SINGULARITY THEORY

Does the view from your window encompass a range of hills or mountains? If you are so blessed you will see immediately both of the persistent singularities that can be observed generically; they are indicated on the mountains sketched in Figure 1, a section of a remote but important range called the Transversal Alps. The points marked "country" are known as fold singularities. Actually the outline of the mountains, which in reality is a mapping of a surface contour onto the retina of the eye, is composed of an infinite number of fold singularities. The two points marked "western" are known as cusp singularities. They occur where a line of folds along the edge of a mountain meets another line of folds along the gully between the mountains.

If the view from your office is more urban in character—of faces in the street, say—you will probably see many more of these two fundamental singularities. You must search a lot harder to find other types of singularities, or create one purposefully, because all other singularities dissolve under the slightest perturbation into either a fold or a cusp. This remarkable fact was first proved by Whitney (1955) whose fundamental discoveries about singularities of differentiable mappings were developed into catastrophe theory and toolkits for treating bifurcation problems with parameters, by mathematicians such as Mather (in a series of very technical papers from 1968 to 1971), Thom (1972), Martinet (1982), Arnold et al. (1985), and Golubitsky & Schaeffer (1985).

Singularity theory is not secret mathematicians' business though, and a more apt name for the whole business would be "theory and applications of singularities." It is one of the more accessible entry points to both highly abstract areas of mathematics and to applied fields such as dynamical systems and bifurcations, because singularities can arise in almost any problem. The prerequisites are standard fare in first- and second-year maths courses: knowledge of Taylor's formula, the implicit function theorem, the theorem of existence and uniqueness, and some basic group theory, and willingness to learn some terminology.

Naturally, we should begin with a definition of singularity, this is (after Lu, 1976): Let f be a differentiable mapping from $\mathcal{M}$ to $\mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds. A point $x_0 \in \mathcal{M}$ is a singular point of f if rank $\mathrm{d}f(x_0) < \min\{\dim \mathcal{M}, \dim \mathcal{N}\}$, where $\mathrm{d}f(x_0)$ is the Jacobian matrix of f at x_0. Otherwise x_0 is a regular point of f.

Singularity theory solves three key related problems: (1) Given a mapping f it determines what types of singularities any good approximation $\bar{f}$ to f must have, (2) it tells us how can we perturb f slightly to obtain a nicer and simpler, but in some sense equivalent, mapping, and (3) it provides a taxonomy of singular

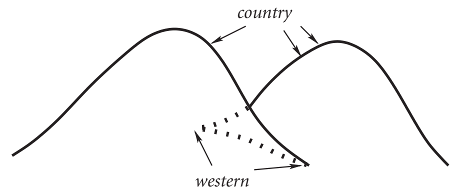

Figure 1. The Transversal Alps.

objects and a binary key to identify them, hence it is a *classification* science.

At the heart of singularity theory is a concept that is profound and yet somehow ingenuous, it is the concept of transversality. The naive (but not impercipient) version says that two curves intersect transversally if a small deformation of either one would not change the type of intersection. It is transversality that allows us to boil things down to classified normal forms.

Example 1: The mapping f from $\mathbb{R}^2$ into $\mathbb{R}^2$

$$u = x^2, \quad v = y \tag{1}$$

has a fold along $x = 0$ in the xy plane, the set of singular points is $\{(x, y) \in \mathbb{R}^2 \mid y = 2x\}$. Points along the fold remain fixed as y changes, that is, under perturbation. Equations (1) are called the *normal form* for a fold.

Example 2: The equations

$$u = xy - x^3, \quad v = y \tag{2}$$

define a mapping g from $\mathbb{R}^2$ into $\mathbb{R}^2$ for which the set of singular points is $\{(x, y) \in \mathbb{R}^2 \mid y = 3x^2\}$. The arms of the parabola are two lines of folds that come together and disappear at the cusp $(0, 0)$. Equations 2 are the normal form for the cusp and Whitney proved that any other mapping containing a regular point satisfying the conditions

$$\begin{aligned} &u_x = u_y = v_x = 0, \quad v_y = 1, u_{xx} = 0, \\ &u_{xy} \neq 0, u_{xxx} - 3u_{xy}v_{xx} \neq 0 \end{aligned} \tag{3}$$

can be transformed by coordinate changes into the normal form (2). In catastrophe theory this normal form becomes the *universal unfolding* $G(x, y, u)$ of the germ $g(x) = x^3$:

$$G(x, y, u) \equiv x^3 - yx + u, \tag{4}$$

where G is the gradient of a governing potential V.

One may understand the cusp by studying the surface $G(x, y, u) = 0$ shown in Figure 2. By taking slices of this surface at constant y we recover the three qualitatively different bifurcation diagrams, as shown.

