Renormalized Lie perturbation theory

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A Lie operator method for constructing action-angle transformations continuously connected to the identity is developed for area preserving mappings. By a simple change of variable from action to angular frequency, a perturbation expansion is obtained in which the small denominators have been renormalized. The method is shown to lead to the same series as the Lagrangian perturbation method of Greene and Percival, which converges on KAM surfaces. The method is not superconvergent but yields simple recursion relations which allow automatic algebraic manipulation techniques to be used to develop the series to high order. It is argued that the operator method can be justified by analytically continuing from the complex angular frequency plane onto the real line. The resulting picture is one where preserved primary KAM surfaces are continuously connected to one another.

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I. INTRODUCTION

The construction of action-angle transformations for Hamiltonian systems is an important goal since it represents an essentially complete solution of the dynamics. An integrable system is one where these transformations are defined almost everywhere in phase space, with the possible exception of the separatrices between regions where the topologies of the orbits are different (in general a set of zero measure).

The Kolmogorov--Arnold--Moser theorem guarantees that for a wide class of nonintegrable systems sufficiently close to integrability a privileged set of invariant surfaces exists (the KAM surfaces). This set is of finite measure, and on each KAM surface a form of action-angle variables is well defined. As we increase the perturbation parameter which takes us away from integrability, the isolating integrals (actions) cease to exist locally. Accordingly, KAM surfaces are broken when action is no longer well defined. When, in a given region of phase space, the last KAM surface with the same topology as the unperturbed surfaces is broken, connected stochasticity prevails throughout that region. The purpose of this paper is to examine methods that make use of the notion of action as an isolating integral. Our aim is to develop perturbative methods that will be practical tools for determining the transition to stochastic behavior, but with a more solid foundation than the empirical observations that lead to the “overlap-of-resonances” criterion. The transition to global stochasticity can be determined if we have a method for calculating the action-angle transformation and a rule for selecting the most robust surfaces.

In this paper we compare two perturbative approaches to the problem of constructing action-angle variables on the primary KAM surfaces. We limit ourselves to iterated two-dimensional area-preserving mappings, in particular to the Standard mapping of Chirikov and Taylor and a variant, the Semistandard mapping of Greene and Percival. These mappings can be regarded as return maps or surfaces of section of Hamiltonian flows. The perturbative approach can be readily extended to continuous flows and systems of dimensionality higher than two.

In Sec. II we set up the general problem of computing the action-angle transformation for primary KAM curves. The first method we examine is based on the assumption that there exists a global canonical transformation continuously connected to the identity, such that the primary KAM surfaces (KAM curves in 2D) are reduced to straight lines. Lie methods are used to construct a formal power series in the perturbation parameter, \( \epsilon \), which is justified by an argument based on analytic continuation in the complex action plane. In Sec. III we review the Lie transform formalism. In Sec. IV we generate a perturbation theory in the canonical action-angle variables using Lie methods. The theory is affected by the well known problem of small denominators, even on preserved KAM surfaces, and therefore can provide only an asymptotic series.

In Sec. V we review a different approach, the Lagrangian perturbation theory of Greene and Percival. The theory has the remarkable property of avoiding resonant denominators on primary KAM surfaces. In fact, this Lagrangian theory is convergent on a strip around the real axis of the angle variable, for small enough \( \epsilon \). We call such a theory a renormalized theory.

In Sec. VI we alter the theory of Sec. IV to avoid the resonant denominators through the use of a noncanonical set of variables. We thus construct a renormalized Lie perturbation theory which shares the advantages of the Lagrangian renormalized method while being of wider applicability than it. In Sec. VII we show that the perturbation method can be used to provide an alternative and more justifiable version of the overlap of resonances picture. In Sec. VIII we examine the convergence of the perturbation series, while in Sec. IX we use the Fourier convergence method of Greene and Percival to map out the region of convergence in the complex angular frequency plane.

II. ACTION-ANGLE TRANSFORMATIONS FOR PRIMARY KAM CURVES

Suppose we have an area preserving \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) (i.e., of a 2D real vector space onto itself). We denote the \( n \)th iterate
of the 2-vector \((\theta_n, I_n)\) by \((\theta_n, I_n)\). That is
\[
(\theta_n, I_n) = T(\theta_{n-1}, I_{n-1}) = T^n(\theta_{n-2}, I_{n-2}) = \ldots ,
\]
where \(f \circ g(x) = f(g(x))\) denotes the composition of any two functions \(f\) and \(g\).

