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Understanding critical behaviour through visualization: A walk around the pitchfork ☆

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Abstract

Computed 3-dimensional surfaces of critical points, or limit-point shells, are visualized for bifurcation problems that contain a pitchfork as an organizing centre. It is shown by comparison of notionally equivalent problems how the ranges of discontinuous behaviour in nonlinear dynamical models (and the physical systems they purport to represent) are determined by other singularities that shape this surface. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Discontinuous behaviour in dissipative dynamical systems is often ascribed to the propinquity of a pitchfork, a degenerate point that requires two auxiliary parameters for a universal unfolding and is therefore described as a codimension 2 singularity [1]. The persistent singularities within the pitchfork manifold — i.e., the limit-points — cover a surface in a space labeled by the principal bifurcation parameter and the two auxiliary parameters, called a limit-point shell [2]. In this work computed bifurcation surfaces are used to visualize the pitchfork [1] and its surroundings in parameter space.

Pitchforks are often reported in idealized models possessing $\mathbf{Z}_2$ or reflectional symmetry, e.g., [3–8]. In these so-called imperfect bifurcation problems, symmetry-breaking perturbations dissolve the pitchfork leaving persistent limit points. In other problems a non-symmetric, fully perturbed pitchfork is intrinsic, e.g., reacting chemical systems.

Bifurcation surfaces can illuminate dramatically the truism that what you see depends on where you view the object from. Originally the pitchfork was described from an orthogonal point of view as the generic cusp, and in Section 2 we work around to the prototypic pitchfork from the more familiar cusp manifold. Moving up a dimension, the limit-point shell L_p of the prototypic pitchfork is presented as a fundamental, generic object. In Section 3 the limit-point shells of several problems containing pitchforks are compared. Section 4 concludes with a brief discussion of what these results imply for experimental dynamical systems.

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[1] "Bifurcation" simply means "fork", so it seems unnecessary to refer to a pitchfork fork.

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2. From cusp manifold to limit-point shell

In the original exposition of singularity theory [9] conditions were derived for a regular point p of a smooth mapping f from $\mathbb{R}^2$ into $\mathbb{R}^2$ to be a cusp. In coordinates (u, v), (x, y) these are

$$u_x = u_y = v_x = 0, \quad v_y = 1,$$
$$u_{xx} = 0, \quad u_{xy} \neq 0, \quad u_{xxx} - 3u_{xy}v_{xx} \neq 0, \tag{1}$$

at p. Any mapping containing a point satisfying (1) can be transformed by coordinate changes into the normal form for the cusp, $u = xy - x^3$, $v = y$. In catastrophe theory [10] this normal form becomes the universal unfolding $G(x, u, v)$ of the germ $g(x) = x^3$:

$$G(x, u, v) = x^3 - vx + u, \tag{2}$$

where G is the gradient of a governing potential V. The familiar cusp surface, $G(x, u, v) = 0$ is shown in Fig. 1, along with the three qualitatively different constant-v slices or bifurcation diagrams.

Although the cusp is generic in the sense that all other singularities may be perturbed to either a fold or a cusp [11], the surface in Fig. 1 is not a unique manifold of the cusp. Since all paths through the unfolding (2) are equally valid, we may choose a path in the x, v plane. Any such path unfurls laterally into u to form a *different* surface, shown from two points of view in Fig. 2 along with the three qualitatively different constant-u bifurcation diagrams. Thus we arrive at the classical pitchfork. Returning to Eq. (1) we see that, from a different point of view, these conditions can also define a pitchfork.

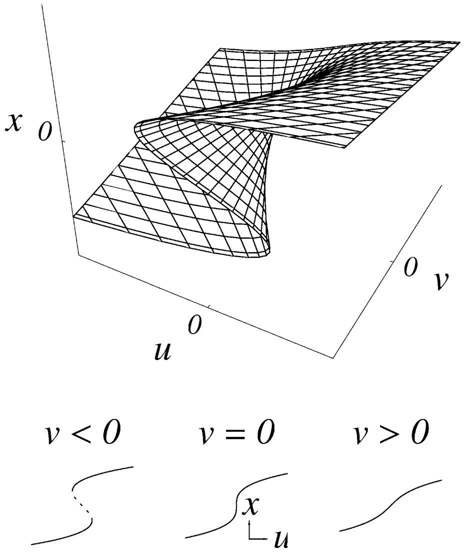

Fig. 1. The cusp catastrophe.

