Forward propagation of acoustic pressure pulses in 3D soft biological tissue

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Motivation

Ideal

Ultrasound image of a tissue-mimicking phantom.
Motivation

Ultrasound image of a tissue-mimicking phantom.
Overview

- Modelling
- Numerical solution
- Parallelisation
- Examples
Some notation

Movement about equilibrium \( r \):

\[
    r_E(r, t) = r + \Psi(r, t)
\]

Deformation gradient tensor

\[
    F = I + \frac{\partial \Psi}{\partial r} = \begin{pmatrix}
    1 + \frac{\partial \Psi_1}{\partial r_1} & \frac{\partial \Psi_1}{\partial r_2} & \frac{\partial \Psi_1}{\partial r_3} \\
    \frac{\partial \Psi_2}{\partial r_1} & 1 + \frac{\partial \Psi_2}{\partial r_2} & \frac{\partial \Psi_2}{\partial r_3} \\
    \frac{\partial \Psi_3}{\partial r_1} & \frac{\partial \Psi_3}{\partial r_2} & 1 + \frac{\partial \Psi_3}{\partial r_3}
\end{pmatrix}
\]

Jacobian determinant

\[
    |F| \equiv \det F
\]
Fundamental relations

Conservation of mass:

\[
\int_{V_0} \rho_0(r) \, dr = \int_{V(t)} \rho(r_E, t) \, dr_E = \int_{V_0} \rho(r, t) |F| \, dr
\]

Pressure forces:

\[
- \int_{V_t} \nabla_E p \, dr_E = - \int_{V_0} (F^{-1})^T \nabla p |F| \, dr
\]

 Constitutive relation:

\[
p(\rho) = A \left( \frac{\rho - \rho_0}{\rho_0} \right) + \frac{B}{2} \left( \frac{\rho - \rho_0}{\rho_0} \right)^2
\]
Acoustic wave equation

Newton’s law of motion with conservation of mass:

$$\rho_0 \frac{\partial^2 \Psi}{\partial t^2} = -|F| (F^{-1})^T \nabla p$$

Constitutive relation with conservation of mass:

$$p(|F|) = A \left( \frac{1 - |F|}{|F|} \right) + \frac{B}{2} \left( \frac{1 - |F|}{|F|} \right)^2$$
Acoustic wave equation

Newton’s law of motion with conservation of mass:

\[ \rho_0 \frac{\partial^2 \Psi}{\partial t^2} = -|F| (F^{-1})^T \nabla p \]

Constitutive relation with conservation of mass:

\[ 1 - |F| = \kappa p - \beta_n (\kappa p)^2 \]

\[ \kappa = 1/A \quad : \text{compressibility} \]
\[ \beta_n = 1 + B/2A \quad : \text{coefficient of nonlinearity} \]
Acoustic wave equation

Newton's law of motion with conservation of mass:

$$\rho_0 \frac{\partial^2 \Psi}{\partial t^2} = -|F| (F^{-1})^T \nabla p$$

Constitutive relation with conservation of mass:

$$1 - |F| = \kappa p - \beta_n (\kappa p)^2 - \kappa \mathcal{L} p$$

$$\kappa = 1/A : \text{compressibility}$$

$$\beta_n = 1 + B/2A : \text{coefficient of nonlinearity}$$
Acoustic wave equation (II)

Observations:

\[ |F| \approx 1 + \nabla \cdot \Psi \]

\[ |F| (F^{-1})^T \nabla p \approx \nabla p \]

A generalised Westervelt equation

\[ \kappa \ddot{p} - \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) = \frac{d^2}{dt^2} \left( \beta_n \kappa^2 p^2 + \kappa \mathcal{L}_p \right) \]
Acoustic wave equation (III)

Retarded time : \( \tau = t - z \)
Scaled pressure : \( p_* = p \sqrt{\rho} \)
Speed of sound : \( 1/c^2 = 1 - 2c_1(r) \)
Mass density : \( g = \sqrt{\rho} \Delta (1/\sqrt{\rho}) \)

