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Forward propagation of acoustic pressure pulses in 3D soft biological tissue

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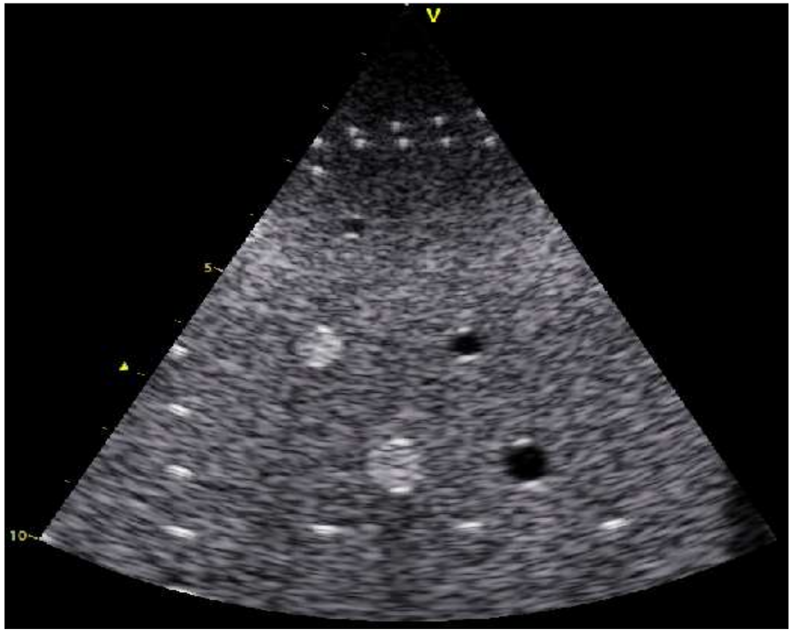
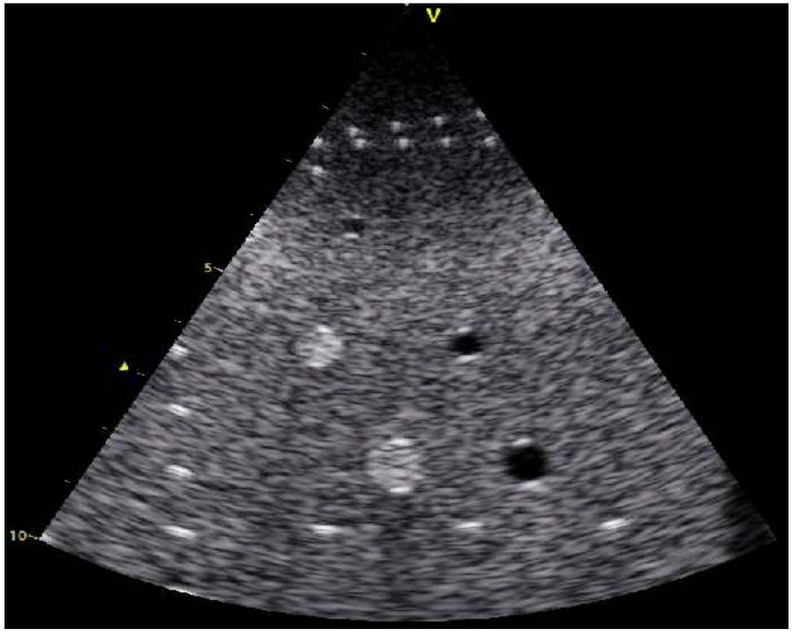
Motivation



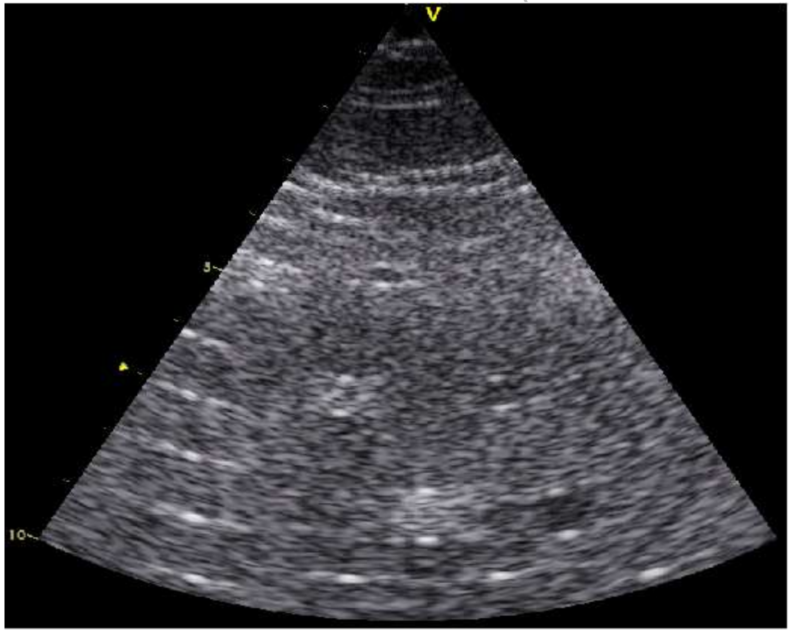
Ideal

Ultrasound image of a tissue-mimicking phantom.

Motivation



Ideal



Aberrated

Ultrasound image of a tissue-mimicking phantom.

Overview

- Modelling
- Numerical solution
- Parallellisation
- Examples

Some notation

Movement about equilibrium r:

$$r_{\mathrm{E}}(r,t) = r + \Psi(r,t)$$

Deformation gradient tensor

$$F = I + \frac{\partial \Psi}{\partial r} = \begin{pmatrix} 1 + \frac{\partial \Psi_1}{\partial r_1} & \frac{\partial \Psi_1}{\partial r_2} & \frac{\partial \Psi_1}{\partial r_3} \\ \frac{\partial \Psi_2}{\partial r_1} & 1 + \frac{\partial \Psi_2}{\partial r_2} & \frac{\partial \Psi_2}{\partial r_3} \\ \frac{\partial \Psi_3}{\partial r_1} & \frac{\partial \Psi_3}{\partial r_2} & 1 + \frac{\partial \Psi_3}{\partial r_3} \end{pmatrix}$$

Jacobian determinant

$$|F| \equiv \det F$$

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Fundamental relations

Conservation of mass:

$$\int_{V_0} \rho_0(r)\mathrm{d}r = \int_{V(t)} \rho(r_\mathrm{E}, t)\mathrm{d}r_\mathrm{E} = \int_{V_0} \rho(r,t)|F|\mathrm{d}r$$