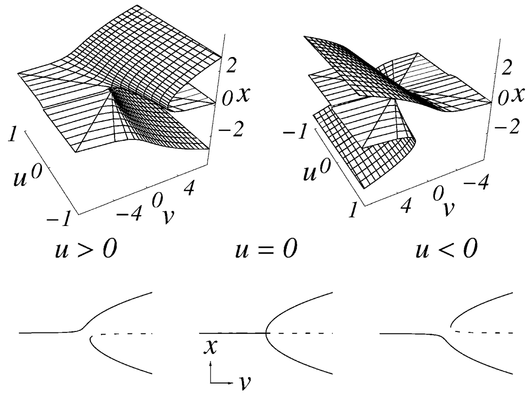

Fig. 2. An orthogonal path through the cusp unfolding opens into a manifold around the pitchfork.

However, the surface in Fig. 2 does not represent *all* qualitative information about the pitchfork. In [1] it was proved that a universal unfolding of the pitchfork must include a fourth variable. The contextual setting is that of autonomous dynamical systems dependent on parameters: $\frac{d\mathbf{x}}{dt} = G(\mathbf{x}, \lambda, \alpha_i) = 0$, where λ is the principal bifurcation parameter and the α_i are auxiliary or unfolding parameters. Assuming henceforth this context and notation, the pitchfork conditions P are given in Table 1. A bifurcation problem which sat-

Table 1
Conditions on a bifurcation problem $G(x, \lambda, \alpha_i)$ for the pitchfork P, hysteresis H, transcritical T, isola I, and asymmetric cusp A singularities. The primary singularity conditions $G = G_x = 0$ are assumed. d^2G is the Hessian matrix of second partial derivatives. Q_3 is a third-order directional derivative (see Ref. [1])

	P	H	T	I	A
G_λ	0	$\neq 0$	0	0	0
G_{xx}	0	0	$\neq 0$	$\neq 0$	$\neq 0$
$G_{x\lambda}$	$\neq 0$				
G_{xxx}	$\neq 0$	$\neq 0$			
$\det d^2G$			< 0	> 0	0
Q_3					$\neq 0$
codimension	2	1	1	1	2

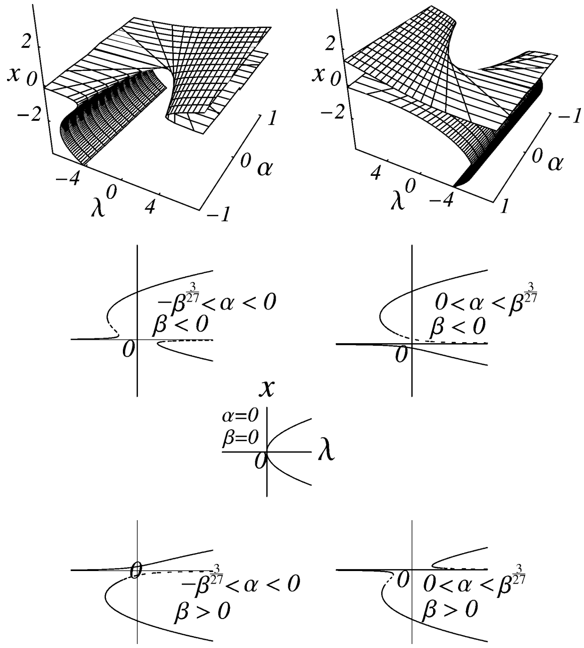

Fig. 3. Slices of this bifurcation manifold of P_p for $\beta > 0$ (shown from two angles) yield the two lower bifurcation diagrams. The two upper bifurcation diagrams would be obtained from a bifurcation manifold of P_p for $\beta < 0$ (not illustrated).

isfies P is said to be locally equivalent to the normal form $g(x, \lambda) = \pm x^3 \pm \lambda x$. The prototypic universal unfolding of the pitchfork, P_p, is given as

$$G(x, \lambda, \alpha, \beta) = x^3 - \lambda x + \alpha + \beta x^2. \qquad (P_p)$$

For $\beta \neq 0$ the bifurcation surface of P_p, shown from two angles in Fig. 3, does *not* include the pitchfork. If the fold lines or loci of limit-points are projected on the λ, α plane two codimension 1 singularities become evident: a hysteresis point H and a transcritical point T (defined in Table 1). Thus the pitchfork is the limiting degeneracy of H and T for $\beta = 0$. A unique manifold L_p around P_p is obtained by unfurling the *fold* lines in the λ, α plane into β. This forms a limit-point shell, the surface of limit points of an unfolding. L_p is shown from four vantage points in Fig. 4.

