Power coupling to helicon and Trivelpiece–Gould modes in helicon sources

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The effects of finite electron mass on the helicon mode and the role of the Trivelpiece-Gould (TG) mode in helicon sources are considered. In an unbounded plasma these waves are commonly referred to as whistler waves. A simple cold plasma antenna-wave coupling code gives results which compare favorably with experimental observations of the whistler wave resonance cone. The connection between these cones and the Trivelpiece-Gould eigenmodes of bounded plasmas is discussed. The theory is applied to the study of antenna-wave coupling in helicon sources. It reveals that finite electron mass effects do not significantly improve antenna-wave coupling for \( \omega/c_e < 0.5 \). It is also suggested that, because of the resonance cone dispersion, eigenmode cavity resonances of the Trivelpiece-Gould mode could be difficult to excite in practice. This precludes geometric resonances of the antenna impedance due to the existence of the TG mode as an explanation for the high plasma production efficiency of helicon sources. © 1998 American Institute of Physics. [S1070-664X(98)00603-X]

I. INTRODUCTION

Much theoretical work has been undertaken to evaluate the importance of finite electron mass on helicon waves.\(^\text{1–6}\) This work has tended to concentrate on the influence of finite electron mass on the wave dispersion, field profiles and wave damping. Recently however, calculations have also been made of the antenna radiation resistance for uniform density helicon sources in the cold plasma approximation with finite electron mass.\(^\text{5b}\)

In practice, there are two areas where finite electron mass effects need to be considered in the wave dynamics. On the one hand, there is the effect of finite electron mass on the lowest order radial mode of the helicon wave most commonly observed in plasma sources. For high density non-uniform sources, \( n_e \approx 10^{19} \text{ m}^{-3} \), \( B_0 \approx 0.1 \text{ T} \) and frequency in the range 7–30 MHz, the simple cold plasma model which neglects electron mass \( (E_z = 0) \) appears to be adequate for a description of antenna-wave coupling and dispersion.\(^\text{7–9}\) For these conditions, \( \omega_p/c_e \approx 10 \) and \( \omega/c_e \approx 0.002 \text{–} 0.01 \approx 1 \). Good qualitative agreement has been obtained between this model and experiment; even in small plasma radius devices where the first radial mode of the \( m = +1 \) helicon wave dominates the excited spectrum and accounts for most of the antenna radiated power. The neglect of finite electron mass for small radius devices is a surprising result and has been discussed in detail by Kamen-

Secondly, when finite electron mass is taken into account in the cold plasma model, for frequencies above the lower hybrid frequency, another solution to the wave equation appears in bounded geometries. This wave is referred to as the electrostatic TG (Trivelpiece–Gould) mode and was identified by Trivelpiece and Gould as the cavity eigenmode of a cold plasma, space charge wave in a cylinder.\(^\text{10}\) The TG mode has been observed in plasmas at high ratios of frequency to electron cyclotron frequency, \( \omega/c_e \approx 0.1 \text{–} 2 \).

In space plasma physics, waves propagating below the electron cyclotron frequency and above the ion cyclotron frequency are called whistlers. Commonly they are generated by a lightning flash at high latitudes. Some of the frequency components of this electromagnetic pulse are trapped in density formations in the magnetosphere called ‘‘ducts’’ and bounce back and forth between the two hemispheres along a magnetic field line. When propagating parallel to the field they are purely electromagnetic but in the presence of density variations can change their propagation angle and acquire some electrostatic component \( E||k \). In this frequency range the plasma is highly anisotropic and above the lower hybrid frequency there are certain angles of propagation that are forbidden. As the wave approaches this forbidden region it becomes more electrostatic and is purely electrostatic at the propagation angle defining the forbidden zone. Hence the whistler wave can be purely electromagnetic when propagating along the magnetic field or purely electrostatic on the so called ‘‘resonance cone angle.’’ At intermediate angles it shows both characteristics.