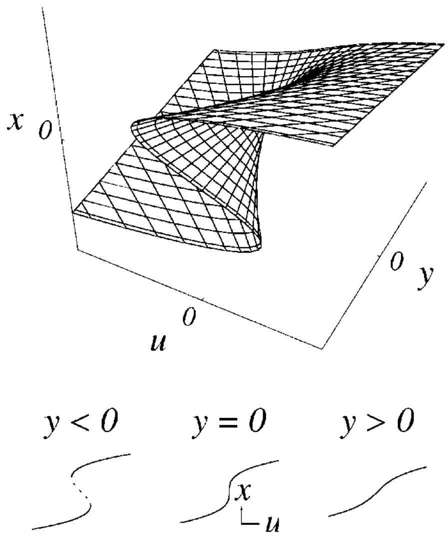

Figure 2. Slices through the cusp manifold (top) yield the three possible bifurcation diagrams (bottom).

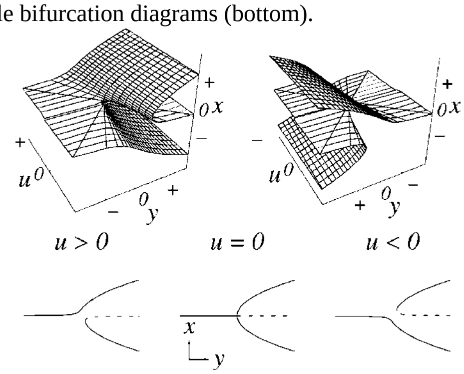

Figure 3. An orthogonal path through the cusp opens into a manifold around the pitchfork.

Now visualize a *projection* of this surface onto the (u, y) plane. We find that the two lines of folds meet at a cusp singularity, the very same that we saw in the Transversal Alps in Figure 1.

An instructive and fascinating lesson on the properties and classification of simple singularities is to to parameterize the surface $G(x, y, u) = 0$ differently. In a study of singular surfaces by Ball (2001) it was observed that, although the cusp is generic in the sense that all other singularities may be perturbed to either a fold or a cusp, the surface in Figure 2 is not a unique manifold of the cusp. Since all paths through the unfolding (4) are equally valid, we may choose a path in the (x, y) plane. Any such path unfurls laterally into the u-dimension to form a different surface. It is shown from two points of view in Figure 3. Constant-u slices show up the *pitchfork* singularity (center bifurcation diagram) and two of its perturbations. From this point of view Equation (4) is a *partial* unfolding of the pitchfork (but it is not a universal unfolding of the pitchfork).

In applications, the singularity theory approach has been most successful in qualitative studies of the equilibria of dynamical systems dependent on parameters. Given a dynamical system that can be reduced to a set of ordinary differential equations (by a procedure such as Lyapunov–Schmidt reduction), a general approach is to apply defining algebraic criteria systematically to the equilibria until one discovers the highest-order or most degenerate singularity, defined by its normal form. (The binary key given in Golubitsky & Schaeffer (1985, p. 201) is extremely useful for this task.) From a universal unfolding of this *organizing center* one can "read off" all of the possible qualitatively different bifurcation behavior of the equilibria. A great many dynamical models of physical systems have been given the singularity theory treatment, often yielding results having important implications for the prediction and control of such systems. For a few but varied examples see Ball (1999) (chemical reactions), Ball et al. (2002) (plasma physics), and Broer et al. (2003) (periodic dynamics).

ROWENA BALL

See also **Bifurcations; Catastrophe theory; Development of singularities**

Further Reading

Arnold, V.I., Gusein-Zade, S.M. & Varchenko, A.N. 1985. *Singularities of Differentiable Maps*, vol. 1. Boston and Basel: Birkhäuser

Ball, R. 1999. The origins and limits of thermal steady state multiplicity in the continuous stirred tank reactor. *Proceedings of the Royal Society of London Series* A, 455: 141–161

Ball, R. 2001. Understanding critical behaviour through visualisation: a walk around the pitchfork. *Computer Physics Communications*, 142: 71–75

Ball, R., Dewar, R.L. & Sugama, H. 2002. Metamorphosis of plasma shear flow–turbulence dynamics through a transcritical bifurcation. *Physical Review* E, 66: 066408-1–066408-9

Broer, H.W., Golubitsky, M. & Vegter, G. 2003. The geometry of resonance tongues: a singularity theory approach. *Nonlinearity*, 16: 1511–1538

Golubitsky, M. & Schaeffer, D.G. 1985. *Singularities and Groups in Bifurcation Theory*, vol. 1, New York: Springer

Lu, Y.-C. 1976. *Singularity Theory and an Introduction to Catastrophe Theory*, Berlin and New York: Springer

Martinet, J. 1982. *Singularities of Smooth Functions and Maps*, Cambridge and New York: Cambridge University Press

Thom, R. 1972. *Stabilité structurelle et morphogénése: Essai d' une théorie générale des modèles*, Reading, MA: W.A. Benjamin

Whitney, H. 1955. On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane. *Annals of Mathematics*, 62: 374–410

Manuscript Queries

Title: Encyclopedia of Non-linear Sciences
Alphabet S: Singularity theory

Page	Query Number	Query
No Query		