The Pursuit of Perfect Packing. By Tomaso Aste and Denis Weaire. Institute of Physics Publishing, Bristol, U.K., 2000, xi + 136 pp., ISBN 0-7503-0648-3, $32.30.

Kepler's Conjecture. By George G. Szpiro. Wiley, Hoboken, NJ, 2003, viii + 296 pp., ISBN 0-471-08601-0, $24.95.

Reviewed by **Charles Radin**

In 1990, from news flashes on television and in newspapers, the whole world heard of W.-Y. Hsiang's announcement of a proof of the "sphere packing conjecture" on the most efficient way to pack equal spheres in space. This was followed by the equally well-publicized announcement of another proof by T. Hales in 1998. Definitive versions of their arguments have yet to be published, so I will not make any pronouncements on the status of their work—tempting as that might be!

Mathematical results do not often receive such publicity, so why all the attention to *this* problem? Two reasons come quickly to mind. The first is its long history: the conjecture can be plausibly traced back four hundred years to Kepler and is sometimes designated by his name. The other is that the conjecture can be stated in plain English without the need of esoteric terminology. Those two features are attractive to journalists but do not explain the interest of the mathematical research community in the problem.

To explain this it is useful to note a parallel with another old problem that drew much publicity when Wiles solved it in 1995: Fermat's conjecture (also called his "Last Theorem") on the unsolvability in integers of certain polynomial equations. Again the problem was old and could be stated in simple terms. The parallel is apt because in both cases it is not so much the explicit conjecture that defines the importance of the problem as the expected fallout from the attempt to solve it. A great deal of wonderful number theory has been inspired by Fermat's problem, and this is generally considered of much greater value than the validity of the conjecture. Likewise, merely having a proof of the most efficient way to pack spheres would definitely be worthwhile, but the real interest is in what we expect to learn from studying the broader circle of questions that the problem suggests.

The fertility of the Fermat problem was noted by Hilbert in 1900 in the introduction to his famous list of problems, where he used it to illustrate the quality he sought in choosing his problems. The sphere packing problem was number eighteen in that list. Paraphrasing the problem in appropriate generality, Hilbert asked for a classification of the symmetries of the densest packings, in Euclidean and hyperbolic spaces, of unit-radius spheres or unit-edge simplices. (Note the shift in emphasis from *the packings* that optimize something to *the symmetries* of those packings.) The densest packings for these shapes in $\mathbb{E}^2$, two-dimensional Euclidean space, are closely related through their symmetries, as one sees from Figures 1 and 2. The densest known packings for spheres in $\mathbb{E}^3$ are related to the optimal packing for $\mathbb{E}^2$: if we imagine Figure 1 indicating a planar layer of spheres as seen from above, a similar layer can be nestled in the valleys created by those spheres, as shown in Figure 3. Note that the spheres in the new layer lie in only half the valleys. The position of the second layer is not unique: there are two choices for it as well as for each subsequent layer. Kepler's conjecture is that no sphere packing fills space more densely than one made by any such infinite layering, which fills up the fraction $\pi/\sqrt{18} \approx 0.74$ of space. Kepler did not address the question of symmetry, which is central to Hilbert's version of the sphere packing problem.

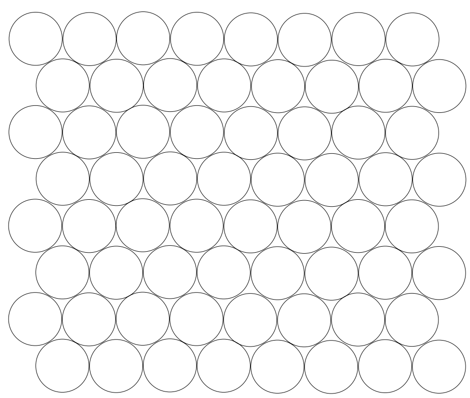

Figure 1. The optimal circle packing in the Euclidean plane.

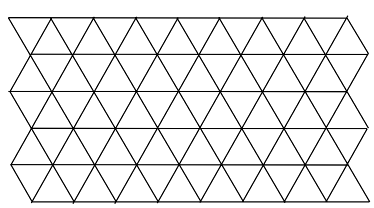

Figure 2. The triangular lattice.

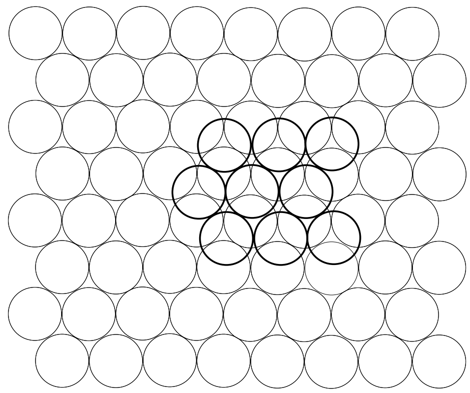

Figure 3. One layer of a sphere packing and the start of another layer.