We assume \(T\) to be analytic in both \(\theta\) and \(I\), so that it may be extended to a complex mapping \(T : \mathbb{C} \rightarrow \mathbb{C}^2\), where \(\mathbb{C}^2\) is a 2D complex vector space. By the area preserving property, the Jacobian of \(T\) is unity on \(\mathbb{R}^2\), and hence on \(\mathbb{C}^2\) by analytic continuation:
\[
\frac{\partial \theta_{n+1}}{\partial \theta_n} \frac{\partial I_{n+1}}{\partial I_n} - \frac{\partial \theta_{n+1}}{\partial I_n} \frac{\partial I_{n+1}}{\partial \theta_n} = 1 .
\]

Suppose further that \(T\) is the vector sum of an unperturbed part \(T_0\) and a perturbation \(\varepsilon T_1\), where \(\varepsilon\) is a continuously variable parameter. We assume that \((\theta, I)\) are action-angle variables for the unperturbed mapping, i.e.,
\[
T_0(\theta, I) = (\theta + \Theta_0(I), I).
\]
We also assume \(T_1\) to be \(2\pi\)-periodic in \(\theta\).

In this paper we shall be applying the formalisms to an especially simple class of mappings whose unperturbed angular frequency is given by
\[
\Omega(I) = I .
\]
The perturbed part of the mapping is assumed to be derivable from a potential, i.e.,
\[
T_1(\theta, I) = -V(\theta) \cdot (1, 1) .
\]
Chirikov\(^*\) gives physical motivations for discussing this class, and has termed the choice
\[
V(\theta) = -\cos \theta ,
\]
the Standard mapping (we have changed the sign to put the potential well at \(\theta = 0\)). This is the potential which describes, in a continuous time formulation, the motion of a particle in an electrostatic wave; in this case, the action-angle transformation can be called an oscillation-center transformation.\(^7\)

The slow transition to ergodic behavior that the KAM theorem implies occurs only for potentials whose Fourier expansion \(\Sigma_m \sim V_m \exp(i m \theta)\) decays rapidly (e.g. exponentially) as \(m \rightarrow \infty\). For less smooth potentials, KAM curves may not exist at all for nonzero \(\varepsilon\).\(^11\) The standard mapping has a finite Fourier series, and is therefore a good case to study.

An even simpler representative of this class is obtained by deleting the \(m = -1\) component of \(\cos \theta\):
\[
V(\theta) = -\frac{1}{2} \varepsilon \theta^2 .
\]
This has been called the Semistandard mapping by Greene and Percival,\(^5\) and has a meaning only within the context of mappings on \(\mathbb{C}^2\).

The mapping \(T = T_0 + \varepsilon T_1\) is said to be \textit{integrable} if phase space is covered by curves invariant under the application of \(T\). In this case the invariant curves can be labelled by a new action, \(J\), constant on each curve, and parametrized along their length by a new angle \(\Theta\), so that
\[
(\Theta, J) = C(\Theta, J) ,
\]
where \(C\) is a canonical (unit Jacobian) transformation \(2\pi\)-periodic in \(\Theta\). Since we are interested in constructing \(C\) by a perturbation expansion in \(\varepsilon\), we require \(\mathbb{C} : \mathbb{C}^2 \rightarrow \mathbb{C}^2\) to lie within the group of canonical transformations containing the identity. We also assume the inverse transformation to exist:
\[
(\Theta, J) = C^{-1}(\theta, I) ,
\]
and assume \(C\) and \(C^{-1}\) to be differentiable. That is, \(C\) is a \textit{diffeomorphism}. These restrictions on \(C\) mean that \(\theta\) and \(J\) will correspond to action-angle variables, as ordinarily understood, only when the invariant curves \(J = \text{const.}\) have the same topology as the unperturbed curves \(I = \text{const.}\). For example, in the case of the physical pendulum, with gravity regarded as the perturbation, \(C\) is an ordinary action-angle transformation only in the region of phase space corresponding to rotational motion.\(^1\) The interpretation of \(C\) for libratory motion will be discussed elsewhere.

