Under Attack!
# Science of Complex Systems

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In which kind of space networks live?
Jumping on trees

Coordination sequence, or “shell map”.

\[ V(t + 1) = \sum_{i \in \{t \text{ shell}\}} (k_i - 1) \]

\[ V(t + 1) = \sum_{k} (k - 1) \frac{k \ p(k)}{\langle k \rangle} \ V(t) \]

\[ V(t) = \left( \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right)^t \ V(0) \propto \exp(\lambda t) \]

with \( \lambda = \log \left( \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right) \)

Exponential growth \( \Rightarrow \) hyperbolic space!
Euclidean space:

\[ V(t) \propto t^{d-1} \]

Hyperbolic space:

\[ V(t) \propto \exp(\lambda t) \]
In hyperbolic spaces all volume is in the boundary

\[ V(t) = \text{# of vertices in shell } t \propto \exp(\lambda t) \]

\[ V_T(t) = \text{total # of vertices within 1 and } t \propto \exp(\lambda t) \]
Under attack: Network Resilience

Real-world networks are often highly resilient to attacks concerning random deletion of their vertices.

Resilience can be measured in different ways, but perhaps the simplest indicator of resilience in a network is the variation in the fraction of vertices in the largest component (giant component). We know [Lecture 8] that such quantity varies as:

$$\langle s \rangle = 1 + \frac{\langle k \rangle^2}{2\langle k \rangle - \langle k^2 \rangle}$$

A phase transition occurs when $\langle k^2 \rangle = 2\langle k \rangle$ and a ‘giant component’ first appears (or disappears).  [Lecture 8]
Under attack II - percolation

In a percolation process, vertices or edges are either “occupied” or “unoccupied” (present or non-present) and one asks how the resulting graph is connected.

One of the main original motivations for the percolation theory was the study of spreading of diseases. But in general it applies to a very vast class of phenomena involving transport through networks.

Among the applications there are the study of percolation of petroleum and natural gas through semi-porous rock (CO$_2$ sequestration being an “hot” contemporary topic).

Flow of electricity through complex conducting material is another typical example but examples are also found in economic and social systems as, for instance diffusion of innovation through firms.
One can study two kinds of percolation processes

1) **Site percolation**: where vertices are “occupied” or “unoccupied”;

2) **Bond percolation**: where edges are “occupied” or “unoccupied”;
**Under attack IV - site percolation**

The resilience to random failure (or removal) of vertices in a network is equivalent to a site percolation problem.

Let us take an existing graph (generated with configuration model) with degree distribution $p(k)$ and let us keep only a fraction $q$ of "occupied" vertices. We want to study the properties of the new graph made of the occupied vertices only.

Under attack V - site percolation

the degree distribution counting the number of edges from a given occupied vertex to other occupied vertices becomes

\[ \tilde{p}(k') = \sum_{k'=k}^{\infty} p(k) \binom{k}{k'} q^{k'} (1 - q)^{k-k'} \]

- Probability that a vertex has \( k \) incident edges.
- All the possible ways to pick \( k' \) vertices among the \( k \) neighboring vertices
- Probability of \( k-k' \) unoccupied vertices
- Probability of \( k' \) occupied vertices

\[ \langle k' \rangle = q \langle k \rangle \]
\[ \langle k'^2 \rangle = q^2 \langle k^2 \rangle + q(1 - q) \langle k \rangle \]
Largest component size: \( \langle s \rangle = 1 + \frac{\langle k \rangle^2}{2\langle k \rangle - \langle k^2 \rangle} \)

Largest component size: \( \langle s' \rangle = 1 + \frac{\langle k' \rangle^2}{2\langle k' \rangle - \langle k'^2 \rangle} \)

Percolation threshold: \( \langle s' \rangle \rightarrow \infty \quad \Rightarrow \quad \frac{\langle k'^2 \rangle}{\langle k' \rangle} = 2 \)
Under attack VII - site percolation: threshold

\[ \frac{\langle k'^2 \rangle}{\langle k' \rangle} = 2 \]

\[ q \langle k^2 \rangle = (1 + q) \langle k \rangle \]

For Poisson random graph we have \( \langle k^2 \rangle = \langle k \rangle (1+\langle k \rangle) \) and such transition is at \( q \langle k \rangle = 1 \).