Pressure forces:

$$-\int_{V_t} \nabla_\mathrm{E} p \mathrm{d}r_\mathrm{E} = -\int_{V_0} \left(F^{-1}\right)^T \nabla p |F| \mathrm{d}r$$

Constitutive relation:

$$p(\rho) = A\left(\frac{\rho - \rho_0}{\rho_0}\right) + \frac{B}{2}\left(\frac{\rho - \rho_0}{\rho_0}\right)^2$$

Acoustic wave equation

Newtont's law of motion with conservation of mass:

$$\rho_0 \frac{\partial^2 \Psi}{\partial t^2} = -|F| \left(F^{-1}\right)^T \nabla p$$

Constitutive relation with conservation of mass:

$$p(|F|) = A\left(\frac{1-|F|}{|F|}\right) + \frac{B}{2}\left(\frac{1-|F|}{|F|}\right)^2$$

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Acoustic wave equation

Newtont's law of motion with conservation of mass:

$$\rho_0 \frac{\partial^2 \Psi}{\partial t^2} = -|F| \left(F^{-1}\right)^T \nabla p$$

Constitutive relation with conservation of mass:

$$1 - |F| = \kappa p - \beta_n (\kappa p)^2$$

$\kappa = 1/A$: *compressibility*

$\beta_n = 1 + B/2A$: *coefficient of nonlinearity*

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Acoustic wave equation

Newtont's law of motion with conservation of mass:

$$\rho_0 \frac{\partial^2 \Psi}{\partial t^2} = -|F| \left(F^{-1}\right)^T \nabla p$$

Constitutive relation with conservation of mass:

$$1 - |F| = \kappa p - \beta_n (\kappa p)^2 - \kappa \mathcal{L} p$$

$\kappa = 1/A$: *compressibility*

$\beta_n = 1 + B/2A$: *coefficient of nonlinearity*

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Acoustic wave equation (II)

Observations:

$$|F| \approx 1 + \nabla \cdot \Psi$$

$$|F| \left(F^{-1}\right)^T \nabla p \approx \nabla p$$

A generalised Westervelt equation

$$\kappa \ddot{p} - \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\beta_n \kappa^2 p^2 + \kappa \mathcal{L} p \right)$$

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Acoustic wave equation(III)

Retarded time : $\tau = t - z$

Scaled pressure : $p_* = p\sqrt{\rho}$

Speed of sound : $1/c^2 = 1 - 2c_1(r)$

Mass density : $g = \sqrt{\rho}\Delta(1/\sqrt{\rho})$

Wave equation rewritten:

$$\frac{\partial^2 p}{\partial\tau\partial z} = \frac{1}{2}\left(\Delta - g\right)p - \epsilon_t \ddot{p} + \frac{\epsilon_n}{2}\frac{\partial^2 p^2}{\partial\tau^2} + \epsilon\frac{\partial^2 Lp}{\partial\tau^2}$$

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Acoustic wave equation(III)

- Retarded time : $\tau = t - z$
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- Speed of sound : $1/c^2 = 1 - 2c_1(r)$
- Mass density : $g = \sqrt{\rho}\Delta(1/\sqrt{\rho})$

Wave equation rewritten:

$$\frac{\partial p}{\partial z} = A_d p + A_n p + A_l p$$

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Splitting schemes

Equations:

$$\frac{\partial p}{\partial z} = A_d p = \frac{1}{2}\int_0^{\tau} (\Delta - g)\, p \mathrm{d}\tau$$

$$\frac{\partial p}{\partial z} = A_n p = (\epsilon_n p - \epsilon_t)\, \dot{p}$$

$$\frac{\partial p}{\partial z} = A_l p = \epsilon \frac{\partial L p}{\partial \tau}$$

“First-order scheme”: $p(z+h) = e^{hA_d} e^{hA_n} e^{hA_l} p(z)$.

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Implementation A_l

Attenuation operator in frequency domain:

$$p(z+h,\omega) = e^{-h\epsilon(1-i\mathcal{H})|\omega|^b} p(z,\omega)$$

Observations:

1. Accurate solution using FFT
2. Each spatial location (x, y) solved separately

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Implementation A_n

Nonlinear operator

$$p(z+h,\tau) = p(z,\tau - h[z,p(z,\tau)])$$

$$[z,p(z,\tau)] = p(z,\tau)\int_z^{z+h}\epsilon_n(\xi)\mathrm{d}\xi - \int_z^{z+h}\epsilon_t(\xi)\mathrm{d}\xi$$

Observations:

1. Possible problem if stepsize is large (shock)
2. Accurate solution using method of characteristics and linear interpolation.
3. Each spatial location (x,y) solved separately

Implementation A_d

Wave equation with constant propagation speed and variable mass density

$$\frac{\partial p}{\partial z} = A_d p = \frac{1}{2}\int_0^\tau (\Delta - g)\, p \mathrm{d}\tau$$

Implementation A_d

Wave equation with constant propagation speed

$$\frac{\partial p}{\partial z} = A_d p = \frac{1}{2}\int_0^{\tau} (\Delta \qquad) \, p \, \mathrm{d}\tau$$

Propagation in positive z-direction:

$$p(x, z+h, \omega) = U(k_x, k_y, \omega, h) p(k_x, k_y, z, \omega),$$

$$U(k_x, k_y, \omega, h) = \begin{cases} e^{-ih\omega\left(\sqrt{1-\left(k_x^2+k_y^2\right)/\omega^2}-1\right)} & , \omega^2 > k_x^2 + k_y^2 \\ e^{ih\omega\left(i\sqrt{\left(k_x^2+k_y^2\right)/\omega^2-1}+1\right)} & , \text{otherwise.} \end{cases}$$

Implementation A_d

Propagation in positive z-direction:

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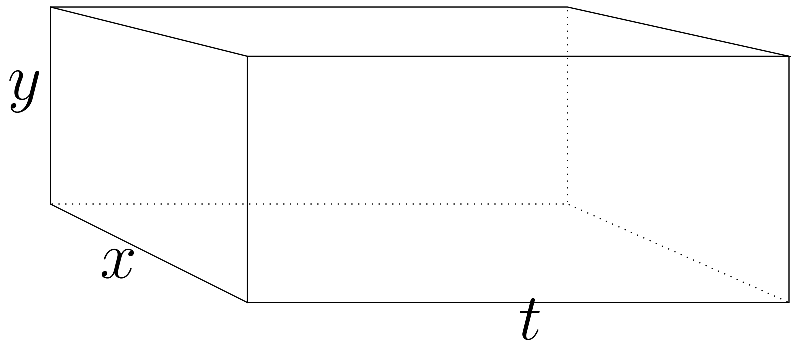
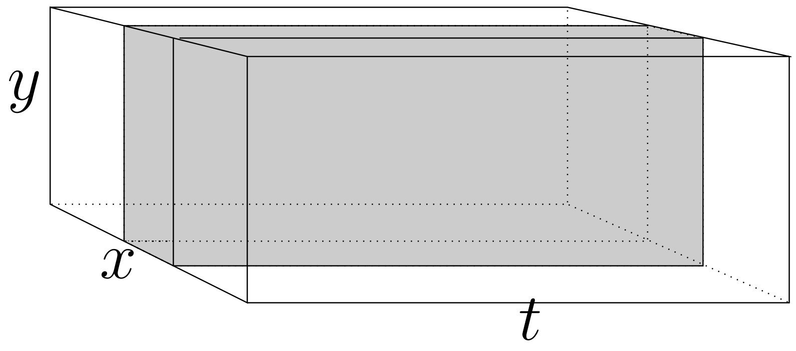
1. Accurate solution using FFT
2. Solve each frequency separately

Parallellisation (initial steps)



$$p(x, y, 0, t) = p_0(x, y, t)$$

Parallellisation (initial steps)

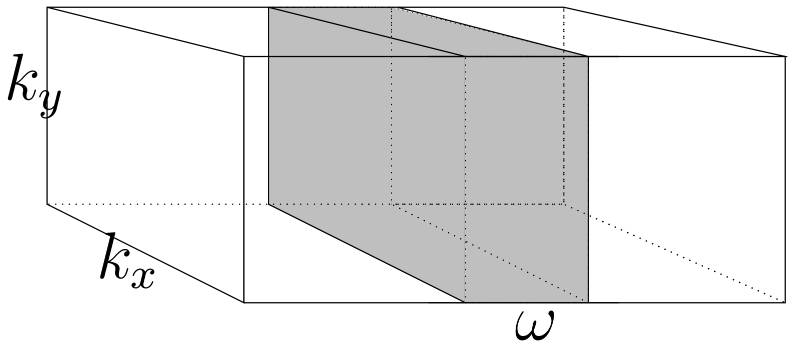


$$p(x, y, 0, t) = p_0(x, y, t)$$

Initial step 1:

Distribute $p_0(x, y, t)$) and medium parameters, $\epsilon_n(x, y, z)$, $\epsilon_t(x, y, z)$, $\epsilon(x, y, z)$ and $b(x, y, z)$, over P processors in slices along the x-axis ranging from x_n to x_{n+1} on processor n.

Parallellisation (initial steps)

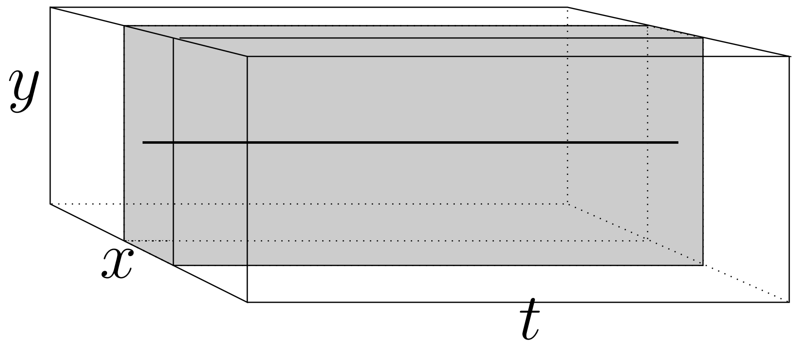


$$p(x, y, 0, t) = p_0(x, y, t)$$

Initial step 1:
Distribute $p_0(x, y, t)$) and medium parameters, $\epsilon_n(x, y, z)$, $\epsilon_t(x, y, z)$, $\epsilon(x, y, z)$ and $b(x, y, z)$, over P processors in slices along the x-axis ranging from x_n to x_{n+1} on processor n.

Initial step 2:
Distribute propagation factor $U(k_x, k_y, \omega, h)$ over P processors in slices along the ω-axis ranging from ω_n to ω_{n+1} on processor n.

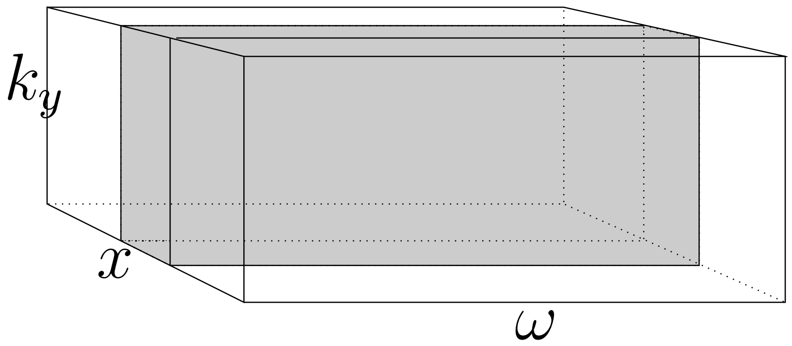
Parallellisation (solution)



$$e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

Step 1: Apply e^{hA_l} and e^{hA_n}. Local operation at each processor, only requiring the pre-distributed material parameters.

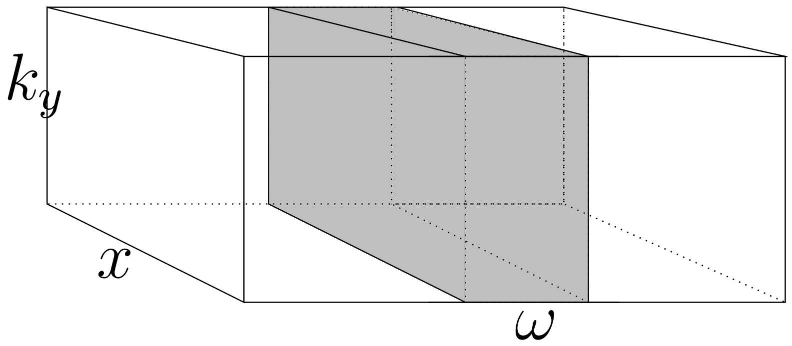
Parallellisation (solution)



$$e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

Step 2: Perform FFT along y and t-axis.