In the new action-angle representation, the \(n\)th iterate of the point \((\Theta_n, J_n)\) is given by composing \(n\) times a mapping \(S\), which is similar to \(T_0\), but with a different angular frequency function \(\Omega(J)\). That is:
\[
(\Theta_n, J_n) = S(\Theta_{n-1}, J_{n-1}) ,
\]
\[
S(\Theta, J) = (\Theta + \Omega(I, \varepsilon), J) .
\]
From Eqs. (1), (4), and (6) we see that
\[
T = C \circ S \circ C^{-1} .
\]
The relation between \(T\) and \(S\) is represented by the following commutative diagram:
\[
\begin{array}{ccc}
(\theta_0) & \rightarrow & (\theta_1) & \rightarrow & \cdots & \rightarrow & (\theta_n) \\
I_0 & \rightarrow & I_1 & \rightarrow & \cdots & \rightarrow & I_n \\
\downarrow & & \downarrow & & \cdots & \downarrow & \downarrow \\
\downarrow & & \downarrow & & \cdots & \downarrow & \downarrow \\
(\Theta_0) & \rightarrow & (\Theta_1) & \rightarrow & \cdots & \rightarrow & (\Theta_n) \\
J_0 & \rightarrow & J_1 & \rightarrow & \cdots & \rightarrow & J_n \\
\end{array}
\]
In this diagram horizontal direction corresponds to the discrete "time" flow, while the vertical direction corresponds to a continuous flow in \(\varepsilon\).

The case when \(T\) is integrable for finite \(\varepsilon\) is highly exceptional; in general, no \(C\) exists such that \(S\) is everywhere of the form given by Eq. (11). However, the KAM theorem, for sufficiently small \(\varepsilon\) there remains an infinity of invariant curves characterized by Eq. (11) with \(J = \text{const.}\).\(^2\) Because \(C\) is a diffeomorphism, these invariant KAM surfaces are topologically equivalent to the straight line invariant curves of \(T_0\). We term these the \textit{primary} KAM curves to distinguish them from other invariant curves surrounding elliptic fixed points of \(T\).

In Sec. IV we construct a perturbation theory using \(\varepsilon\) as an expansion parameter by proceeding formally as if \(T\) were integrable. If \(T\) is restricted to \(\mathbb{R}^2\) this expansion can at best converge only for values of \(J\) on the primary KAM curves. Unfortunately, there is no way of telling \textit{a priori} which values of \(J\) to examine, since KAM curves are actually characterized by an irrational value of the winding number \(2\pi/\Omega(I, \varepsilon)\). We follow Greene\(^12\) in assuming that the most robust KAM surfaces, those which disrupt last as \(\varepsilon\) increases, are those possessing inverse winding numbers which we can call \textit{generalized golden means}. That is, the most robust surfaces
will be those with values of \( \Omega \) whose partial fraction expansion is terminated by an infinite number of ones, i.e.,
\[
\Omega = 2\pi \{ a, b, \ldots, y, z, 1, 1, 1, \ldots \},
\]
with \( a, b, \ldots, y, z \) integers. Since \( \Omega \) selects the most stable KAM surfaces, it would seem preferable to treat \( \Omega \), rather than \( J \), as the independent variable. Two such methods are discussed in Secs. V and VI.

From the above discussion it would appear that \( C \) is defined only on a highly pathological, perhaps not even dense, set of values of \( J \) or \( \Omega \) on which differentiation cannot be defined. This is true if \( \Omega \) is restricted to be real, but by continuing \( T \) and \( C \) into \( \mathbb{C} \), we shall find that \( C \) is analytic in a large connected region of the complex \( \Omega \) plane. By proceeding formally on the assumption that \( C \) exists we are implicitly working in \( \mathbb{C} \). The transformation is obtained on primary KAM curves by analytic continuation back to the real \( \Omega \) axis. The analytic continuation across the real \( \Omega \) axis is justified by the theory of monogenic functions.  

### III. LIE THEORY

There is a one to one correspondence between canonical transformations, such as \( T \) and \( C \) which map phase space onto itself, and certain unitary operators which map the space of functions defined on phase space onto itself. For instance, given the transformation \( A \), we define the corresponding unitary operator \( A \) by

\[
Af(\theta, J) = f(A(\theta, J)),
\]

where \( f \) is any function on phase space. \( Af(\theta, J) \) is known as the pullback of \( f \) under \( A \).

