The 13th problem

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In the year 2000, exactly one hundred years after David Hilbert posed his now famous list of 23 open problems, The Clay Mathematics Institute (CMI) announced its seven Millennium Problems. (http://www.claymath.org/millennium). The Gazette has asked leading Australian mathematicians to put forth their own favourite 'Millennium Problem'. Due to the Gazette's limited budget, we are unfortunately not in a position to back these up with seven-figure prize monies, and have decided on the more modest 10 Australian dollars instead.
In this final instalment, Tomaso Aste and Tiziana Di Matteo will explain their favourite open problem that should have made it to the list.

In this note we describe our favourite problem in discrete geometry: *how many equal spheres can be packed inside a larger sphere?*

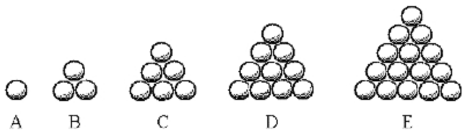
This problem is related with the long standing 'greengrocers dilemma': *which is the most space-efficient way of placing vegetables in a market stand?* Such a dilemma might have intrigued a few greengrocers (Fig. 1) but it has certainly attracted several mathematicians becoming one of the best-known problems in discrete geometry. This problem is often referred as the Kepler conjecture and it was included at the 18th place in the Hilbert's list.

Figure 1. The greengrocers dilemma: which is the most space-efficient way of placing vegetables on a market stand?

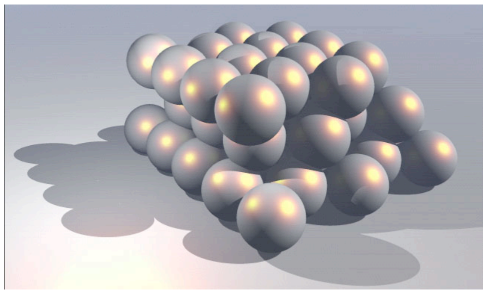
In the winter 1611, Kepler was fascinated by the origin of the hexagonal shapes of the snowflakes and he hypothesized that they could be originated by the tight packing of small spherical ice pieces. For this reason he investigated the *'most compact'* way to arrange *'equal pellets in any vessel'* [1]. Kepler conjectured that the closest packing is achieved by placing the pellets in stacks of hexagonal layers (Fig. 2). He wrote: *'This arrangement will be the tightest possible, so that in no other arrangement could more pellets be packed into the same container'* [1, 2].

There are several possible ways to place the hexagonal layers one upon the other (Barlow packings) but in all these packings the fraction of the volume occupied by the balls divided by the total volume is $\rho = \pi/\sqrt{18} \simeq 0.74048$.

The so called 'Kepler conjecture' affirms that this is the maximum density attainable. Such a conjecture resisted challenges for almost 400 years, but in 1998 T. Hales produced a proof [2, 3, 4]. Such a proof is still under examination (and verifying the Hales proof has become a challenge per se...), however it is widely believed in the scientific community that $\rho = 0.74048...$ is indeed the maximum density achievable for a packing of equal spheres in an infinitely large container with no boundary.



Equal spheres are packed tightly in the plane making hexagonal layers figure from the original Kepler's paper [1].



Hexagonal layers can be placed in stacks one upon the other forming the so called Barlow packings.

Figure 2. The densest packing of spheres in a infinitely large container with no boundary.

The Kepler problem might be solved for an *infinite* number of spheres in *infinitely* large containers but it rests still open for *finite arbitrary containers*. Indeed, the effect of the container on the packing density of a finite number of spheres can be dramatic. For instance, if we let the shape of the container be deformable (as for the case of a rubber balloon), the problem becomes the search for the configuration that minimizes the volume of the convex figure containing all the spheres. In this case, for small n (eventually up to 56) the shape that maximizes packing is a 'sausage' [5] (spheres placed in a straight line).

In general the *finite* sphere packing problem is addressing the maximization of the *local density*. But local density is intrinsically related to the way in which space is subdivided (i.e. the container's shape) and different ways of dividing space lead to different answers. Indeed, such differences between local and global packings are the very sources of all the difficulties encountered in the solution of the Kepler problem. *There exist several local configurations that are denser than the Barlow's packings*, however it appears that these configurations cannot be combined in space in a way to avoid large gaps. The result is that, for large enough systems, the densest packings tend to the density of the Barlow's arrangements.

Once clarified that the solution of the *finite* Kepler problem depends on the container, let us come back to our proposed problem. Our choice for the '13th Millennium Problem' has been to address the case for the simplest possible container: a spherical vessel. An equivalent formulation of our proposed problem is the quest for the smallest diameter $s(n)$ of a spherical container in which interior n spheres with diameter $d = 1$ can be packed. Simple solutions can be found for small n. For instance, it is straightforward to prove that $s(1) = 1$ and it is easy to verify that $s(2) = 2$. It is also established that no more than 13 spheres can be packed in a sphere of diameter $s = 3$ (this is related with the kissing number problem [6]). Very few results are known for larger n. Up to 32 spheres have so far been packed in a sphere of diameter $s = 4$ and up to 67 can be fitted into a sphere with $s = 5$ [7]. Other bounds for the cases up to $n = 40$ where published recently in [8]. On the other

hand, for very large n, the tightest packings must tend to the Barlow's one and the solutions should approach the bound: $s(n) > 2(n\sqrt{18}/\pi)^{1/3}$.

The study of the most effective packings is of great relevance in several physical systems. Atoms can be often conveniently modelled as hard spheres and the pursuit of densest packing is a feature associated with the metallic bond. There is therefore no surprise in discovering that several metals naturally have *fcc* crystalline structure, which is the one described by Kepler and it is also the favourite greengrocers' choice. However, when we look at local arrangements of small clusters of atoms very different scenarios arise. Indeed, it has been observed that atoms in nano-clusters or nano-wires result in arrangements that are different from the one in the larger bulk structures.

The *Gazette*'s prize of 10 Australian dollars might not sound too appealing to some of the readers. However, the solution of the proposed problem could yield to much greater rewards. Indeed, it turns out that a complete enumeration of the densest local packings for different number of spheres in different containers is of great interest in the cutting-edge field of nanotechnology.

References

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