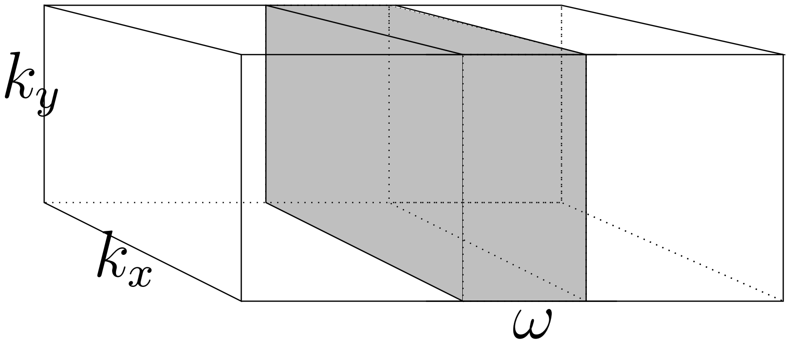
Parallellisation (solution)



$$e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

Step 3: Redistribute solution over P processors in slices along the ω-axis.

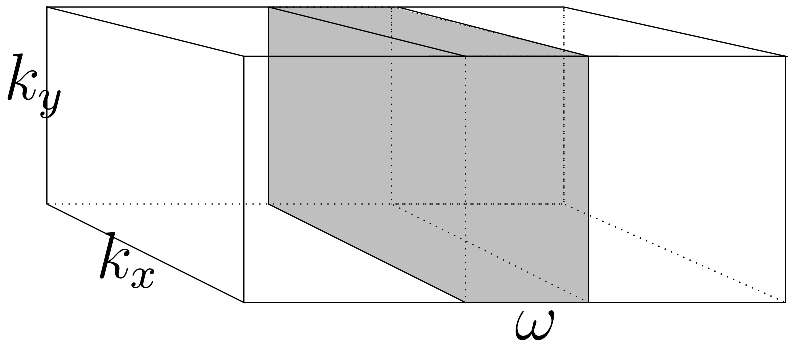
Parallellisation (solution)



$$e^{hA_n}e^{hA_l}p(x,y,0,t)$$

Step 4: Perform FFT along x-axis.

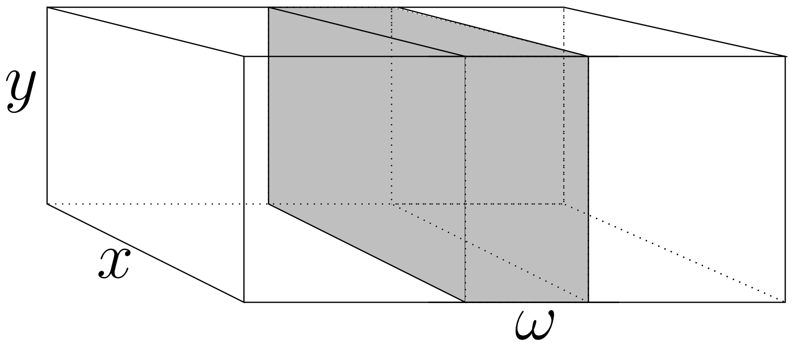
Parallellisation (solution)



$$e^{hA_n}e^{hA_l}p(x,y,0,t)$$

Step 5: Multiply by $U(k_x,k_y,\omega,h)$. Local operation at each processor, only requiring the pre-distributed propagation factor.

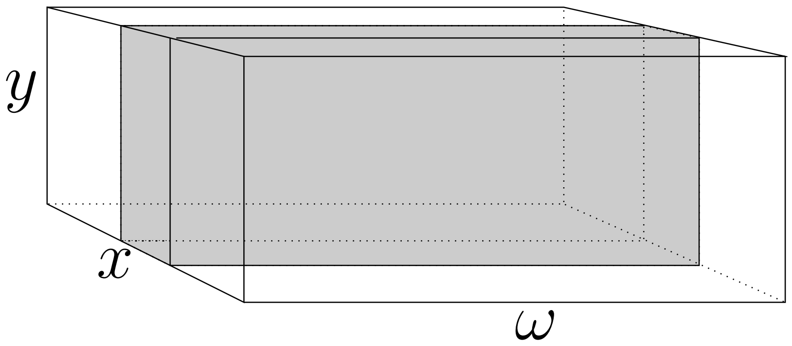
Parallellisation (solution)



$$e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

Step 6: Perform inverse FFT (IFFT) along k_x and k_y.

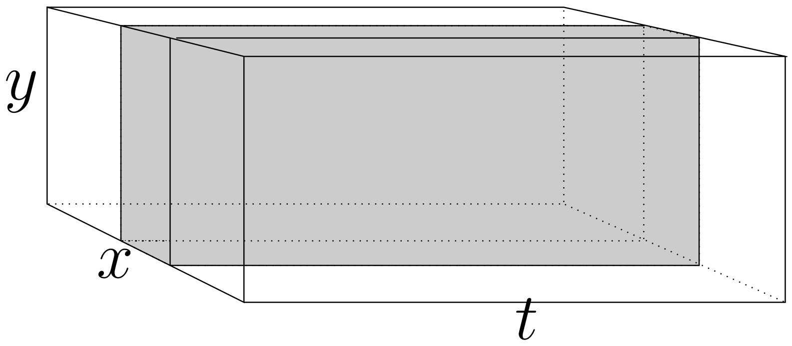
Parallellisation (solution)



$$e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

Step 7: Redistribute solution over P processors in slices along the x-axis.

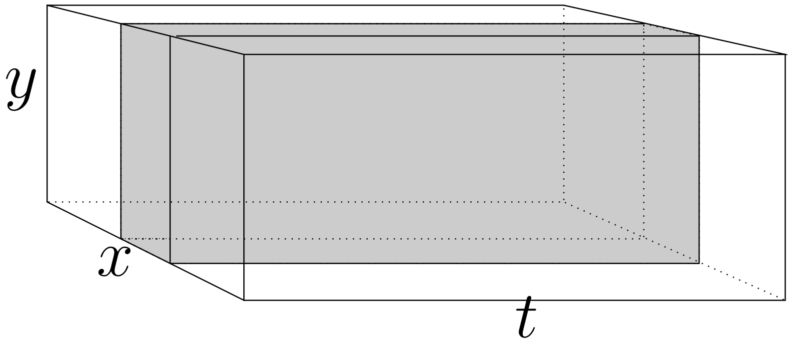
Parallellisation (solution)



$$p(x, y, h, t) = e^{hA_d} e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

Step 8: Perform IFFT along ω. The result on each processor now constitutes the solution one step of length h forward.