We seem to be far from determining the densest packings of unit tetrahedra in $\mathbb{E}^3$, let alone the densest packings of spheres or simplices in the higher dimensional $\mathbb{E}^d$ and their hyperbolic analogues $\mathbb{H}^d$. And, as noted earlier, we are not just looking for tidy answers, but trying to learn about geometric symmetry. (In this sense we may have a solution of Kepler's conjecture but are far from a solution to Hilbert's eighteenth problem.) As noted by Hilbert, the crystallographic groups in $\mathbb{E}^3$ were classified in the nineteenth century, motivated in part by sphere packing and its connection with the possible atomic structure of solids. The crystallographic groups are a natural class of "large" subgroups of the group $\mathcal{G}(\mathbb{E}^d)$ of all isometries of $\mathbb{E}^d$, and Hilbert couched his eighteenth problem in terms of these groups presumably because he expected that densest packings would carry crystallographic symmetry. In response to Hilbert's eighteenth problem, Bieberbach gave a qualitative bound on the number of crystallographic groups for all $\mathbb{E}^d$: he showed there were only finitely many for each d. Of course, there is a great deal more to learn. For instance, contrary to Hilbert's assumption, it is quite possible that in $\mathbb{E}^d$ for large d the densest sphere packings do not have crystallographic symmetry at all. Such lack of crystallographic symmetry in the densest sphere packings has been demonstrated recently in $\mathbb{H}^d$ for all $d \geq 2$. So the sphere packing problem is taking us toward a deeper understanding of geometric symmetry in part through the study of other kinds of structure in the groups $\mathcal{G}(\mathbb{E}^d)$ and $\mathcal{G}(\mathbb{H}^d)$. (One example of this new direction is the analysis of the Penrose tilings of $\mathbb{E}^2$.)

We now turn to the two books under review, both published since Hales's announcement and both addressed to a general audience: *The Pursuit of Perfect Packing* by Aste and Weaire, and *Kepler's Conjec-*

ture by Szpiro.

The book by Aste and Weaire consists of fourteen chapters divided into seventy-nine sections. It is easy to break off reading between sections; that is, the book feels like a collection of individual articles held together by the common theme of packing of shapes in space. The authors are physicists, which perhaps explains the unusually wide range of topics: mathematics (Kepler's problem, Tammes's problem, ...), physics (random close packing, quasicrystals, ...), and biology (cell connectivity, honeycombs, ...). Strictly speaking, many of the topics are not conceptually part of the sphere packing problem: the shared theme is more generally that of geometric optimization of packings. The book is a treasure trove of intriguing examples, often accompanied by useful illustrations. A typical example is the following, drawn from the discussion on random close packing (p. 25):

> When granular material, such as sand or rice, is poured into a jar its density is relatively low and it flows rather like an ordinary fluid. A stick can be inserted into it and removed again easily. If the vessel, with the stick inside, is gently shaken the level of the sand decreases and the packing density increases. Eventually the stick can no longer be easily removed and when raised it will support the whole jar.

Another nice example is the introduction to Tammes's problem (p. 108):

> Many pollen grains are spheroidal and have exit points distributed on the surface. The pollen comes out from these points during fertilization. The position of the exit points is rather regular and the number of them varies from species to species. In 1930 the biologist Tammes described the number and the arrangement of the exit points in pollen grains of many species. He found that the preferred numbers are 4, 6, 8, 12, while 5 never appears. The numbers 7, 9 and 10 are quite rare and 11 is almost never found. He also found that distance between the exit points is approximately constant, and the number of these points is proportional to the surface of the sphere.
>
> Tammes posed the following question: *given a minimal distance between them, how many points can be put on the sphere?*"

Szpiro's book has a very different flavor; he entertains the reader with colorful history, whereas Aste and Weaire intrigue the reader with fascinating mathematical and physical phenomena. For many years now there have been books for the general audience that consist largely of biographical and anecdotal matter interspersed with technical expositions, sometimes directing readers with less interest in the technical sections to simply skip them "on first reading." This technique is employed here, with some of the technical material printed in a different typeface. The historical material is definitely fun to read. I particularly enjoyed the opening discussion on Rayleigh and Harriot and the later discussions on Dürer and Hales. These biographical sketches give life to the mathematics. I can recommend the book wholeheartedly to readers not interested in the technical mathematics for the colorful characterizations in the biographical material.

For those with a mathematical bent, however, the large number of mathematical errors in Szpiro's book is simply unacceptable. The errors that one easily notices, such as the claim that modern logic has shown that "Both a statement and its opposite may be true simultaneously" (p. 119) or the confusion between surface and area or between temperature and heat, are merely irritating. But there are many errors that might lead a nonexpert astray, such as "upon close—very close—inspection, it turns out that the fcc and the hcp are the exact same packings, viewed from different angles! This seems rather unbelievable at first" (p. 8). The fcc and hcp structures are obtained by specific choices in the layerings I described earlier, and the quoted assertion is unbelievable because it is not true. The physics, too, is often incorrect. For instance: "[a] universal principle of nature that says that *any physical system, left to itself, strives to achieve a state in which energy is at its lowest level* ... Water, snow, and ice are no exceptions to the principle of lowest energy, and the arrangement of the molecules will be selected accordingly" (p. 30). In fact, if a physical system is left to itself, its energy will not change in time; this is a basic feature of energy. And although a derivation from basic principles of the internal structure of ice would be of great interest, it is well beyond current physics, in contradiction to the simple discussion in the book. Unfortunately, the nature of layerings of spheres and the mechanisms governing material structure are fundamental to understanding sphere packing in three dimensions. Such errors in Szpiro's book could seriously mislead a reader interested in learning technical facts about sphere packing.

In conclusion, it is natural that books are appearing on the sphere packing conjecture that are aimed at a variety of audiences, from the general public to those with some mathematical or scientific background. (In fact, perhaps here is a good place to lament the fact that Fejes Tóth's *Regular Figures* [1] is out of print, as it is a true classic, though too dry for a general audience.) We are very fortunate to have available the book by Aste and Weaire, a real gem, which can be enjoyed by all audiences with at least some college experience.

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Reference

1. L. Fejes Tóth, *Regular Figures*, Macmillan, New York, 1964.

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