Kinetic Theory of Alfvén Waves in the Auroral Zone
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- Kinetic Alfvén Waves are now well-established as an important contributor to the dynamics of the auroral zone (previous talk), and have been well studied in laboratory experiments (esp. LAPD).

- Auroral field lines couple higher $\beta$ ($>m_e/m_i$) plasma sheet plasma with low $\beta$ ($< m_e/m_i$) cold ionospheric plasma. Strong inhomogeneities along the field line are present, especially in topside ionosphere (“ionospheric Alfvén resonator”).

- Auroral currents are often filamented, with strong perpendicular gradients, leading to mode structuring and phase mixing.

- Outline:
  - Comments on local Alfvén wave dispersion relation.
  - Effects of perpendicular gradients.
  - Propagation of waves in ionospheric Alfvén resonator with density cavity.
  - Generation of Alfvén waves on plasma sheet boundary layer.
Kinetic Alfvén Wave: Local Theory

- Kinetic Alfvén wave dispersion relation can be written as:

\[
\frac{\omega}{k} \approx k^2 a_A^2 \left( 1 + k^2 a_A^2 + k^2 \rho_i^2 \left[ 1 + i \frac{\pi}{2} \left( \frac{\omega}{k a_e} \right) \right] \right)
\]

- Dispersion relation is then solved to read:

\[
\frac{\omega}{k} \approx k^2 a_A^2 \left( 1 + k^2 a_A^2 + k^2 \rho_i^2 \left[ 1 + i \frac{\pi}{2} \left( \frac{\omega}{k a_e} \right) \right] \right)
\]

- In cold electron limit:

\[
\frac{\omega}{k} \approx k^2 a_A^2 \left( 1 + k^2 a_A^2 + k^2 \rho_i^2 \left[ 1 + i \frac{\pi}{2} \left( \frac{\omega}{k a_e} \right) \right] \right)
\]

assuming

Alternate form (e.g., Gekelman et al., 1997):

\[
\left( \frac{k \cdot c}{\omega_{pe}} \right)^2 = Z' \left( \xi \right) \left\{ \frac{V_A^2}{a_e^2} \frac{\mu}{1 - \Gamma_0 (\mu)} - \xi^2 \right\}
\]
Results from Low-frequency Kinetic Model

(Lysak and Lotko, 1996; Lysak, 1998)

\( \beta = \begin{cases} 
0.01 \\
0.001 \\
0.0001 
\end{cases} \)

\( \frac{\omega}{k_e c_A}, \frac{T_i}{T_e} = 2.0 \)

\( \frac{\gamma/\omega}{T_i/T_e} = 2.0 \)

\( \frac{k_e E_x}{k_e E_x}, \frac{T_i}{T_e} = 2.0 \)

\( \frac{E_x}{c_A B_y}, \frac{T_i}{T_e} = 2.0 \)
Finite Frequency Effects

- Previous model neglects $\omega/\Omega_i$ effects; not appropriate in lab and in some space contexts.
- Modification of perpendicular ion term, neglecting cyclotron damping, can be written as:

$$\varepsilon_{xx} = 1 + \frac{\omega_{pl}^2}{\omega^2} \omega \sum_{n=-\infty}^{\infty} \frac{n^2 \Gamma_n (\mu)}{k_{\perp} a_i} Z \left( \frac{\omega - n\Omega_i}{k_{\perp} a_i} \right) \approx 1 + \frac{c^2}{V_A^2} G (\mu, \omega / \Omega_i)$$

- For low frequency: $G = (1 - \Gamma_0 (\mu)) / \mu$
- At finite frequency, $G$ becomes:

$$G (\mu, \omega / \Omega_i) = \frac{1 - \Gamma_0 (\mu)}{\mu} + \frac{2 \Gamma_1 (\mu)}{\mu} \frac{\omega^2 / \Omega_i^2}{1 - \omega^2 / \Omega_i^2}$$

- Note alternate expression (not as accurate):

$$G (\mu, \omega / \Omega_i) = \frac{1 - \Gamma_0 (\mu)}{\mu \left(1 - \omega^2 / \Omega_i^2\right)}$$