Parallellisation (solution)

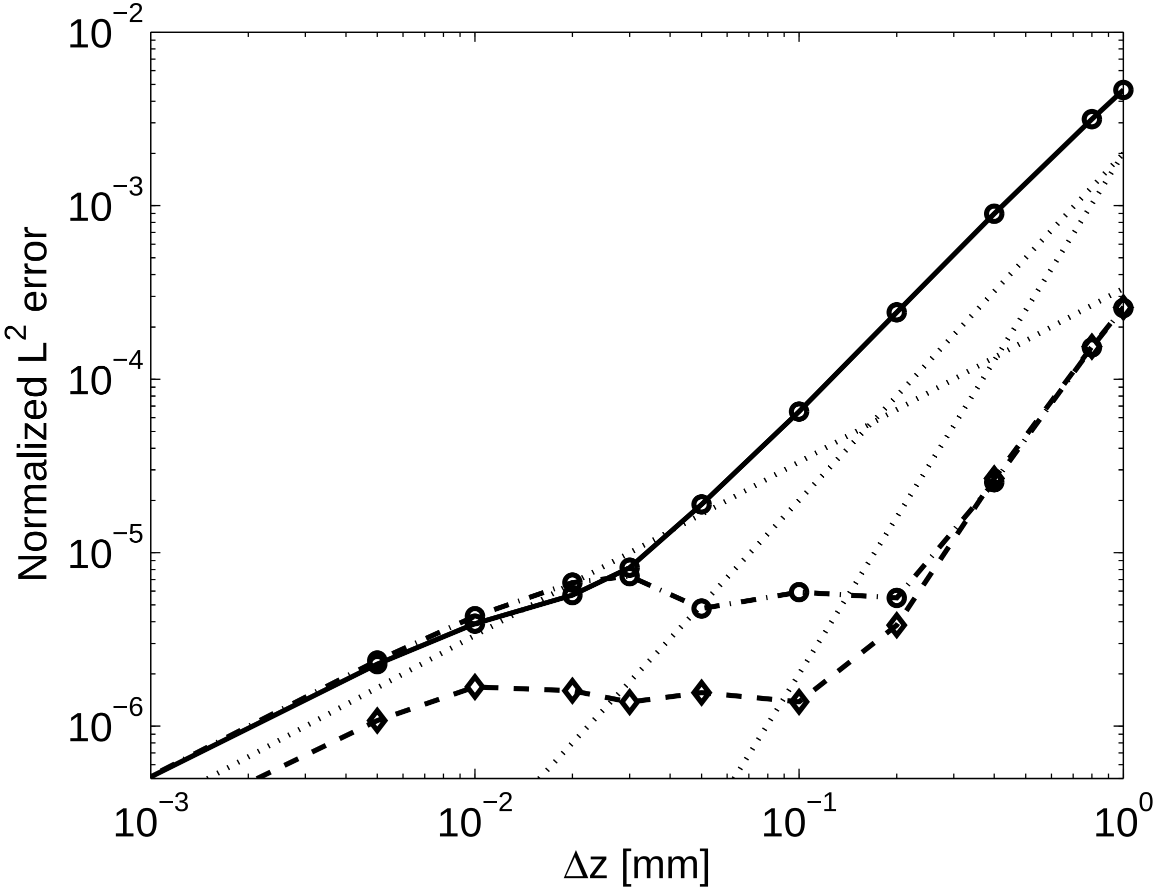


$$p(x, y, h, t) = e^{hA_d} e^{hA_n} e^{hA_l} p(x, y, 0, t)$$

- Repeat until the desired depth z.
- Close to perfect speed-up on a dual SMP-machine.