We have chosen to use a nomenclature as close as possible to that employed by the space and laboratory plasma communities. Hence we use the name whistler for waves in unbounded plasmas and helicon and TG modes for cavity eigenmodes in bounded plasmas. The TG mode cannot exist in an unbounded plasma although whistlers can propagate on the resonance cone. The propagation of waves in a bounded plasma is governed by the same refractive index surface and is modified by the boundaries. Helicon modes are related to whistlers propagating along the field lines and TG modes are related to whistlers propagating along the resonance cone. At first glance these two modes of a bounded plasma, the heli-
con and the TG mode appear independent. The helicon is mainly electromagnetic and the TG mode is electrostatic. However it has been known for some time that the electron mass can play a dominant role in the dispersion of higher radial modes of the helicon\(^3\) and the TG mode dispersion is dominated by the electron inertial term \((m_e/e^2 \partial j/\partial t)\) and the boundary conditions. In the lower density and field devices where \(n_e = 3 - 5 \times 10^{17} \text{ m}^{-3}\) and \(B_0 \approx 0.002 - 0.004 \text{ T}\), it has been demonstrated\(^3\) that the finite electron mass cold plasma model gives distinctly better agreement with dispersion observations in non-uniform plasmas than the simple cold plasma model which negleects electron mass. For these conditions, \(\omega_{pe}/\omega_{ce} = 44 - 113\) and \(\omega/\omega_{ce} = 0.1 - 0.5\), where \(\omega\) is the wave frequency, \(\omega_{pe}\) is the plasma frequency and \(\omega_{ce}\) is the electron cyclotron frequency. Since these conditions are similar to those of low density helicon sources, one needs to know how the wave and discharge physics is qualitatively affected by electron mass. Arguably, the most important question in this regard is whether finite electron mass can lead to improved antenna-wave coupling (antenna radiation resistance). Finite electron mass could improve coupling, for example, by significantly improving the antenna radiation resistance (of an infinite plasma) per se. Alternatively, TG cavity eigenmodes in bounded plasmas could produce geometric antenna coupling resonances.\(^5\) In addition, power deposition by the TG mode may be strongly localised near the resonance cone. These questions have not previously been considered.

The main purpose of this paper is to show how these modes of propagation are related by examining the refractive index surfaces and how the waves are launched by a point source. The modifying effects of boundaries will be introduced by considering a linear array of point sources and finding points of constructive interference where waves on the resonance cone intersect to form an electrostatic mode which is the basis of the TG mode. The paper is structured as follows. In Sec. II, we review the theory of the TG mode and discuss its relation to the helicon mode. For completeness we also include a discussion of the TG mode relation to the lower hybrid slow mode when the TG mode propagates in the vicinity of the lower hybrid resonance. In Sec. III, we develop an antenna-coupling theory based on the cold plasma model with finite electron mass in an unbounded plasma slab and compare the results with experiment. In Sec. IV, we employ this theory to determine how the TG mode affects the antenna-plasma coupling and the spatial distribution of wave energy in helicon sources. We compare the antenna coupling results with the simple cold plasma model that neglects electron mass. In Sec. V we conclude.

II. THEORY

A. The relation between the Trivelpiece–Gould and the helicon mode

Plasma waves are described in the cold plasma limit by the dielectric tensor,

\[
\begin{pmatrix}
\epsilon_x & -i \epsilon_z & 0 \\
i \epsilon_z & \epsilon_x & 0 \\
0 & 0 & \epsilon_||
\end{pmatrix}
\]

(1)

If ion mass is neglected then we may describe helicon waves by the dielectric tensor elements,

\[
\begin{align*}
\epsilon_x &= 1 + \frac{\omega_{pe}^2/\omega_{ce}^2}{1 - f_e^2}, \\
\epsilon_z &= 1 - \frac{\omega_{pe}^2}{\omega_{ce}^2},
\end{align*}
\]

where \(f_e = \omega/\omega_{ce}\).\(^1\) This tensor describes the relationship of the plasma wave driven currents to the wave electric field through the relation,

\[
\vec{J} = -i \omega \epsilon_0 (\epsilon - 1) \vec{E},
\]

and is obtained from the equations of motion for electrons and ions in the wave electric field.

The dispersion relation obtained for \(\omega_{pe}^2 \gg \omega_{ce}^2\) is given by,

\[
k^2 = \frac{\omega_{pe}^2 \omega/c^2}{\omega_{ce} \cos \theta - \omega},
\]

where \(\theta\) is the angle between \(\mathbf{k}\) and \(\mathbf{B}_0\). The simple helicon wave dispersion relation is obtained for \(\epsilon_\perp = -\infty\) and \(\epsilon_z = 0\) so that \(E_z = 0\),

\[
k^2 = \frac{\omega_{pe}^2 \omega/c^2}{\omega_{ce} \cos \theta},
\]

(3)

It is clear that electron inertia can be neglected provided that \(k^2 \ll \omega_{pe}^2/c^2\) (long wavelength) which in turn implies that \(\omega/\omega_{ce} \ll \cos \theta\) where \(k\) is the modulus of the wave vector. These dispersion relations are plotted in Fig. 1a for the conditions \(B_0 = 0.01 \text{ T}, n_e = 10^{18} \text{ m}^{-3}\) and frequency, 13.56 MHz so that \(\omega_{pe}/\omega_{ce} = 32\) and \(\omega/\omega_{ce} = 0.048\). Note that Fig. 1a is in effect a plot of the wavenormal or refractive index \((k/c)/\omega\) surface.\(^1\) Equation (3) diverges considerably from Eq. (2) at high \(k_\perp\). It may be shown from Eq. (2) that the minimum of the curve in Fig. 1a occurs at \(k = \omega_{pe}/c\) or \(k_\perp^2 = (\omega_{pe}^2/c^2)(1 - 4 \omega^2/\omega_{ce}^2)\). For \(\omega/\omega_{ce} > 0.5\) the minimum therefore occurs at \(k_\perp = 0\) and finite electron mass effects dominate the dispersion relation for all values of \(k_\perp\).