- Dispersion relation becomes:

$$\left( \frac{\omega}{k_{\perp} V_A} \right)^2 = \frac{1}{V_A^2 / c^2 + G (\mu, \omega / \Omega_i) + \frac{k_{\perp}^2 \rho_s^2}{1 + \xi Z (\xi)}}$$
Model Results, $\beta = 2 \, m_e/m_i$
Comparison with full dispersion relation (WHAMP)

Strong ion cyclotron damping: No WHAMP solution
Effects of inhomogeneities: Perpendicular gradients

- FAST observations show formation of cavities on many scales in auroral zone.
- Consider two-fluid dispersion relation for fixed parallel phase velocity in such a cavity, assuming cylindrical geometry:

\[-i\omega b_\varphi = -ik_z E_r - \frac{\partial E_z}{\partial r}, \quad j_r = -\frac{1}{\mu_0} ik_z B_\varphi\]

\[-i\omega E_r = V_A^2 \left(j_r - \rho_i^2 \frac{1}{r} \frac{\partial}{\partial r} (rj_r)\right), \quad E_z = -i\omega \left(\kappa_e^2 - \frac{k_A^2 V_A^2}{\omega^2 \rho_s^2}\right) \mu_0 j_z\]

- These can be combined into a Bessel type equation for $b_\varphi$:

\[\frac{\partial}{\partial r} \left[ \left(\kappa_e^2 - \frac{k_A^2 V_A^2}{\omega^2 \rho_s^2}\right) \frac{1}{r} \frac{\partial}{\partial r} (rb_\varphi) \right] + \left(\frac{k_A^2 V_A^2}{\omega^2} - 1\right) b_\varphi = 0\]

\[$\left(\rho^2 = \rho_s^2 + \rho_i^2\right)$]
Solutions for a density cavity

- Consider cavity with piecewise-constant parameters. Then solutions depend on a parameter:
  \[ \kappa^2 = \frac{(k_z V_A / \omega)^2 - 1}{\lambda_e^2 - (k_z V_A / \omega)^2 \rho^2} \]
  (Note \( \kappa \) is wave number in local case)
- For \( \kappa^2 > 0 \), solutions are propagating, \( b_\varphi \sim J_1(\kappa r) \)
- For \( \kappa^2 < 0 \), solutions are evanescent, \( b_\varphi \sim K_1(|\kappa|r) \)
- At boundary, tangential components are conserved: \( \Delta b_\varphi = 0 \), \( \Delta E_z = 0 \)
- For waves propagating in both regions, eigenvalue equation is:
  \[
  \kappa_{in} \left( \lambda_e^2 - \frac{k_z^2 V_A^2}{\omega^2} \rho^2 \right)_{in} J_0 \left( \kappa_{in} R \right) J_1 \left( \kappa_{out} R \right) - \kappa_{out} \left( \lambda_e^2 - \frac{k_z^2 V_A^2}{\omega^2} \rho^2 \right)_{out} J_1 \left( \kappa_{in} R \right) J_0 \left( \kappa_{out} R \right) = 0
  \]
- For waves propagating inside and evanescent outside:
  \[
  \kappa_{in} \left( \lambda_e^2 - \frac{k_z^2 V_A^2}{\omega^2} \rho^2 \right)_{in} J_0 \left( \kappa_{in} R \right) K_1 \left( \kappa_{out} R \right) + \kappa_{out} \left( \lambda_e^2 - \frac{k_z^2 V_A^2}{\omega^2} \rho^2 \right)_{out} J_1 \left( \kappa_{in} R \right) K_0 \left( \kappa_{out} R \right) = 0
  \]
Propagation across interface

- Waves propagate at phase velocity above Alfvén speed for warm plasma, below for cold plasma
- Propagation across boundary depends on dispersion on each side:

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<th>Cavity</th>
<th>Enhancement</th>
<th>Propagation from Cavity</th>
<th>Propagation from Enhancement</th>
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Model Results

- Parameters: $V_{A,\text{in}}/V_{A,\text{out}} = 2$, $\lambda_{\text{in}}/\lambda_{\text{out}} = 2$, $\rho_s = \rho_i = 0.2 \lambda_{\text{out}}$ inside and 0.1 outside.
- Note waves are inertial in both regions, so $\omega/k_z < V_A$.
- Longer wavelength modes are evanescent outside, shorter wavelengths propagate.
Modes 1, 3, 5, 7