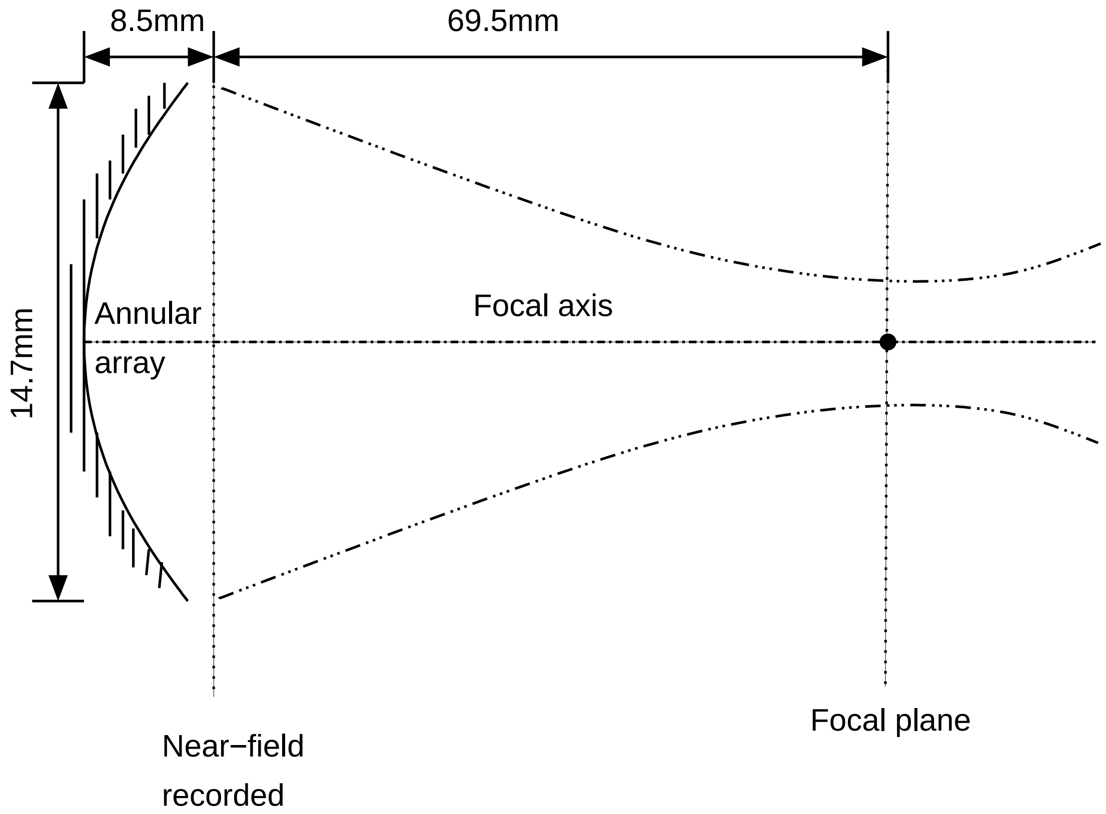
Accuracy

Local relative error compared to analytic reference solution



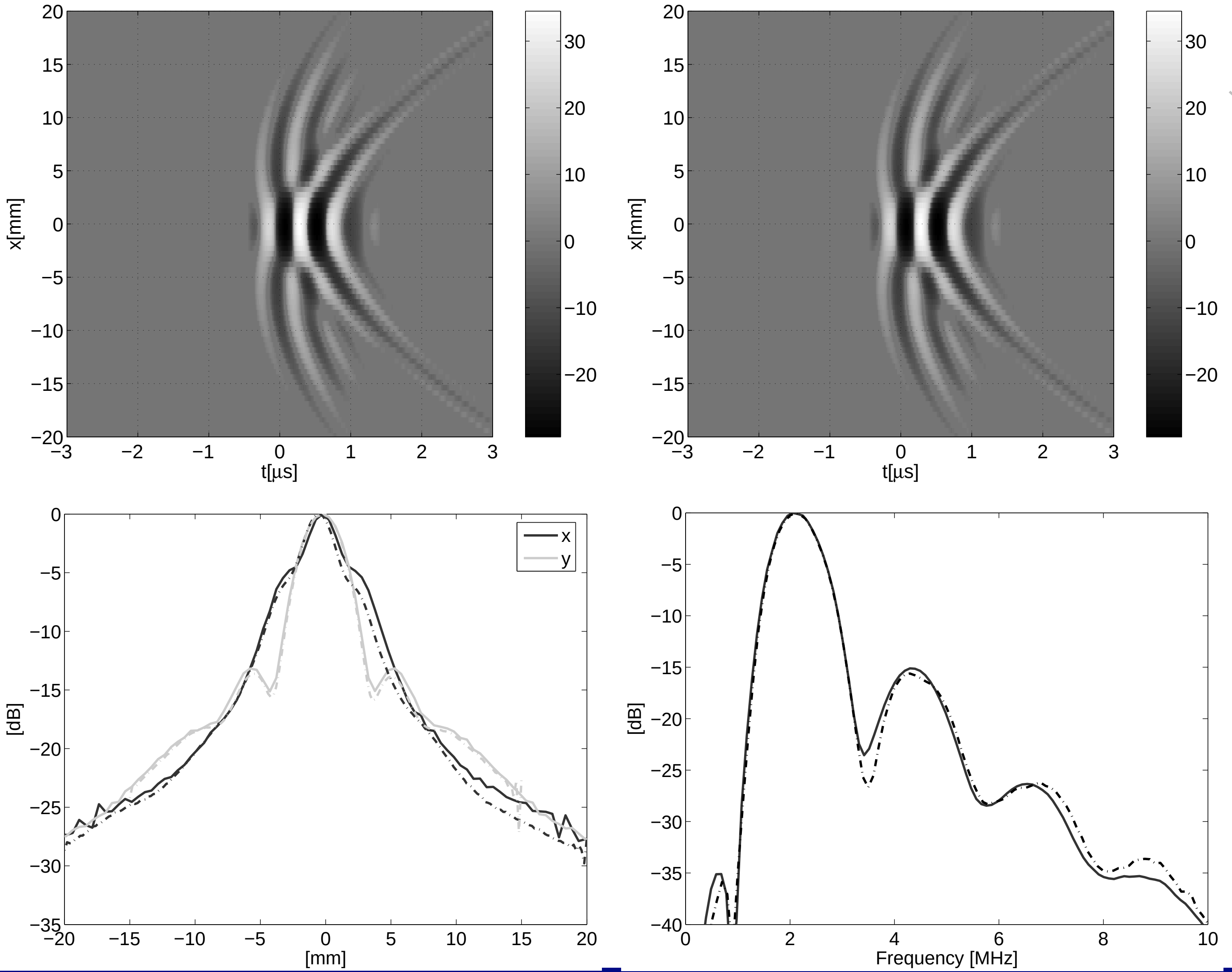
Accuracy (II)