The cold plasma model without finite electron mass is a good approximation to the complete model for \(\omega/\omega_{ce} < 0.5\) and for \(k_\perp\) lower than that at which the minimum in \(k_\perp\) occurs \((k_\perp < 150 \text{ m}^{-1}\) in Fig. 1a). The complete model shows that \(k_\perp\) is proportional to \(k_\perp\) at high \(k_\perp\) indicating a phase velocity resonance cone. The wavenumbers which constitute the TG eigenmodes in a bounded plasma are determined by
the cone in the limit of high \( k_z \). It is a simple matter to demonstrate that TG eigenmodes are electrostatic.

Electrostatic waves are described by \( \nabla \cdot \mathbf{D} = \rho_e \) where \( \rho_e \) is the external charge density (due to an antenna) and \( \mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E} \) is the electric displacement. For an electrostatic wave, \( \mathbf{E} = -\nabla \Phi \) so that on taking a Fourier transform we obtain \( \mathbf{k} \cdot \mathbf{E} = 0 \). This is the well known electrostatic wave dispersion relation. For the dielectric tensor (1) and the above dielectric tensor elements we obtain,

\[
\frac{k_z^2}{k_p^2} = \frac{\omega^2 (\omega_{pe}^2 + \omega_{ce}^2 - \omega^2)}{(\omega_{ce}^2 - \omega^2)(\omega_{pe}^2 - \omega^2)},
\]

which is the dispersion relation of Fisher and Gould\(^{12}\) and it describes whistler waves that propagate on the resonance cone. For the approximations leading to Eq. (2), Eq. (4) reduces to \( k_z / k_p = \omega / \omega_{ce} \). This electrostatic limit is however an approximation to Eq. (2) valid in the limit of high \( k_z \). Equation (4) can also be employed to derive the dispersion relation of the space charge waves of Trivelpiece and Gould.\(^{10}\) The experiments of Trivelpiece and Gould\(^{10}\) were performed in cylindrical plasma sources. For the case of a conducting cylindrical boundary, parallel wavelengths of the TG eigenmodes were measured which agreed with those calculated from Eq. (4) with \( k_z = p_n v_b / b \), where \( b \) is the cylinder radius, \( n \) is the order and \( \nu \) the zero, of the Bessel function eigenmode solution.

Finite electron mass significantly changes the form of the wavenormal surface. We see from Fig. 1a and Eq. (3) that the helicon mode still exists at low \( k_z \). Hence it would seem that two waves having the same frequency can propagate in a bounded plasma; an electromagnetic mode having wavevectors close to \( \mathbf{B}_0 \), the helicon mode, and an electrostatic mode having wavevectors near the resonance cone, the TG mode. We attempt to explain qualitatively what happens when an antenna excites these modes in a bounded plasma.

To find the radiation pattern (the ray surface) of a point source, we calculate the phase velocity, \( \omega / k \), surface shown in Fig. 1b (the closed surface at the bottom). For the case of CW (continuous wave) excitation of a point source located at the origin of Fig. 1b in an unbounded plasma, the phase velocity surface is the locus of the tips of the phase velocity vectors of the Fourier components emitted by this source. If one draws perpendiculars (representing wave fronts) to the tips of the chords which join the origin to each point on the phase velocity surface as shown in the figure, then the intersection of all the perpendiculars will form the ray surface, the locus of constructive interference. The ray surface is therefore the surface of constant phase of the disturbance emitted by our source. This locus is indicated by the thick dashed line in Fig. 1b. These constant phase surfaces are contained within a new angle which is that of the group velocity resonance cone. This cone is at an angle complimentary to that of the phase velocity resonance cone. When \( \omega_{pe} \gg \omega_{ce} \), the group velocity resonance cone lies along \( \sin \theta = \omega / \omega_{ce} \).

There are two portions of the ray surface. The flat portion in Fig. 1b originates from the low \( k_z \) portion of Fig. 1a and produces the helicon mode of a bounded plasma. The helicon mode is also described by the portion after the intersection of the ray surface. Toward the top of the figure, the dashed curve approaches an asymptote which defines the group velocity resonance cone describing whistlers on the cone. It originates from the \( k \)-values near the phase velocity resonance cone and the corresponding discrete \( k_z \)'s which produce the TG modes of a bounded plasma. Very crudely, one can say that the helicon mode would be expected to propagate as a plane wave front and be relatively insensitive to the plasma boundary. This observation is in agreement with antenna-wave coupling computations and experimental measurements of the wave dispersion.\(^9\) The TG mode will form eigenmodes in a bounded plasma whose wave fronts lie along the resonance cone of an unbounded plasma. We look at this in more detail in Sec. III.