Note that \( A \) is a nonlinear, vector-valued function of a vector, whereas \( A \) is a linear, scalar operator defined on functions of \( \theta \) and \( J \). Given \( A \), we can conversely construct \( A \) since

\[
A(\theta, J) = (A(\theta), AJ).
\]

We denote this one-to-one correspondence thus: \( A \leftrightarrow A \). Consider now a function \( f(A \circ B(\theta, J)) = f(A(B(\theta, J))) \). According to Eq. (14) we get:

\[
f(A \circ B(\theta, J)) = (A f)(B(\theta, J)) = BA f(\theta, J),
\]

from where the composition rule for unitary operators is given by

\[
AB \leftrightarrow B \circ A.
\]

Similarly, \( ABC \leftrightarrow C \circ B \circ A \), and so on. In particular, the operator equation corresponding to Eq. (12) is

\[
T = C^{-1}SC.
\]

It is well known that a Hamiltonian flow generates a family of canonical transformations parametrized by the time variable. Analogously, by inventing a suitable generator \( W \) which plays the role of the Hamiltonian, we can obtain any canonical transformation that is continuously connected to the identity by "advancing" the coordinates in a new variable which plays the role of time (the Lie parameter, \( \epsilon \)). The canonical transformation \( C \) is thus forced to be continuously connected to the identity at \( \epsilon = 0 \), while \( T \) and \( S \) are assumed to be continuously connected to the unper-turbed mapping, i.e., \( T_0 = T(\epsilon = 0) = S(\epsilon = 0) \). Following Dewar we seek "Lie generating functions"

\[
U(\theta, Je), V(\theta, Je), \text{and } W(\theta, Je)
\]
such that the unitary operators \( S, T, \) and \( C \) obey the following operator equations and boundary conditions:

\[
\partial_\epsilon T = LVT, \quad T(\epsilon = 0) = T_0,
\]

\[
\partial_\epsilon S = LU S, \quad S(\epsilon = 0) = T_0,
\]

\[
\partial_\epsilon C = LWC, \quad C(\epsilon = 0) = 1,
\]

where \( \partial_\epsilon = \partial / \partial \epsilon \). \( L_w \) denotes the Lie derivative or Poisson bracket operator:

\[
L_w = [\partial J, W] \partial \theta - \{ \partial \theta, W \} \partial J,
\]

and \( T_0 \) is the operator \( \exp\{\partial J_0 \partial \theta \} \) Similarly, by Eq. (11)

\[
S = \exp\{\partial J \partial \theta \}
\]

Comparing Eqs. (19), (21) and (22) we see that \( U \equiv U(\theta, Je) \) such that

\[
\partial_\epsilon C^{-1} = -C^{-1}L_W C^{-1}.
\]

In similar fashion, we can determine \( V \), since \( T \) is associated with the prescribed mapping \( T \). For instance, in the special case of mappings defined by Eqs. (6) and (7), the generating function \( V(\theta) \) is readily verified to be the potential \( V(\theta) \). What we need now is to find equations for \( W \) and \( U \). We take the \( \epsilon \) derivative of Eq. (17), and using Eqs. (18)–(20) and the expression:

\[
\partial_\epsilon C^{-1} = -C^{-1}L_W C^{-1},
\]

we get

\[
-C^{-1}L_W S C^{-1}L_U S C^{-1}L_W C = L_V T,
\]

As Dewar shows, the identities

\[
C^{-1}L_C C = L_{C^{-1}}I, \quad SL_S S^{-1} = L_{S_F},
\]

are obeyed for any function \( f \). By inserting the identity operator in the forms \( CC^{-1} \) and \( -S^{-1}\) in the proper places in Eq. (25), Eqs. (26) can be used to show that

\[
-L_{C^{-1}W} T + L_{C^{-1}U} T + L_{C^{-1}W} T = L_V T,
\]

which can be satisfied by choosing the generating function \( W \) and an arbitrary constant in \( U \) so that

\[
\langle S - 1 \rangle W = CV - U.
\]

This is the analogue of the "Hamilton-Jacobi equation for the Lie generating function" derived previously for continuous Hamiltonian flows, with \( U \) playing the role of the "new Hamiltonian," \( K \). In order to determine \( U(\theta, Je) \), observe that

\[
\langle (S - 1)W \rangle = \langle S - 1 \rangle \langle W \rangle = 0,
\]

where the averaging operation \( \langle \cdot \rangle \) is defined for any \( f \) by

\[
\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta f.
\]

The average of Eq. (27) yields

\[
U = \langle CV \rangle.
\]

Eq. (28) ensures that \( W \) will be a generator for an action-angle transformation, but it is not still unique, since we have
not yet specified \( \langle W \rangle \). The simplest choice is to take
\[ \langle W \rangle = 0 \; \text{; instead, to facilitate comparison with the Lagrangian perturbation theory, we can require} \]
\[ \langle \mathcal{O} \mathcal{O} - \Theta \rangle = 0 . \]  
(29)

This is not necessarily equivalent to specifying \( \langle W \rangle = 0 \), but the
difference corresponds simply to a trivial action-dependent phase shift in the \( \Theta \) coordinate. In the case of the Standard and Semistandard mappings, Eq. (29) is equivalent to \( \langle W \rangle = 0 \).