Comparison with experiment in water tank

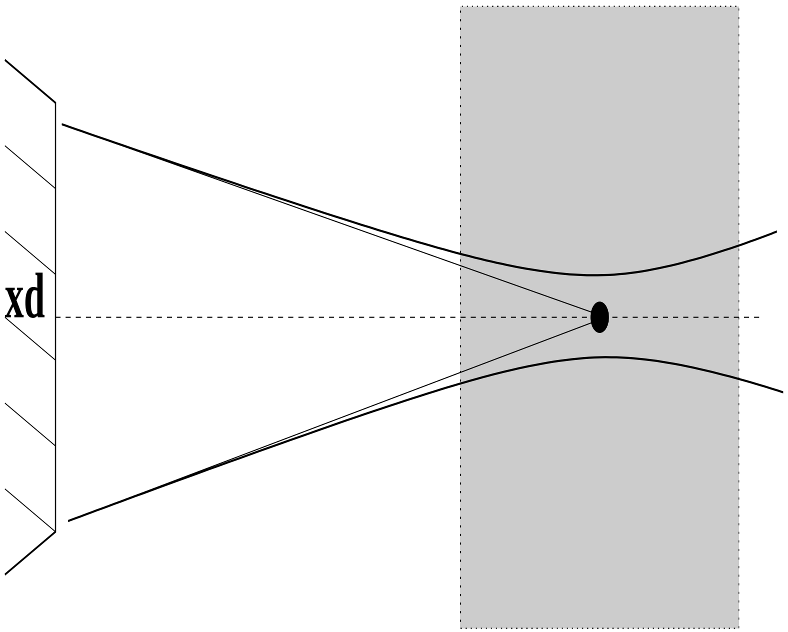


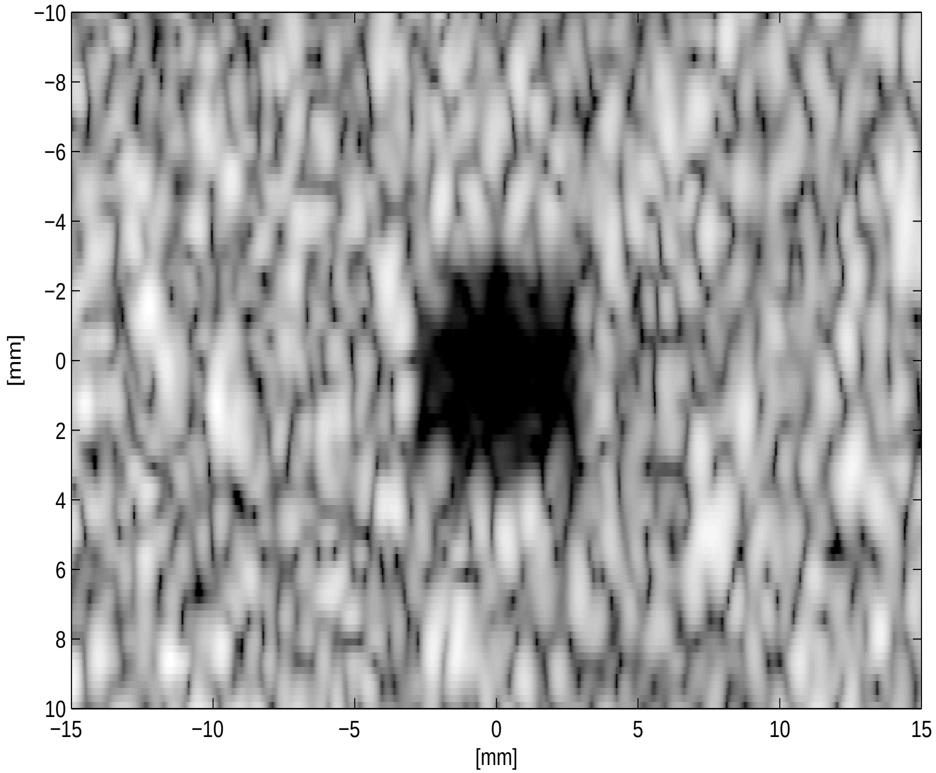
Accuracy (II)