B. Relation of the Trivelpiece–Gould mode to the lower hybrid wave

If the frequency is near the lower hybrid frequency where the influence of ion mass on the dispersion cannot be neglected, the TG mode becomes the slow “lower hybrid” mode whose dispersion relation in the vicinity of the lower hybrid frequency is given by \( \omega^2 = \omega_{lh}^2 [1 + (m_i / m_e)] \times (k_z^2 / k_p^2) \), where in this approximation, \( \omega_{lh}^2 = \omega_{pe}^2 / \omega_{ce}^2 \). For excitation at fixed \( k_z \) and low density with high \( k_p \) this mode connects to the fast mode (helicon wave) at the conversion layer where,\(^{11}\)

\[
n_1 = k_p c / \omega = \omega_{pe} / \omega_{ce} + \sqrt{\varepsilon},
\]

and \( \varepsilon = 1 + \left( \omega_{pe}^2 / \omega^2 \right) \left( \omega_{ce}^2 / \omega_{ce}^2 - 1 \right) \). This picture is relevant to lower hybrid heating in a fusion device and is shown in Fig. 2. The plasma conditions are the same as Fig. 1 except that \( k_p \) is fixed at 20 m\(^{-1}\) and the density is allowed to vary. The form of the dispersion relation here is insensitive to ion mass for frequencies much higher than the lower hybrid frequency at high density given by \( \omega_{lh} = \sqrt{\omega_{ce} \omega_{ce}} \). Below this frequency, the upper branch in Fig. 2 may undergo
a resonance at the lower hybrid frequency somewhere in the plasma if \( \min(\omega_p) < \omega < \max(\omega_p) \). When this is the case, the conversion layer may still lie in the plasma and if it occurs closer to the plasma boundary than the lower hybrid resonance layer then a slow wave launched from the plasma boundary will undergo mode transformation to the fast wave before reaching the resonance layer. We note that mode conversion does not occur because mode conversion always resolves a plasma resonance. The conversion layer of Eq. (5) does not coincide with any plasma resonance, but simply designates the point where the lower hybrid mode launched from the plasma boundary returns to the boundary as a fast wave along the fast wave branch of the dispersion relation.

It is clear from Fig. 2 that the helicon and TG modes also lie on the same branch of the dispersion relation. This implies that, if an antenna excites helicon waves in a plasma with \( \omega / \omega_{ce} \ll 1 \) and the antenna has a peaked \( k_\parallel \) spectrum high enough to excite a high \( k_\perp \) TG radial mode, then the TG mode can in principle transform to a helicon mode if Eq. (5) is satisfied in the plasma. In this case however the TG mode propagates at fixed \( k_\perp \) across the magnetic field with a short perpendicular wavelength. This process (and its reverse) are only possible in an inhomogeneous plasma. In a homogeneous plasma, the modes are independent.

It is also clear that plasma formation and heating should be sensitive to the lower hybrid resonance when it is in the plasma, especially if the conversion layer is absent. This phenomenon can only be treated properly by including the effects of plasma inhomogeneity in the model. However it should be noted that for low field helicon sources which satisfy \( \omega > \sqrt{\omega_p \omega_{ce}} \), lower hybrid heating effects can be neglected. For argon gas this means that lower hybrid effects are unimportant if \( f / (\text{MHz}) / B(T) > 103 \). This is easily satisfied for example by the lower density sources run at 13.56 MHz with \( B_0 < 0.01 \) T.