IV. CANONICAL PERTURBATION THEORY

Adding \( (T_0 - S) \mathcal{W} \) to both sides of Eq. (27) we write it in the
form:
\[ (T_0 - 1) \mathcal{W} = CV + (T_0 - S) \mathcal{W} - U . \]  
(30)

Since \( (T_0 - 1) \) is independent of \( \epsilon \), Eq. (30) is in a form amenable
to solution by simple power-series expansion of \( \mathcal{W}(\Theta, J, \epsilon) \) which we term a "primitive" perturbation expansion.

In order to avoid working with operators, it is convenient to define auxiliary variables:
\[ \omega = CV , \]  
(31)
\[ \Theta = C \Theta , \]  
(32)
\[ \omega = (T_0 - S) \mathcal{W} . \]  
(33)

The variable \( \Theta (\Theta, J, \epsilon) \) is simply the old angle in terms of the
new phase space coordinates \( \Theta \) and \( J \). It is needed to implement Eq. (29). The auxiliary variable \( v(\Theta, J, \epsilon) \) is simply \( V(\Theta) \) in the case of the mappings defined by Eqs. (6) and (7).

We now expand \( \omega \) and \( U \) according to the convention:
\[ \mathcal{W} = \sum_{n = 0}^{\infty} \mathcal{W}_{n+1} \epsilon^n , \]  
(34)
\[ U = \sum_{n = 1}^{\infty} U_{n+1} \epsilon^n \]  
(35)

(we assume \( \langle V \rangle = 0 \) so that Eq. (28) implies \( U_1 = 0 \)). The convention for \( v \) is
\[ v = \sum_{n = 0}^{\infty} v_n \epsilon^n , \]  
(36)

and similarly for \( \Theta \) and \( \omega \). Differentiating Eqs. (31) and (32) with respect to \( \epsilon \), using Eq. (20), and equating coefficients of different powers of \( \epsilon \), we find the following simple nonlinear recursion relations for \( n \geq 1 \):
\[ v_n = \frac{1}{n} \sum_{m = 1}^{n} \{ v_{n-m}, W_m \} , \]  
(37)
\[ \Theta_n = \frac{1}{n} \sum_{m = 1}^{n} \{ \Theta_{n-m}, W_m \} , \]  
(38)

where \( [f, g] \) denotes the Poisson bracket \( L_f g \), and the recursion is initialized with
\[ v_0 = V(\Theta) , \]  
(39)
\[ \Theta_0 = \Theta . \]  
(40)

The recursion for \( w_n \) is much more complicated:
\[ w_n = -T_0 \sum_{k = 1}^{[n/2]} \sum_{l = 0}^{[n/2 - m]} \frac{C_{lk}}{k!} \partial_\Theta W_{n-2k+2l+1} , \]  
(41)

where
\[ C_{lk} = \frac{1}{l!} \sum_{\nu = 1}^{l} [\nu k - l + \nu] A_\nu C_{l-\nu} , \quad \text{for} \ l > 0 , \]
\[ C_{0,0} = A_0 \]
(42)
\[ A_\nu = [1/2(\nu + 1)] \partial_\Theta U_{2(\nu+1)} , \]
and \([f, g] \) denotes the integer part of \( f \). We have assumed that \( V \)
involves only odd Fourier components (as for the Standard and Semistandard map cases), so that \( U_{2n+1} = 0 \) for all integers \( n \). The coefficients \( C_{lk} \) are independent of \( \Theta \), so Eq. (41) does not couple Fourier components.