A slice of L_p at constant $\beta \neq 0$ is simply the lines of the folds or limit-points in Fig. 3. There is a distinct seam of hysteresis points that runs through the pitchfork at $(0, 0, 0)$. The line of transcritical points lies along $\lambda = 0$ in the λ, β plane. The shell is symmetric about two reflections.

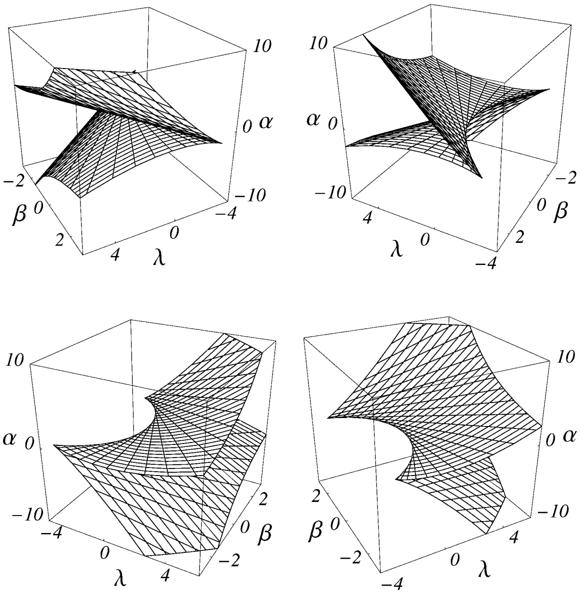

Fig. 4. Four views of L_p.

3. Applications

In dynamical models the limit-point shell outlines the boundary of steady-state multiplicity in parameter space. Therefore, its shape is a guide in the selection of design and operating criteria for an experimental system. However, L_p is a poor metaphor for many models which contain a pitchfork. A particular limit-point shell is shaped by other bifurcations and by the symmetry of the problem. These ideas are illustrated below by example.

3.1. The CSTR problem

The simplest CSTR (continuous stirred tank reactor) model emulates an exothermal chemical reaction in a well-stirred bounded medium. The associated bifurcation problem may be written as

$$G(u, f, \theta, \varepsilon, \ell) = \frac{f e^{-1/u}}{e^{-1/u} + f} + (\varepsilon f + \ell)(\theta - u), \qquad (P_c)$$

where the temperature u is the state variable. For $u \neq 1/2$ it can be shown [2] that the organizing centre is the non-symmetric pitchfork P_c. The limit point shell L_c at a fixed value of ℓ is shown in Fig. 5.

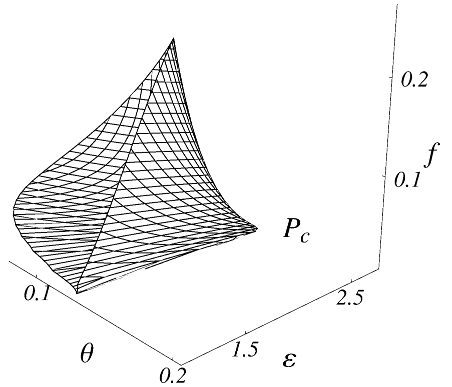

Fig. 5. L_c for $\ell = 0.05$.

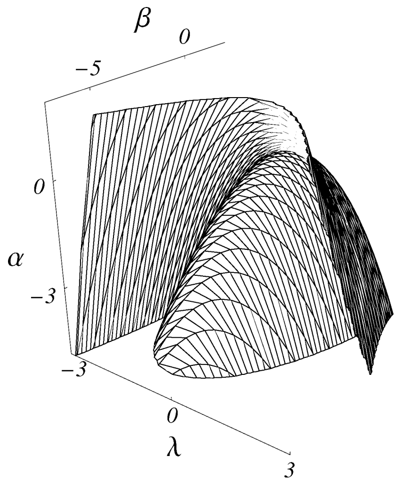

Fig. 6. A view of L_{TI}. P_{TI} occurs at $(0, 0, 0)$.

Contrasted with L_p (Fig. 4) the overall qualities of L_c appear to be *asymmetry* and *convexity*.

The role of co-existing bifurcations: From Table 1 we see that embedded in the pitchfork are the codimension 1 bifurcations H and T or I, or H and T and I. For the prototypic unfolding, P_p, we find that $G = G_x = G_\lambda = 0$ and $\det(d^2G) = -1$, $G_{xx} = -6$ at $x = \lambda = \alpha = 0$, thus T but not I is embedded in P_p. For the CSTR problem both T and I are embedded in P_c. In Table 1 conditions are given for another codimension 2 bifurcation, A, which also occurs in this problem. The form of L_c is strongly sculpted by its presence.