III. ANTENNA COUPLING TO WHISTLER WAVES

On the basis of our understanding of Fig. 1b we see that the energy in the high \( k_\perp \) components of the point source (i.e., those propagating across the magnetic field and in which there is the bulk of the antenna spectral power) constructively interfere along the group velocity resonance cone of Fig. 1b. These wavenumber components are those in Fig. 1a satisfying \( k_\parallel^2 / k_\perp^2 = \omega^2 / (\omega_{ce}^2 - \omega^2) = \omega^2 / \omega_{ce}^2 \) if \( \omega \approx \omega_{ce} \). Let us consider for the sake of simplicity, a filamentary line source lying along the \( x \)-axis with the magnetic field along \( z \) as shown in Fig. 3. The antenna is a \( \delta \)-function in the \( y-z \) plane and therefore consists of a Fourier spectrum of plane waves, all of equal amplitude, of the form \( \exp(i(k_x y + k_z z)) \). Let us suppose further, for the sake of the argument, that the dispersion relation is given only by the straight resonance cone line of Fig. 1a down to \( k_\perp = 0 \). In this case, even the low \( k_\perp \) components of the antenna current lie on the resonance cone. We are then studying antenna excitation of the whistler wave which lies on the resonance cone of Eq. (4) for \( \omega_{pe}^2 \gg \omega_{ce}^2 \). The wavevectors lie on the resonance cone, \( k_\perp^2 / k_\parallel^2 = \omega^2 / \omega_{ce}^2 \). One of the Fourier components is shown schematically in Fig. 3. Each of these components would correspond to a TG mode in a bounded homogeneous plasma if it satisfies the appropriate boundary conditions. We ignore for the moment the actual relationship between the amplitude of this Fourier component and that of the antenna current which excites it and assume that all Fourier components are excited equally. In view of the dispersion relation, one may therefore consider that the resultant wavefield is given by \( \int dk_\parallel \exp(i(k_x y + k_z z)) = 2\pi \delta(z \pm \omega y / \omega_{ce}) \). All components coalesce to form an infinite spike along the resonance cone. In this simple example the entire wave energy lies along the resonance cone at \( z = \pm \omega y / \omega_{ce} \). This is the group velocity resonance cone and it is clearly complimentary to the phase velocity resonance cone which is in the direction of propagation of the Fourier components.

The main result of this appealing argument is that one might expect most of the wave energy to be transported along the resonance cone. If this were the case, then the TG mode would indeed seriously affect the antenna-plasma coupling and play a very important role in the physics of wave-plasma interaction. This simple picture also suggests that cavity resonances of the TG mode in a bounded plasma could be difficult to excite because the resonance cone would be specularly reflected from density gradients and the plasma...
boundary with little chance of selecting out a single mode. We consider this point in more detail later.

In reality the dispersion relation includes more complicated behavior at low $k_\perp$ (the helicon mode) and more importantly, the effect of the coupling of the individual fields to the antenna has not been considered. We now consider this problem in more detail. The question of antenna coupling to the TG mode by a point divergence free current source in an infinite homogeneous plasma has been previously considered by Giles.\textsuperscript{13} Here it is pointed out that the wavevector components that interfere constructively on the ray surface at a particular location in the field are those for which the group velocity vector points toward this location. We therefore have the relation that $v_{\perp}/v_{\parallel} = \tan \psi$ where $v_{\parallel} = \partial \omega_{\parallel}/\partial k$ and $\psi$ is the angle between the point in the field and $B_0$. The perpendicular versus parallel group velocity is shown plotted in Fig. 4 for the conditions of Fig. 1. The most important features to notice are that the ratio of group velocities is limited to an opening angle about $19^\circ$ and that at high $k_\perp$ there is also the whistler wave resonance cone. Along a resonance cone the magnitude of the group velocity approaches zero. According to Giles,\textsuperscript{13} there are three possible wave motions that can be supported for a surface such as that in Fig. 1, (4). These are (1) where the curvature of the wave-normal surface is non-zero (point A), (2) where the curvature of the wavenormal surface is zero and the ratio of group velocities is zero (point B) and (3) the curvature is zero and the ratio of group velocities is an extremum (points C and D). Cases (1) and (3), points A and C correspond to the helicon mode whilst case (3) (point D) corresponds to the TG mode. At point B, $k^2 = \omega_{ce}^2/c^2$ which is a skin depth or cylindrical mode.

We now calculate the radiation fields of the following infinite divergence free line source current pointing in the $x$ direction,

$$J_{\text{ant}}(x) = J_0 \delta(x) \frac{\exp(-y^2/\delta^2)}{\sqrt{\pi} \delta} \delta(z).$$  \hspace{1cm} (6)

The magnetic field is assumed to point along $z$. Equation (6) represents a point source in two dimensions as that shown schematically in Fig. 3. The difference is that the antenna current is assumed Gaussian in the $y$-direction but infinitely thin along the $z$-direction instead of solid and circular in cross-section. The oscillations in Fig. 3 are intended to indicate those of just one Fourier component of this antenna current versus time. The wave fronts of the Fourier component advance as for a propagating wave.