Wave equation rewritten:

\[
\frac{\partial^2 p}{\partial \tau \partial z} = \frac{1}{2} (\Delta - g) p - \epsilon_t \ddot{p} + \frac{\epsilon_n}{2} \frac{\partial^2 p^2}{\partial \tau^2} + \epsilon \frac{\partial^2 Lp}{\partial \tau^2}
\]
Acoustic wave equation (III)

Retarded time : $\tau = t - z$
Scaled pressure : $p_* = p \sqrt{\rho}$
Speed of sound : $1/c^2 = 1 - 2c_1(r)$
Mass density : $g = \sqrt{\rho} \Delta (1/\sqrt{\rho})$
Wave equation rewritten:

$$\frac{\partial p}{\partial z} = \frac{1}{2} \int_0^\tau (\Delta - g) p d\tau - \epsilon_t \dot{p} + \frac{\epsilon_n}{2} \frac{\partial p^2}{\partial \tau} + \epsilon \frac{\partial L \rho}{\partial \tau}$$
Acoustic wave equation(III)

Retarded time : $\tau = t - z$
Scaled pressure : $p_* = p\sqrt{\rho}$
Speed of sound : $1/c^2 = 1 - 2c_1(r)$
Mass density : $g = \sqrt{\rho}\Delta(1/\sqrt{\rho})$

Wave equation rewritten:

$$\frac{\partial p}{\partial z} = A_d p + A_n p + A_l p$$
Splitting schemes

Equations:

\[
\frac{\partial p}{\partial z} = A_d p = \frac{1}{2} \int_0^\tau (\Delta - g) p d\tau
\]

\[
\frac{\partial p}{\partial z} = A_n p = (\epsilon_n p - \epsilon_t) \dot{p}
\]

\[
\frac{\partial p}{\partial z} = A_l p = \epsilon \frac{\partial L p}{\partial \tau}
\]

“First-order scheme”: \[ p(z + h) = e^{hA_d} e^{hA_n} e^{hA_l} p(z) \].
Implementation $A_l$

Attenuation operator in frequency domain:

$$p(z + h, \omega) = e^{-h\epsilon(1-i\mathcal{H})|\omega|^b} p(z, \omega)$$

Observations:

1. Accurate solution using FFT
2. Each spatial location $(x, y)$ solved separately
Implementation $A_n$

Nonlinear operator

\[ p(z + h, \tau) = p(z, \tau - h[z, p(z, \tau)]) \]

\[ [z, p(z, \tau)] = p(z, \tau) \int_{z}^{z+h} \epsilon_n(\xi) d\xi - \int_{z}^{z+h} \epsilon_t(\xi) d\xi \]

Observations:

1. Possible problem if stepsize is large (shock)
3. Each spatial location \((x, y)\) solved separately
Implementation $A_d$

Wave equation with constant propagation speed and variable mass density

$$\frac{\partial p}{\partial z} = A_d p = \frac{1}{2} \int_{0}^{\tau} (\Delta - g) p d\tau$$
Implementation $A_d$

Wave equation with constant propagation speed

$$\frac{\partial p}{\partial z} = A_d p = \frac{1}{2} \int_0^\tau (\Delta \quad ) p d\tau$$

Propagation in positive z-direction:

$$p(x, z + h, \omega) = U(k_x, k_y, \omega, h)p(k_x, k_y, z, \omega),$$

$$U(k_x, k_y, \omega, h) = \begin{cases} 
e^{-i\omega\left(\sqrt{1-(k_x^2+k_y^2)/\omega^2-1}\right)} & , \omega^2 > k_x^2 + k_y^2 \\
e^{i\omega\left(i\sqrt{(k_x^2+k_y^2)/\omega^2-1+1}\right)} & , \text{otherwise.} \end{cases}$$
Implementation \( A_d \)

Propagation in positive \( z \)-direction:

\[
p(x, z + h, \omega) = U(k_x, k_y, \omega, h)p(k_x, k_y, z, \omega),
\]

\[
U(k_x, k_y, \omega, h) = \begin{cases} 
  e^{-i\omega \left( \sqrt{1 - (k_x^2 + k_y^2)/\omega^2} - 1 \right)} & , \omega^2 > k_x^2 + k_y^2 \\
  e^{i\omega \left( i \sqrt{(k_x^2 + k_y^2)/\omega^2} - 1 + 1 \right)} & , \text{otherwise.}
\end{cases}
\]