Mode 1, $V_{ph} = 1.577$, $V_{A,in} = 2.00$, $\lambda_{in} = 2.00$, $\rho_{a,in} = 0.20$

Mode 3, $V_{ph} = 1.092$, $V_{A,in} = 2.00$, $\lambda_{in} = 2.00$, $\rho_{a,in} = 0.20$

Mode 5, $V_{ph} = 0.930$, $V_{A,in} = 2.00$, $\lambda_{in} = 2.00$, $\rho_{a,in} = 0.20$

Mode 7, $V_{ph} = 0.752$, $V_{A,in} = 2.00$, $\lambda_{in} = 2.00$, $\rho_{a,in} = 0.20$

$B\phi, Jz, Ex, Ez$
Parallel Inhomogeneity: Ionospheric Alfvén Resonator

- Alfvén speed rises sharply above ionosphere due to exponential fall of plasma density.

- Alfvén waves are partially reflected from this sharp gradient: wave can bounce between ionosphere and peak in speed: Ionospheric Alfvén Resonator (Periods 1-10 s).

- Waves in this frequency range are commonly observed on ground and from satellites. Field-aligned acceleration can also be modulated at these frequencies.

Profiles of Alfvén speed for high density case (solid line) and low-density case (dashed line). Ionosphere is at r/R_E = 1. Sharp rise in speed can trap waves (like quantum mechanical well). Note speed can approach c in low-density case.
Observational Evidence for 0.1-1.0 Hz waves in the ionospheric Alfvén resonator

Above: Spectrogram from ground magnetic observations from Finland, showing waves at about 0.5 Hz (Koskinen et al., 1993)

Right: Electric field data and spectrum from Viking satellite, showing harmonics of resonator (Block and Fälthammar, 1990)
Dynamics of Aurora on 1-sec scales

Real time imaging of aurora (courtesy of Josh Semeter, Boston U.)
Alfvén Wave Dynamics in Ionospheric Resonator

- Simulation of a density cavity with a 1 Hz wave imposed.
- $64 \text{ km} \times 64 \text{ km} \times 3 \text{ R}_E$ box, pseudo-dipole coordinates ($B \sim 1/r^3$)
- Cold plasma model, $E_z$ due to electron inertia
- Pedersen conductance of 1 mho, Hall conductance 2 mho

See also Lysak and Song, 2000
Evolution of Run: $E_x, B_y$ at $y=0$

- Note structuring of $E_x$ on density gradient. $B_y$ not as structured since short scale waves become electrostatic.
- Evident phase progression away from center line.
- Movie shown in real time.
Structure Perpendicular to Field: $E_x$

$E_x$, Run 80, $z= 1.50$, $t= 0.50$ s, Max= 195.07 mV/m
Structure Perpendicular to Field: $E_z$

$E_z$, Run 80, $z = 1.50$, $t = 0.50$ s, Max $= 0.40$ mV/m
Structure Perpendicular to Field: Poynting flux

Note Poynting flux perpendicular to field is toward center of structure, opposite to phase motion, consistent with dispersion relation and observations.
Model Limitations

• Dynamics of cavity formation not included.
• Possible mechanisms for cavity formation:
  ➢ Perpendicular ion heating plus mirror force (Lysak et al., 1980; Singh, 1984)
  ➢ Ponderomotive force (Li and Temerin, 1993; Rankin et al., 1999)
  ➢ Direct acceleration of ions across field (Knudsen et al., 2004)
• First two mechanisms are slow, $a \sim 1 \text{ km/s}^2$, so time scales of 100 s to travel 1 $R_E$
• Third mechanism is effective only for scales $< \rho_i$ ($< 1 \text{ km for } T_i < 1 \text{ keV}$).
• On fast time scales of ionospheric resonator, density structure probably doesn’t change much.
• Ionospheric conductivity changes (feedback) not included.
• Effects of shear flow (e.g., Kelvin-Helmholtz) also important for dynamics
Observations of Poynting flux from Polar Satellite (Wygant et al., 2000)

Left Panel: From Top to Bottom: Electric Field, Magnetic Field, Poynting Flux, Particle Energy Flux, Density