Fourier transforming Maxwell’s equations leads to the following wave equation,

$$k^2 \tilde{\mathbf{E}} - k \mathbf{k} \cdot \tilde{\mathbf{E}} = i \omega \mu_0 \tilde{\mathbf{J}}_{\text{ant}} + k_0^2 \mathbf{e} \cdot \tilde{\mathbf{E}}.$$  \hspace{1cm} (7)

This is a matrix equation which, for the particular case of our problem where $k_z = 0$, reduces to the following set,

$$[k_c^2 + k_\perp^2 - \epsilon_\perp k_\parallel^2] \tilde{E}_z + i \epsilon_\parallel k_\parallel^2 \tilde{E}_z = i \omega \mu_0 \tilde{J}_{\text{ant}},$$  \hspace{1cm} (7a)

$$-i \epsilon_\parallel k_\parallel^2 \tilde{E}_z + (k_c^2 - \epsilon_\perp k_\parallel^2) \tilde{E}_z - k_\parallel \tilde{E}_z = 0,$$  \hspace{1cm} (7b)

$$-k_\parallel \tilde{E}_z + (k_c^2 - \epsilon_\perp k_\parallel^2) \tilde{E}_z = 0.$$  \hspace{1cm} (7c)

It is immediately obvious from the coefficient of $E_z$ that a $z$-directed antenna current will launch an evanescent field penetrating a collisionless skin depth, $d = \sqrt{(\epsilon_\perp/k_\parallel)}$ into the plasma as previously noted from Fig. 4.

Solving for the $E$-fields from Eqs. (7) and inverting the Fourier transforms with respect to $k_\parallel$ by the residue theorem leads to the following Fourier inversion integrals over $k_\perp$,

$$E_z = \frac{\omega \mu_0 J_0}{4 \pi \epsilon_\parallel k_\parallel^2} \int_{-\infty}^{\infty} dk_\perp \left\{ \frac{(k_c^2 - \epsilon_\perp k_\parallel^2) \exp(-k_\parallel^2 \delta^2/4) \exp(i k_\perp z)}{\Delta} \right\},$$  \hspace{1cm} (8a)

$$E_y = \frac{i \omega \mu_0 \epsilon_\parallel J_0}{4 \pi \epsilon_\parallel} \int_{-\infty}^{\infty} dk_\perp \left\{ \frac{(k_c^2 - \epsilon_\perp k_\parallel^2) \exp(-k_\parallel^2 \delta^2/4) \exp(i k_\perp z)}{\Delta} \right\},$$  \hspace{1cm} (8b)

where $k_\perp$ and $k_\parallel$ are defined by,

$$k_\perp = \sqrt{(-\mu + \Delta)/2}, \quad k_\parallel = \sqrt{(\mu + \Delta)/2},$$  \hspace{1cm} (8c)

where $\Delta = \sqrt{k_c^2 + 4 \nu_\parallel}, \quad \mu = k_\parallel^2 + (1 + \epsilon_\parallel/k_\parallel^2) - 2 \epsilon_\parallel k_\parallel^2, \quad \nu = (k_\parallel^2 - k_c^2 - \epsilon_\parallel \epsilon_0)/(\epsilon_\parallel k_\parallel^2 + \epsilon_0 (\epsilon_\parallel - \epsilon_0)), \quad \gamma_\perp = -\epsilon_\perp k_\parallel^2 + \epsilon_0 k_\perp^2 z/k_\parallel^2 - \epsilon_\parallel k_\parallel^2.$$

It turns out for the Whistler wave that application of the residue theorem to the inversion of the Fourier transform with respect to $k_\parallel$ leads to the complete expressions for both the near fields and the radiation fields. For the present one dimensional antenna, it is therefore better to evaluate the integrals numerically to invert the $k_\perp$-transform rather than by the method of stationary phase used by Giles.\textsuperscript{13} A more compelling reason for numerical evaluation in the present case is that the method of stationary phase is based on a short wavelength approximation being applied between the source and the point of observation in the field. The points of sta-
FIG. 5. Whistler wave excitation for $B_0 = 0.0114$ T, $n_e = 0.4 \times 10^{17}$ m$^{-3}$ and frequency, 150 MHz. (a) Contours of the phase of $E_y$ (top) and a trace of the component of $E_z$ in phase with the antenna current for $y = 0.10$ m (bottom). (b) The experimental data showing the phase fronts (circles), resonance cone (crosses), the reflected resonance cone (triangles) and a trace of the wave fields at $y = 0.10$ m. The magnetic field points along the horizontal axis in these figures. For these conditions, $\omega_pe/\omega_ee = 5.6$ and $\omega/\omega_ee = 0.47$. Reprinted with permission from Nature [Nature 258, 58 (1975)] Copyright 1975 Macmillan Magazines Ltd.

tionary phase for the exponentials in the integrands in Eqs. (8) occur for $d\phi = 0$ where $\phi = k_y z + k_x y$. This produces the result that $v_{xg}/v_{eg} = y/z$ as previously noted. Thus we plainly see that the method of stationary phase is inaccurate unless there are rapid variations in $\phi$ on either side of the point of stationary phase. Rapid oscillations only occur if the wavelength of the wave is much shorter than the distance between the antenna and the point in the field, viz $[k_y, k_x, z] \gg 1$. This also explains how the wave energy may still access regions in laboratory plasmas which are outside the cone of the group velocity.