For scale free graphs with \( P(k) = \begin{cases} 0 & \text{for } k = 0 \\ \frac{k^{-\alpha}}{\zeta(\alpha)} & \text{for } k > 0 \end{cases} \)

For large \( \alpha \), we have \( \langle k^n \rangle = \frac{\zeta\langle k \rangle^n}{\zeta(\alpha)} \) (\( \alpha > n+1 \)) and the transition is at

\[ q = \frac{\zeta(\alpha-1)}{\zeta(\alpha-2) - \zeta(\alpha-1)} \quad \text{for } \alpha > 3 \]
Under attack VIII - site percolation in scale free networks

For smaller $\alpha$ ($\alpha \leq 3$) we have that $\langle k^2 \rangle \to \infty$

therefore $q\langle k^2 \rangle = (1 + q)\langle k \rangle$ only when $q \to 0$

No percolation threshold!

Attempts to put internet down by randomly attacking routers will be unsuccessful unless one puts down 100% of the routers. Analogously epidemics will spread unless 100% of the population is immunized.

FIG. 1. Percolation transition for networks with power-law connectivity distribution. Plotted is the fraction of nodes that remain in the spanning cluster after breakdown of a fraction $p$ of all nodes, $P_s(p)/P_s(0)$, as a function of $p$, for $\alpha = 3.5$ (crosses) and $\alpha = 2.5$ (other symbols), as obtained from computer simulations of up to $N = 10^6$. In the former case, it can be seen that for $p > p_c \approx 0.5$ the spanning cluster disintegrates and the network becomes fragmented. However, for $\alpha = 2.5$ (the case of the Internet), the spanning cluster persists up to nearly 100% breakdown. The different curves for $K = 25$ (circles), 100 (squares), and 400 (triangles) illustrate the finite size effect: the transition exists only for finite networks, while the critical threshold $p_c$ approaches 100% as the networks grow in size.
Under attack IX - site percolation targeted attacks

On the contrary targeted attacks that puts selectively down the the most connected sites will be very successful.

The fraction of vertices that must be removed from a network to destroy the giant component, if the network has the form of a configuration model with a power-law degree distribution of exponent $\alpha$, and vertices are removed in decreasing order of their degrees.
Under attack X - site percolation: cascading failures

In some networks, such as electrical power networks, the operation of the network is such that the failure of one vertex or edge results in the redistribution of the load to other nearby vertices or edges. If vertices or edges fail when the load on them exceeds some maximum capacity, then this mechanism can result in a cascading failure or avalanche in which the redistribution of load pushes another vertex or edge over its threshold and causes it to fail, leading to further redistribution. (Examples: Northeast Blackout of 2003 USA-Canada, 2003 Italy blackout).

Model: a vertex fails if a fraction $\Psi_i$ of its neighbors have failed. The model starts from an initial (small) density of failures and takes the $\Psi_i$ as iid numbers drawn from a distribution $f(\Psi)$. The onset of the cascading process can be mapped onto the percolation problem.

The population is divided into three classes:
(S) susceptible, they don’t have the disease but can catch it if exposed to someone who does;
(I) infective, they have the disease and they can pass it on;
(R) recovered, they have recovered from the disease and have permanent immunity, so that they can never get it again or pass it on.

The model (and more complex variations) can be mapped onto bond percolation on a network with edge occupancy:

\[ T = 1 - \int_{0}^{\infty} p(\beta)Q(\gamma)e^{-\beta / \gamma} d\beta d\gamma \]

The percolation transition corresponds to the “epidemic threshold”. For scale free networks there is no threshold!

Scale free networks (from configuration model) are impossible to immunize and easy to destroy with targeted attacks. Likely, configuration models are not realistic models for real networks. For instance, one finds finite percolation thresholds for scale free graphs with negative degrees correlations. Furthermore, one gets non zero threshold by incorporating “geographical effects” (local community structures), low dimensional embeddings or high transitivity.
Under attack XIII - Vaccinations

The same properties that make the network extremely vulnerable can make vaccinations extremely effective.

If the virus spreading can be mapped onto a bond percolation problem the vaccination is a site percolation problem therefore one must study the combination of the two.

Vaccination of a few targeted vertices can stop the spreading of the disease.