplest explanation for this is that the ES wave is more easily damped by collisions than the EM wave; hence the point of equipartition of amplitude between the two waves occurs at a lower density value as the collision frequency is increased.

The ingredients included in the model are the cold plasma dielectric tensor (including the collision frequency) and the antenna geometry. The dielectric tensor specifies the wave dispersion relation, while the antenna geometry largely defines the antenna power spectrum. It is tempting to consider whether these two components alone provide further insight to the transition in wave characteristics. Figure 17(a) shows a series of wave \( k \) surfaces for various densities at constant \( B_0 \), obtained from the dispersion relation, and Fig. 17(b) shows the power spectrum of the antenna. Note that with \( R=0 \) in the model, the antenna is a \( \delta \) function in the axial direction; hence the only variations in the power spectrum occur in the perpendicular direction \( k_\perp \). This figure illustrates that at low densities all of the power from the antenna is coupled to wave numbers that contribute to the resonance cone. As the density increases the electromagnetic part of the \( k \) surface moves to higher \( k_\perp \) values, receiving more power from the antenna, while the electrostatic part of the \( k \) surface moves to wave numbers beyond the range of the antenna.

While this illustrates that the wave should evolve from electrostatic to electromagnetic in character as the density is increased, it does not indicate that an abrupt transition should occur. It appears that this behavior is only reproduced correctly when the wave equation is solved by taking the inverse Fourier transform, as in the model. This picture does, however, illustrate why the wave propagating parallel to \( B_0 \) above the transition corresponds to the Gendrin mode. As mentioned earlier, there are two locations on the wave \( k \) surface that correspond to a group velocity direction parallel to \( B_0 \): one at \( k_\perp = 0 \) where the \( k \) surface has a local maximum and the other at the Gendrin angle where the \( k \) surface has a local minimum. Figure 17(b), however, shows that no power is transmitted from the antenna into waves with \( k_\perp = 0 \), which leaves the Gendrin mode as the only alternative.

VI. CONCLUSIONS

A transition from electrostatic to electromagnetic whistler wave characteristics has been observed as the plasma density is increased in a test wave experiment using an electrostatically coupled disk antenna. At low densities below the transition, the wave amplitude maximizes along the group velocity resonance cone angle extending from the antenna, and above the transition an amplitude maximum extends from the antenna along the axis of the antenna for \( B_z \) and off axis for \( B_x \) and \( B_y \). The dispersion of these waves agrees with the so-called Gendrin mode, which corresponds to the local minimum in the whistler wave \( k \) surface. The density value at which the transition occurs is found to decrease with magnetic field and decrease strongly with the neutral filling pressure (collisionality). The wave fields launched from the antenna used in the experiment are calculated for an unbounded, homogeneous plasma, and a similar transition is found as the density is increased. The characteristics of the launched waves at densities below and above the transition are reproduced by the model, as is the scaling of the density at which the transition occurred with magnetic field and collision frequency.

ACKNOWLEDGMENTS

The authors would like to thank in particular Xavier Llobet, Laurie Porte, and John Scharer for many useful discussions on various aspects of this work. This work is partly funded by the Swiss National Science Foundation.