The inverse Fourier integrals (8a) and (8b) for the electric fields as a function of $y$ and $z$ were evaluated by numerical integration with respect to $k_y$. Figure 5a shows the $E_y$ field component for the same conditions as the experimental results displayed in Fig. 2 of Boswell$^{14}$ and reproduced in Fig. 5b. The plasma conditions are, $B_0 = 0.0114$ T, $n_e = 0.4 \times 10^{17}$ m$^{-3}$ and frequency, 150 MHz so that $\omega_pe/\omega_ee = 5.6$ and $\omega/\omega_ee = 0.47$. The antenna in this calculation carries 100 Amperes, has a Gaussian radius $\delta$ of 2 mm and is located at the origin of coordinates. In the figure the magnetic field points along the horizontal axis and the antenna current is directed into the plane of the page ($x$-axis) to simulate the point electric source used in the Boswell experiment. The resonance cone is evident in both figures and the wave front plot clearly shows that the cone lies at an angle to the magnetic field given approximately by $\theta = \sin^{-1}(\omega/\omega_ee) = 28^\circ$. There is no dissipation in the model so that the decrease of the amplitude of $E_y$ along the axis inside the resonance cone is a purely geometrical effect and agrees well with the observations of Boswell. The reflection of the resonance cone observed by Boswell is not observed in the simulation as the effect of a boundary has not been included. This reflection is an example of the specular reflection of the resonance cone previously mentioned.

To see the relationship of this result to the experimental measurements of the resonance cone let us return to Fig. 3. Conducting wall boundaries located parallel to the magnetic field at equal distances $\pm a$ along the $y$-axis could be simulated by placing an infinite series of alternately phased line sources at $\pm na$ along the $y$-axis where $n$ is an even number. The distance between reflected cones along $z$ equals the distance between the crests of the fundamental TG eigenmode of the same cavity. The higher radial modes of the TG mode all propagate at the same angle to $B_0$ and coalesce to form a specularly reflected cone. Thus we see that the TG eigenmodes of a bounded plasma correspond to whistler waves lying on specularly reflected resonance cones in a bounded plasma.

It is worth noting that, contrary to the above simple arguments on the nature of the resonance cone, the TG mode energy is quite diffuse and is not confined to the resonance cone region. For an infinitely thin antenna however, a singularity in the wavefields does in fact occur along the resonance cone in the cold plasma results of Fig. 5a at the bottom. The resonance occurs in the imaginary part of $E_y$ and is smeared out by the 2 mm half-width of the antenna. The axial component of the Poynting flux for the calculation in Fig. 5a is also diffuse and shows no singularity near the resonance cone, even for a filamentary antenna.

IV. POWER COUPLING TO TG AND HELICON MODES

Given the agreement between the cold plasma theory and the experiments of Boswell, it is worthwhile trying to estimate the importance of TG mode coupling for the conditions of some typical helicon plasma sources above the lower hybrid frequency. We now consider whether the TG mode leads to enhanced power coupling per se and whether the resonance cone is a location of resonant power flux in helicon reactors. We consider the spatial distribution of the radiated power in our simple infinite homogeneous plasma model and assume that our line source of Eq. (6) is a representation of a typical transversely directed element in an antenna. Poynting fluxes for conditions $B_0 = 0.1$ T, $n_e = 10^{18}$ m$^{-3}$ and frequency 28 MHz typical of a high density helicon source for $z = 0.25$ m and 0.50 m from the antenna are shown in Fig. 6. The density was chosen deliberately low so that coupling to the edge of the plasma was considered. These fluxes are shown plotted as a function of the distance transverse to the direction of the antenna or parallel to the $y$-axis in Figs. 4 and 5. The thick line is the main, axial component, of the Poynting flux calculated from the full cold model with the antenna used for the calculations of Fig. 5a and the thick dashed line is the value calculated for the simple $E_z = 0$, $\epsilon_e = -\infty$ model. Only the axial component of the Poynting flux is shown in Figs. 6 and 7 because the $x$ and $y$ directed fluxes are smaller as expected for helicon and TG modes. More-
over, the total power radiated by the antenna can be calculated by integrating the axial component of the Poynting flux over $y \in [-\infty, \infty]$ at any axial location. For $z < 0.50$ m in Fig. 6 the axial component of the Poynting fluxes in the two models are very similar. For such a high density source, the TG mode is unimportant.