Comparison with experiment in water tank



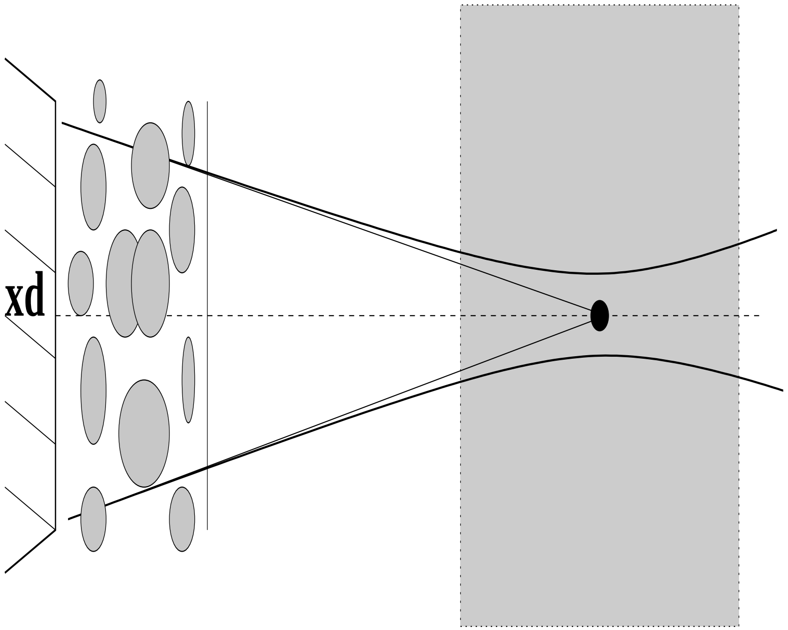
Simulated ultrasonic imaging

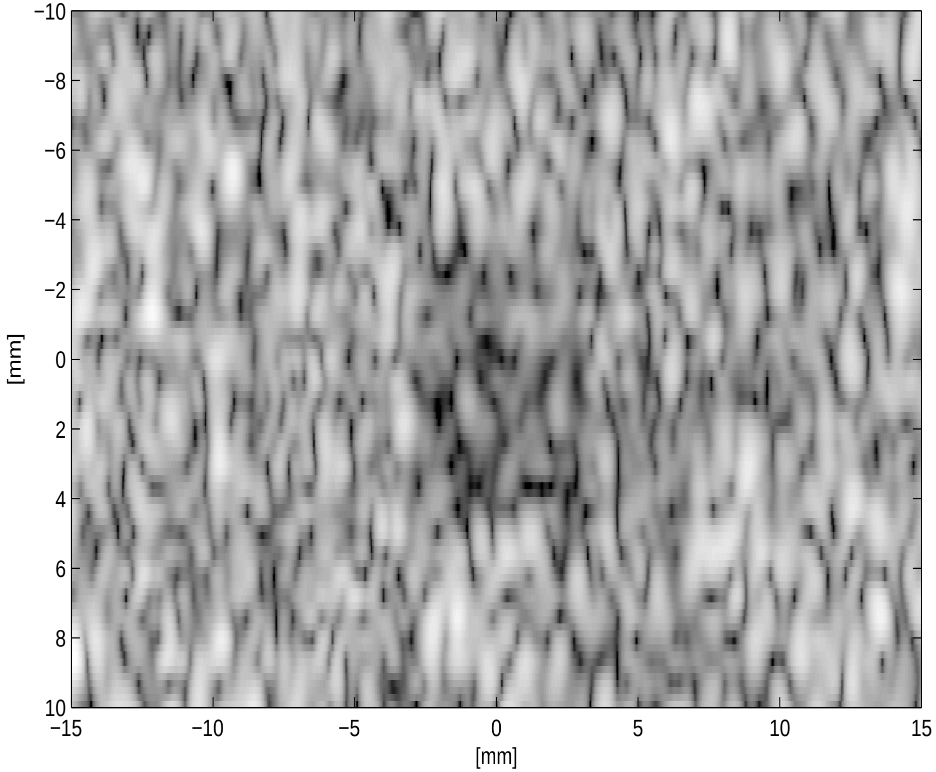




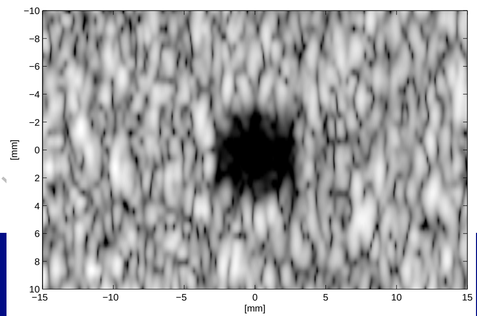
Ideal situation

Simulated ultrasonic imaging

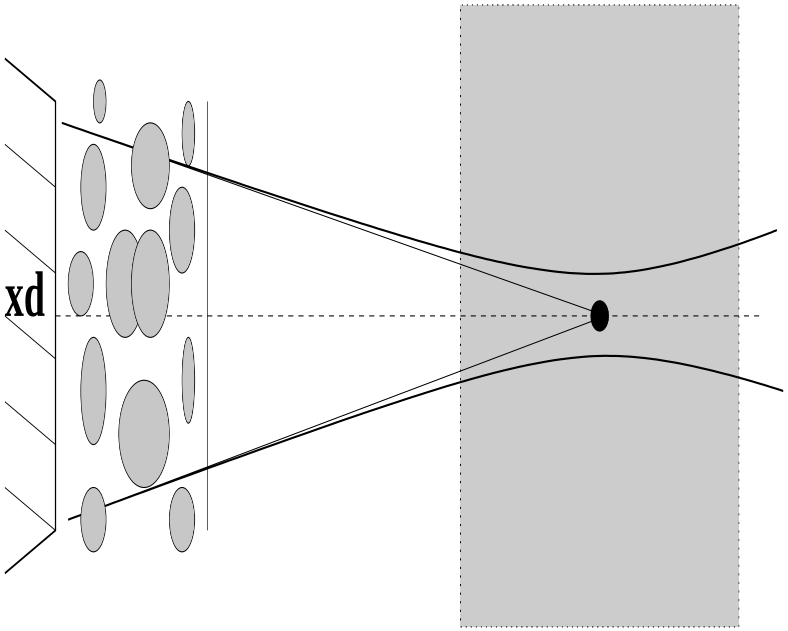


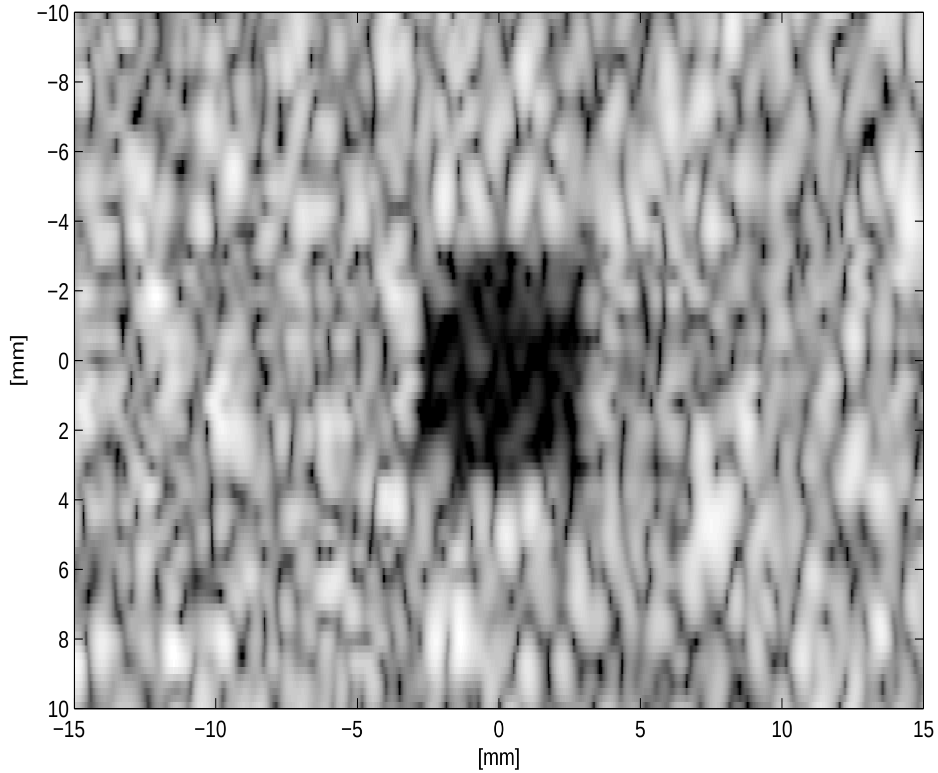


Standard imaging

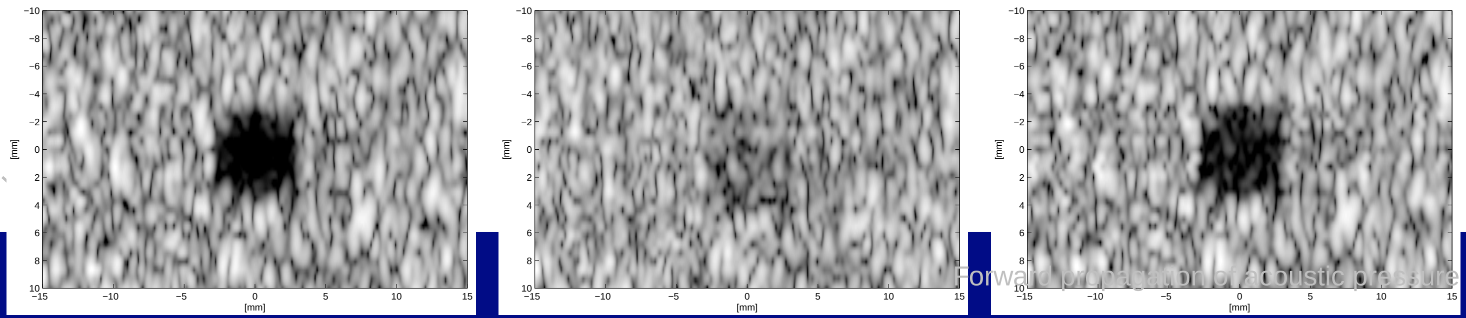


Simulated ultrasonic imaging





Corrected imaging



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