Is the normal form for the CSTR problem an adequate proxy? Singularity theory criteria [1] tell us that the CSTR problem is qualitatively equivalent to the simplest universal unfolding of a bifurcation problem containing P, H, T, I, and A. This is designated P_{TI}:

$$G(x, \lambda, \alpha, \beta) = x^3 + \lambda(\lambda - x) + \alpha + \beta x. \qquad (P_{TI})$$

Evaluation of T and I embedded in P_{TI} and A indicates that the bifurcation behaviour of the CSTR problem is encapsulated in the simpler problem P_{TI} — but the limit point shell L_{TI} of P_{TI}, shown in Fig. 6, tells a different story. All of the qualitatively distinct bifurcation diagrams of P_c can indeed be recovered from various slices of L_{TI} However, the differences between the two limit-point shells are quite striking.

3.2. A tale of three pitchforks

An important issue in the physics of magnetically confined plasmas is the spontaneous jump to an improved confinement régime known as the L–H transition. A dynamical model for this behaviour was analyzed in [8] and found to contain a partially unfolded pitchfork. The simplest universal unfolding is

$$G(u, q, d, \alpha) = \frac{(dq - u^2)(b + au^{5/2})}{u^{5/2}} + \frac{q(u^2 - dq)}{u^2} + \alpha, \qquad (P_{LH})$$

where the state variable u is the pressure gradient.

Unusually, P_{LH} contains *two* pitchforks in physical space. This introduces a formidable global aspect to what hitherto has been a purely local focus on the structure of the limit-point shell around a *unique* pitchfork. A bifurcation analysis of P_{LH} and construction of the limit-point shell is clearly not for the faint-hearted.

Another complication is the existence of a *third* pitchfork P_{3LH} in the unphysical region $q < 0$, $d < 0$. P_{3LH} is important because part of its limit-point manifold intrudes into the physical parameter region and is connected to the limit-point manifold of P_{1LH} by a seam of hysteresis points. This curved seam can be seen in Fig. 7, a fragment of the limit-point shell of P_{LH} around the connection. From a series of slices

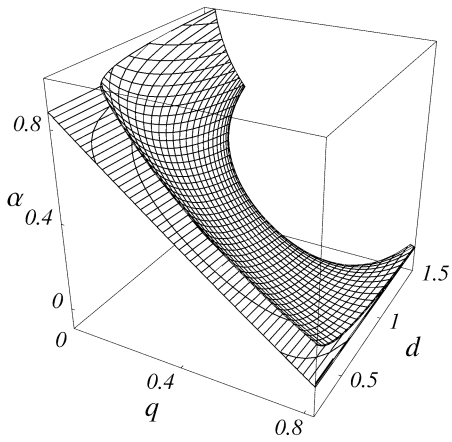

Fig. 7. Part of L_{LH} showing how P_{1LH} and P_{3LH} are connected.

in the q, α plane it can be seen that the connection occurs through point-to-point contact of two hysteresis points. The putative conditions for this singularity, designated tentatively as E_2, are

$$G = G_u = G_{uq} = G_{uu} = 0,$$
$$G_q > 0, \quad G_{uuu} < 0, \tag{3}$$

which yield a single exact E_2 point when applied to P_{LH}. The conditions (3) for the E_2 singularity seem pathological, but they can be understood by referring to Fig. 7 and Table 1.

4. Discussion and conclusion

The above analysis highlights some pitfalls in accepting qualitative equivalence of a bifurcation problem to a normal form as, in some sense, a "solution". In the case of the CSTR problem and its normal form, the boundaries of multiplicity are profoundly different. The analysis of the L–H problem also hints at the bizarre features that a limit-point shell can have while still remaining continuous. Although L_{LH} is locally equivalent to L_p around each of the three pitchforks, the global definition of L_{LH} involves at least one new singularity, the conjectured E_2.

This is largely an interpretive and exploratory work, investigating the environment of the pitchfork through visualizations of bifurcation manifolds of example problems. It turns out that bifurcation problems containing such a simple organizing centre can have rather complex boundaries of multiplicity. For this reason, and with advances in 3-dimensional computer visualization, the limit-point shell has potential as a design and control aid for experimental dynamical systems.

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