The Poynting fluxes for a low density plasma source with conditions $B_0 = 0.01$ T, $n_e = 10^{17}$ m$^{-3}$ and frequency, 13.56 MHz are shown in Fig. 7. These conditions are typical of the WOMBAT (Waves On Magnetised Beams and Turbulence) device$^{15}$ or helicon sources used for plasma processing.$^{16}$ The curve designations are the same as for Fig. 6. Once again the low density chosen is typical only of the edge plasma. The calculated radial locations of the resonance cone are $y = 0.024$ m at $z = 0.50$ m and $y = 0.048$ m for $z = 1.0$ m, respectively. This agrees with the features observed in the profiles of the full cold plasma Poynting fluxes at these locations. As expected, there is no resonance cone evident in the simple $E_z = 0$, $\epsilon_1 = -\infty$ curve. Consequently, the Poynting flux for the TG mode is visible for $y < 0.024$ m at $z = 0.50$ m and for $y < 0.048$ m for $z = 1.0$ m. The Poynting fluxes of the two models agree approximately only for values of $y$ greater than the positions of the resonance cone positions. This is to be expected because the helicon wave propagates at radii located outside the resonance cone.

Although there are some qualitative differences for the two models for $z = 1.0$ m, the main conclusion remains that the TG mode does not lead to improved antenna coupling and the resonance cone is not a locus of resonant power absorption. In fact, the power radiated per unit length of our infinite line source, $P = \frac{1}{2}Re(\int_{-\infty}^{\infty} f_{\text{antx}} \cdot (\mathbf{E}_x) dz)$, is given by,

$$P = \frac{\omega \mu_0 \delta_0^2}{8 \pi \epsilon_0^2} \int_{-\infty}^{\infty} dk_z \gamma_+ \left( \frac{k_z^2 - \epsilon_1(k_z^2)}{k_z^2 + \Delta} \right) \exp \left( -\frac{k_z^2 \delta^2}{2} \right).$$

(9)

If one sets $\epsilon_1 = -\infty$ in this expression then one obtains the corresponding expression for the radiated power in the $\epsilon_1 = -\infty$ or zero electron mass model. The ratio of the radiated powers for the cold plasma zero electron mass model to that with electron mass are shown in Fig. 8 for a wide range of densities and magnetic fields and frequency 13.56 MHz. This range includes most helicon sources. It is clear that the antenna radiated power is not affected by finite electron mass.

It is also apparent from our previous results that eigenmode resonances of these waves could be difficult to excite in helicon sources.$^{5(a)}$ The propagation angle of a TG mode with respect to $B_0$ is independent of the density and consequently TG eigenmodes cannot be selected by varying the density. Only a spatially broad antenna with a highly discrete parallel wavenumber spectrum that fixes $k_z$ at a high value would be needed to force a cavity resonance of the TG mode.
Moreover from each of Figs. 6 and 7, the wavefields toward the center of the plasma are predominantly those of the helicon mode. Partial standing waves of the helicon mode have however been observed but not enough to alter significantly the antenna radiation resistance. The conclusions are supported by the fact that other slow wave modes that form resonance cones are generally not observed to form cavity resonances. A similar example of this is the shear Alfvén mode excited by a localized antenna. This mode is described by the dispersion relation \( k_z^2 = k_x^2 + \alpha^2 k_y^2 \) where \( k_z = \omega / v_A \) is the Alfvén wavenumber and \( \alpha = \omega / \sqrt{(\omega_{ce} \omega_{ce})} \). This dispersion relation is similar to that of the helicon wave with finite electron mass.

One caveat on these results is that the linear antenna assumed in the present model launches a two dimensional wave with no phase variation in the \( z \)-direction. In cylindrical sources, the actual launched modes in general have both radial and azimuthal mode number variations. The present model is therefore analogous to an \( m = 0 \) mode in a cylinder. Moreover, \( z \)-directed antenna elements that are present in antennas such as the double saddle coil and helical antennas have been ignored in the present analysis.

The inclusion of \( z \)-directed antenna elements is not expected to alter the present conclusions for several reasons. In the \( \epsilon_z = -\infty \) limit they do not excite the helicon wave at all \( (E_z = 0) \). For the TG mode the ratio of \( E_z \) to \( E_x \) can be obtained approximately from the electrostatic wave dispersion relation so that, \( E_z / E_x \approx \sqrt{(\epsilon_z / \epsilon)} \). The TG mode \( E_z \) amplitude is generally smaller than that of \( E_x \) so that \( z \)-directed current elements are not expected to couple more efficiently to the wave than transversely directed elements.

V. DISCUSSION

The interpretation of experimentally observed resonance cones of the whistler wave excited by a line element current has been discussed using cold plasma theory with finite electron mass. We have demonstrated within the limitations of a simple line-element model that the TG mode does not lead to a significant increase in antenna coupling in helicon sources and the resonance cone is not a location of spatially resonant power flux. It is suggested that eigenmode cavity resonances of the TG mode would be difficult to excite so that resonance enhanced antenna radiation resistances would not be expected. However, the TG mode may enhance wave damping due to its high amplitude electric field near the resonance cone in the presence of high electron collision rates.

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