Denoting \( \langle f - \langle f \rangle \rangle \) by \( \tilde{f} \), we split Eq. (30) into its average and fluctuating parts, and equating powers of \( \epsilon \), we obtain two more recurrence relations:
\[ U_{n+1} = \langle v_n \rangle , \]  
(43)
\[ \tilde{W}_{n+1} = (T_0 - 1)^{-1} \{ \tilde{v}_n + \tilde{w}_n \} . \]  
(44)

By Eqs. (29) and (40) we require
\[ \langle \Theta_n \rangle = 0 \quad \text{for} \ n \geq 1 . \]

Equation (38) then gives a recursion relation for \( \langle W_{n+1} \rangle \):
\[ \partial_\Theta \langle W_{n+1} \rangle = -\sum_{m = 1}^{\infty} \{ \langle \Theta_{n+1}, m \rangle, \tilde{W}_m \} . \]  
(45)

Equations (37), (41)–(45) make up a complete set of recurrence relations for all unknowns. The inversion of \( T_0 - 1 \) in Eq. (44) is most easily accomplished in Fourier space:
\[ \tilde{W}_{n+1} = \{ \langle w_n + w_n^* \rangle / \exp[im\Omega_0(J)] - 1 \} , \]  
(46)

for \( m \neq 0 \). Here, and in the following sections, the Fourier representation \( f^m \) of any function \( f(\Theta) \) is defined by:
\[ f(\Theta) = \sum_{m = -\infty}^{\infty} f^m \exp im\Omega_0 . \]  
(47)

Clearly, Eq. (46) will be undefined for some value of \( m \)
whenever \( \Omega_0(J)/2\pi \) is a rational fraction. Since \( \Omega_0(J) \neq \Omega(J, J, \epsilon) \) in general, we can expect this primitive perturbation series to be divergent even if \( J \) is chosen so that \( \Omega \) is a "generalized golden mean". Also, the complicated nature of the recursion for \( w_n \) suggests that adding \( (T_0 - S) \mathcal{W} \) to Eq. (27) was not the best thing to do. In Sec. VI we present a method which allows \( (S - 1) \) to be inverted directly.

V. LAGRANGIAN PERTURBATION THEORY

The work of Greene and Percival² based on the averaged Lagrangian variational principle of Percival³ shows that there is indeed an alternative and simple perturbation method for primary KAM surfaces; this method is convergent (for \( \epsilon \) small enough) within a strip containing the real axis in the complex angle variable \( \Theta \) plane. Their starting point is the discretization of the Lagrangian differential equations of motion. The use of canonical variables and Poisson brackets is avoided, simplifying the recursion relations significantly. The method does not give rise in general to simple recursion relations. For this reason, we restrict ourselves in this section to an examination of the Standard and the Semistandard maps.

From Eqs. (1)–(5) and (8)–(11) we can show that:
\[ \delta^2 \Theta \equiv \Theta(\Theta + \Omega) - 2\Theta(\Theta) + \Theta(\Theta - \Omega) = -\epsilon V(\Theta) . \]  
(48)
Since $J$ and $\epsilon$ can be regarded as constants in Eq. (48), we have suppressed the last two arguments of $\theta \Omega, J, \epsilon$.

For a concrete example let us consider the Standard mapping, given by Eq. (6). Defining auxiliary variables:

$$f(\theta, \Omega, \epsilon) = -V'(\theta) = -\sin \theta,$$

and

$$v(\theta, \Omega, \epsilon) = V(\theta) = -\cos \theta,$$

we get

$$\partial_\theta f = -v \partial_\theta \theta, \quad \partial_\theta v = f \partial_\theta \theta.$$

(51)

The derivatives with respect to $\epsilon$ are taken at fixed $\Omega$, rather than at fixed $J$. We now assume that $\epsilon$ is a small parameter, and expand $f$, $v$, and $\theta$ as in Eq. (36). From Eqs. (48) and (51), by equating powers of $\epsilon$, we obtain simple nonlinear recursion relations [c.f. Eqs. (37)–(44)]:

$$n v_n = \sum_{-1}^{n} v f_{n+v},$$

$$n f_n = -\sum_{-1}^{n} v v_{n+v},$$

$$\delta^2 \theta_n = f_{n-1}.$$

(52)

Note that the existence of a finite set of quadratically nonlinear recursion relations is critically dependent on the specific form of $V$, in contrast to the canonical perturbation theory.