1. Accurate solution using FFT
2. Solve each frequency separately
Parallellisation (initial steps)

\[ p(x, y, 0, t) = p_0(x, y, t) \]

Forward propagation of acoustic pressure pulses in 3D soft biological tissue
Parallellisation (initial steps)

Initial step 1:
Distribute $p_0(x, y, t)$ and medium parameters, $\epsilon_n(x, y, z)$, $\epsilon_t(x, y, z)$, $\epsilon(x, y, z)$ and $b(x, y, z)$, over $P$ processors in slices along the $x$-axis ranging from $x_n$ to $x_{n+1}$ on processor $n$. 

$p(x, y, 0, t) = p_0(x, y, t)$
Parallellisation (initial steps)

\[ p(x, y, 0, t) = p_0(x, y, t) \]

**Initial step 1:**
Distribute \( p_0(x, y, t) \) and medium parameters, \( \epsilon_n(x, y, z), \epsilon_t(x, y, z), \epsilon(x, y, z) \) and \( b(x, y, z) \), over \( P \) processors in slices along the \( x \)-axis ranging from \( x_n \) to \( x_{n+1} \) on processor \( n \).

**Initial step 2:**
Distribute propagation factor \( U(k_x, k_y, \omega, h) \) over \( P \) processors in slices along the \( \omega \)-axis ranging from \( \omega_n \) to \( \omega_{n+1} \) on processor \( n \).
Parallellisation (solution)

Step 1: Apply $e^{hA_l}$ and $e^{hA_n}$. Local operation at each processor, only requiring the pre-distributed material parameters.

$$e^{hA_n} e^{hA_l} p(x, y, 0, t)$$
Parallellisation (solution)

\[ e^{\hbar A_n} e^{\hbar A_l} p(x, y, 0, t) \]

**Step 2:** Perform FFT along \( y \) and \( t \)-axis.
Parallellisation (solution)

\[ e^{hA_n} e^{hA_l} p(x, y, 0, t) \]

**Step 3:** Redistribute solution over \( P \) processors in slices along the \( \omega \)-axis.
Parallellisation (solution)

Step 4: Perform FFT along $x$-axis.

$$e^{hA_n}e^{hA_l}p(x, y, 0, t)$$
Parallellisation (solution)

\[ e^{hA_n} e^{hA_l} p(x, y, 0, t) \]

Step 5: Multiply by \( U(k_x, k_y, \omega, h) \). Local operation at each processor, only requiring the pre-distributed propagation factor.
Parallellisation (solution)

\[ e^{hA_n} e^{hA_l} p(x, y, 0, t) \]

**Step 6:** Perform inverse FFT (IFFT) along \( k_x \) and \( k_y \).
Parallellisation (solution)

\[ e^{hA_n} e^{hA_l} p(x, y, 0, t) \]

**Step 7:** Redistribute solution over \( P \) processors in slices along the \( x \)-axis.
Parallellisation (solution)

\[ p(x, y, h, t) = e^{hA_d}e^{hA_n}e^{hA_l}p(x, y, 0, t) \]

**Step 8:** Perform IFFT along \( \omega \). The result on each processor now constitutes the solution one step of length \( h \) forward.
Parallellisation (solution)

\[ p(x, y, h, t) = e^{hA_d} e^{hA_n} e^{hA_l} p(x, y, 0, t) \]

- Repeat until the desired depth \( z \).
- Close to perfect speed-up on a dual SMP-machine.
Accuracy

Local relative error compared to analytic reference solution

Forward propagation of acoustic pressure pulses in 3D soft biological tissue
Accuracy (II)

Comparison with experiment in water tank

Forward propagation of acoustic pressure pulses in 3D soft biological tissue
Accuracy (II)

Comparison with experiment in water tank

Forward propagation of acoustic pressure pulses in 3D soft biological tissue
Simulated ultrasonic imaging

Ideal situation
Simulated ultrasonic imaging

Standard imaging

Forward propagation of acoustic pressure pulses in 3D soft biological tissue
Simulated ultrasonic imaging

Corrected imaging

Forward propagation of acoustic pressure pulses in 3D soft biological tissue
Thank you for your attention!