Right Panel: Particle Data. Top 3 panels are electrons, bottom 3 are ions. Panels give particles going down the field line, perpendicular to the field, and up the field line.
Alfvén Waves on Polar Map to Aurora and Accelerate Electrons

Left: Ultra-violet image of aurora taken from Polar satellite. Cross indicates footpoint of field line of Polar (Wygant et al., 2000)

Right: Electron distribution function measured on Polar. Horizontal direction is direction of magnetic field. Scale is ±40,000 km/s is both directions (Wygant et al., 2002)
Alfvén wave generation in magnetotail

- Alfvén waves can be generated by flows in tail induced by reconnection, followed by mode conversion at plasma sheet boundary layer.
- Ideal nonlinear MHD model initiated with Harris equilibrium, including plasma sheet boundary layer at \( z = \pm 1.8 \ R_E \).
- Gaussian perturbation in three dimensions included, reducing cross-tail current to zero at origin.
- Box size is \( 10 \ R_E \times 6 \ R_E \times 6 \ R_E \)
- Plasma sheet has \( B_x = 15 \ \text{nT}, \ n = 1 \ \text{cm}^{-3}, \ \beta = 4 \)
- Lobe has \( B_x = 30 \ \text{nT}, \ n = 0.1 \ \text{cm}^{-3}, \ \beta = 0.25 \)
- Initial value problem: no driving. \( V_x = 0 \) imposed at \( x=0 \).
Wave Speed Profiles

Magnetosonic speed

Sound speed

Alfvén speed

$\text{t} = 0.00, \text{x} = 2.00, \text{y} = 0.00, \text{va, cf, cs}$
**Initial Conditions**

- Perturbations added to Harris sheet to disrupt current sheet, giving localized $B_z$

Above: Magnetic field vectors in $xz$ plane
Above right: focus on region in red box
Right: Current density in $xz$ plane
Evolution of localized reconnection run

V on z=0 plane

Total pressure on y=0 plane

J_x on y=1.5 plane

J_x on z=1.8 plane (boundary layer)
Currents at Plasma Sheet Boundary Layer

Currents with central part filtered out to bring out currents on plasma sheet boundary layer.
Summary

• Kinetic Alfvén wave dispersion relation can be approached through models of varying complexity: Two-fluid, low-frequency kinetic, finite-frequency corrections, non-local effects.

• Perpendicular structure (e.g., density cavities) can give mode structure in perpendicular direction. Waves propagating in cavity can be evanescent outside.

• Simulations of waves in realistic density profiles show development of fine scale structure due to phase mixing, Hall currents in ionosphere.

• Alfvén waves may be produced by tail reconnection processes, with mode conversion at plasma sheet boundary layer.
Test of gyroradius correction

Solid curve: Full sum \((k_{||} \to 0\) limit):
\[
G(\mu, \omega) = \sum_{n=1}^{\infty} \frac{2\Gamma_n(\mu)}{\mu} \frac{1}{1 - \omega^2 / n^2 \Omega_i^2}
\]
(100 terms)

Dotted curve: Previous approximation:
\[
G(\mu, \omega / \Omega_i) = \frac{1 - \Gamma_0(\mu)}{\mu \left(1 - \omega^2 / \Omega_i^2\right)}
\]

Dashed curve: New approximation:
\[
G(\mu, \omega / \Omega_i) = \frac{1 - \Gamma_0(\mu)}{\mu} + \frac{2\Gamma_1(\mu)}{\mu} \frac{\omega^2 / \Omega_i^2}{1 - \omega^2 / \Omega_i^2}
\]
Observations From FAST satellite

30 seconds of data from the Fast Auroral SnapshoT (FAST) satellite are shown.

Top 4 panels give energy and pitch angle of electrons and ions (red is most intense; 180 degrees is upward). Bottom panel perpendicular electric field.