The second difference operator is inverted by Fourier expanding $\theta$, $v$, and $f$ in $\theta$, getting:

$$v_n = \frac{1}{n} \sum_{\mu = -\infty}^{\infty} v f_{n-\mu} \theta_{\mu},$$

$$f_n = -\frac{1}{n} \sum_{\mu = -\infty}^{\infty} v v_{n-\mu} \theta_{\mu},$$

$$\theta_n = -\frac{\exp i m \Omega}{2 \exp [i |m \Omega|] - 1} f_{n-1}.$$

(53)

Equations (53) furnish us with a closed recursion procedure, provided we start with finite Fourier series in $\theta$ for $f_0$ and $v_0$, because the $\mu$ summations truncate after a finite number of terms. Note that the recursion formulas are completely algebraic in nature, involving no differential operators. In particular, for the Standard mapping we start with:

$$f_0 = 1/2, \quad f_{-1} = -1/2,$$

$$v_0 = 1/2, \quad v_{-1} = 1/2,$$

(54)

and $f_m = v_m = 0$ for $m \neq 1, -1$. The Fourier–power coefficients form a triangular array, such that the $n$th power in $\epsilon$ contains Fourier modes ranging from $-n$ through $n$, and only those terms where $n$ has the same parity as $m$ are non-zero, i.e.,

$$\theta_0 = 0 \theta_1^{-1} \theta_2^0 \theta_3^{-1} \cdots \theta_{n-2}^{0} \theta_{n-1}^{-1} \theta_n.$$

(55)

A similar procedure can be followed for the Semistandard mapping. In this case, all harmonic coupling raises the Fourier index; the only terms that survive in the Fourier–power coefficient matrix are those on the left edge of the triangular array (55). Remarkably, for this case the Fourier series is the perturbation series. This allows for a very efficient recursion procedure.

VI. RENORMALIZED CANONICAL PERTURBATION THEORY

A simple renormalization of the resonant denominators in Eq. (46) can be effected by changing from the canonical set $\{\theta, J, \epsilon\}$ to the noncanonical set $\{\theta, \Omega, \epsilon\}$ of independent variables. The procedure amounts to a simple change of variables. Therefore $\partial_\theta$, and $\partial_\theta$ will henceforth imply that $\Omega$, rather than $J$, is to be held fixed. Where $\partial_\theta$ and $\partial_\theta$ occurred previously, they must be replaced according to the rule:

$$\partial_\theta J \rightarrow (\partial_\theta J)^{-1} \partial_\theta \Omega,$$

$$\partial_\theta \epsilon \rightarrow (\partial_\theta J)^{-1} (\partial_\theta J \partial_\theta \epsilon - \partial_\theta J \partial_\theta J).$$

(56)

(57)

We again need the auxiliary variables $v$ and $\theta$, but $w$ does not have to be used since $S = 1$ is now independent of $\epsilon$, and Eq. (27) is now in suitable form for recursively generating $W(\theta, \Omega, \epsilon)$, again assuming power series expansions of the forms in Eqs. (34) and (36).

An equation for $v$ is obtained by taking the $\epsilon$ derivative of Eq. (31) at constant $J$ and using Eqs. (20), (56) and (57):

$$[\partial_\theta J \partial_\theta \epsilon - (\partial_\theta J \partial_\theta J)] v = [v, W],$$

(58)

where $[,]$ denotes the modified Poisson bracket $[f, g] = (\partial_\theta f)(\partial_\theta g) - (\partial_\theta f)(\partial_\theta g).$

(59)

Similarly, $\theta$ obeys:

$$[\partial_\theta J \partial_\theta \epsilon - (\partial_\theta J \partial_\theta J)] \theta = [\theta, W].$$

(60)

An equation for $J$ is obtained by differentiating Eq. (23) with respect to $\epsilon$ at constant $J$ and using Eqs. (56) and (57):

$$J(\Omega, \epsilon) = J(\Omega, \epsilon) - \int_{\Omega}^{\epsilon} d\epsilon' \partial_\epsilon U(\Omega, \epsilon'),$$

(61)

where $J(\Omega, \epsilon) \Omega$ is the solution of the equation

$$\Omega_{J(\Omega)} = \Omega,$$

(62)

We insert Eq. (61) in Eq. (58) having expanded all functions as in Eqs. (34)–(36) and the additional expansions:

$$J = J(\Omega) + \sum_{n=1}^{\infty} J_n \epsilon^n,$$

$$\theta = \theta + \sum_{m=1}^{\infty} \theta_m \epsilon^m.$$

(63)

(64)