Large-scale inverted-V electrons indicate quasi-static $E_\parallel$; field-aligned bursts at lower energies associated with Alfvénic acceleration

(McFadden et al., 1998)
Non-local theory on auroral field lines
(e.g., Rankin et al., 1999; Tikhonchuk and Rankin, 2000)

- Idea is to integrate Vlasov equation over past history of a particle. History is defined by considering constants of motion: magnetic moment

\[
\frac{1}{2} m v^2 + \mu B (z) + q \Phi (z)
\]

- Linearized Vlasov equation can then be written as

\[
\left( \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial z} \right) f \pm (t, z, \nu, W) + W \frac{\partial f_0 \pm}{\partial W} = 0
\]

- Note that positive and negative velocities are separated, and...
Wave Equations Including Non-local Conductivity

- The non-local conductivity must be coupled with Maxwell’s equations to find a self-consistent solution. System of equations become:

\[-i\omega \varepsilon_0 E_z + \int dz' \sigma(z, z') E_z(z') = i k_{\perp} B_0 B_y\]

\[\frac{\partial B_y}{\partial z} = i\omega \left( 1 + \frac{c^2}{V_A^2} \left( 1 - \Gamma_0 \left( \mu_i \right) \right) \right) E_x\]

\[\frac{\partial E_x}{\partial z} = i\omega B_y + i k_{\perp} E_z\]

- Initially, two fluid wave equations used to calculate $E_x$ and $B_y$ profiles. This gives $j_z$, and integral equation is solved to get $E_z$. New $E_z$ is then used to recalculate $E_x$ and $B_y$. Iterations continue to convergence. Solution is checked for energy self-consistency:

$$S_{inc} = S_{ref} + \frac{1}{2} \sum_{p} |E_{xl}|^2 + \frac{1}{2} \int dz \text{Re} \left( j_z^* E_z \right)$$
Non-local conductivity kernel

- Integration of Vlasov equation leads to the conductivity:

\[
\sigma(z, z') = -\frac{2\pi q^2 B(z)}{m^2} \left\{ \int_{-e\Phi_i}^\infty dW \left[ \Theta(z - z') \int_0^{\mu_i} d\mu \frac{\partial f_{0I}}{\partial W} e^{i\omega(z', z)} \right] - \int_{\mu_0}^{\mu_i} d\mu \Theta(z_m - z) \frac{\partial f_{0I}}{\partial W} \left( e^{i\omega(z_m, z)} + i\omega(z_m, z') - \Theta(z' - z) e^{i\omega(z, z')} \right) \right\}
\]

Upgoing ionosphere elec.

\[
-\int_{\mu_0}^{\mu_i} d\mu \Theta(z_m - z) \frac{\partial f_{0I}}{\partial W} \left( e^{i\omega(z_m, z)} + i\omega(z_m, z') - \Theta(z' - z) e^{i\omega(z, z')} \right)
\]

Reflected ionospheric elec.

\[
+ \int_{-e\Phi_o}^\infty dW \left[ \Theta(z' - z) \int_0^{\mu_0} d\mu \frac{\partial f_{0M}}{\partial W} e^{i\omega(z, z')} \right] - \int_{\mu_i}^{\mu_0} d\mu \Theta(z - z_m) \frac{\partial f_{0M}}{\partial W} \left( e^{i\omega(z, z_m)} + i\omega(z, z') - \Theta(z' - z) e^{i\omega(z, z')} \right)
\]

Downgoing magnetospheric elec.

Mirroring magnetospheric elec.

\[
+ \frac{2}{\pi} \int_{W_{\text{min}}}^{W_{\text{max}}} dW \int_{\mu_{\text{min}}}^{\mu_{\text{max}}} d\mu \frac{\partial f_T}{\partial W} \sum_{n=-\infty}^{\infty} \frac{i\omega_B}{\omega - n\omega_B} \sin n\theta(z) \sin n\theta(z')
\]

Trapped electrons

Note here \( \mu_0 = \min_{z' > z} \frac{W - U(z')}{B(z')} \) and \( \mu_1 = \min_{z' < z} \frac{W - U(z')}{B(z')} \)

and \( W_{\text{min}}, W_{\text{max}}, \mu_{\text{min}} \) and \( \mu_{\text{max}} \) give boundary of trapped region
Parallel electric fields for T=1, 10, 100, 1000 eV

Parallel electric fields (real, solid; imaginary, dotted) compared with cold plasma result (real, dashed, imaginary, dot-dash) for electron temperature of 1, 10, 100, 1000 eV.