After expanding all functions in Fourier series over $\theta$ we obtain a recursive expression for the coefficients of the double expansion for $v$, given by Eqs. (36) and (47):

$$v_n = \frac{1}{n} \sum_{m=0}^{n-1} \left( \sum_{-m}^{m} (i-j) \left[ v_{n-\mu} \partial_\mu W_{m+1} + \partial_\mu v_{n-\mu} W_{m+1} \right] \right) - \partial_\mu v_n \mu U_m + \frac{n-m+1}{m} v_{n-m} \partial_\mu U_m.$$

(65)
The Fourier analysis in $\Theta$ permits the explicit evaluation of $(S - 1)W$ in the fluctuating part of Eq. (27), yielding

$$W_{n+1}^m = \begin{cases} \frac{v_n^m}{\exp(i m \Omega) - 1}, & m \neq 0, \\ 0, & m = 0 \text{ (by assumption)}, \end{cases}$$

Eqs. (65)–(66) constitute a closed set of recursion formulas, since $U$ is given by the average of $v$.

Similarly, starting from Eq. (60), we find that $\theta$ can be computed from

$$\theta_n^j = \frac{1}{m} \sum_{m=0}^{n-1} \left[ \sum_{m=0}^{n-1} m^j - \partial_m \theta_{n-m}^j W_{m+1}^j \right] \left( \partial_m \theta_{n-m}^j U_m + \frac{n-m+1}{m} \partial_m U_m \right).$$

The recursion formulas are fairly straightforward, and the resonant denominators now involve only $\Omega$ directly. This achieves the desired renormalization. In fact, Eq. (67) reproduces the results of the Lagrangian theory of Sec. V for the Standard and Semistandard mappings.

**VII. PERTURBATION SERIES AND A RESONANCE OVERLAP CRITERION**

We have constructed different perturbation theories which share certain characteristics: they are recursive, do not involve any approximations beyond those inherent in the perturbative formulation itself, and have relatively simple recursion formulas. Their practical value depends strongly on our ability to obtain high-order results. The recursion relations we have described so far are simple enough to permit this, in contrast to other possible avenues of attack, such as superconvergent expansions. 2

The primitive perturbation theory yields, at best, an asymptotic series. While reasonable results can often be obtained from such series, 17 a convergent series is clearly preferable (for the special case of the Semistandard mapping, the primitive perturbation theory is equivalent to the renormalized theory, since in this case $J = \Omega$ identically). The renormalized theory is simple and convergent on KAM surfaces for small enough values of $\epsilon$, in both its canonical and Lagrangian formulations. The formulas of the renormalized canonical theory are of wider applicability than those of the Lagrangian formulation, since the recursion relations are closed through the use of a generating function instead of the properties of the potentials used. We pay a price for this, however: the canonical theory is more difficult to use, because it involves differential operators in its recursion formulas, as opposed to purely algebraic recursive relations for the Lagrangian theory.

The recursive formulations of perturbation theories are ideally suited to computer implementation. We can obtain analytic results to relatively high order in $\epsilon$ through the use of an automatic algebraic manipulator. 18 We have used MACSYMA to obtain results to order $\epsilon^{10}$ before the complexity of the expressions involved made it impractical to proceed further. The analytic results provide us with significant insight into the resonance-resonance interaction mechanism. The first few orders in the primitive perturbation theory for $W$ and $U$ are given by

$$W = \epsilon \frac{\Xi}{2(\gamma - 1)} + \epsilon^2 \frac{\Xi^2}{4(\gamma - 1)(\gamma + 1)^2} + \epsilon^3 \left( \frac{\gamma(\gamma^2 + 1\gamma + 1)^2}{16(\gamma - 1)(\gamma + 1)^2} + \frac{\gamma^3}{16(\gamma - 1)^2(\gamma + 1)^2(\gamma^2 + 1\gamma + 1)} \right) + O(\epsilon^4),$$

$$U = \frac{\epsilon \gamma}{2(\gamma - 1)^2} \frac{\epsilon^2 \gamma}{8(\gamma - 1)^2(\gamma + 1)^2} + O(\epsilon^4),$$

where $\Xi = \exp(i \Theta)$ and $\gamma = \exp(i J)$. By using Eq. (23) we can attempt to determine the angular frequency for a given value of $J$, but the denominators in $U$ will produce undefined values at the resonances. A selective resummation of the primitive perturbation series could provide us with an alternative renormalization, 19 but that requires a much better picture of the structure of the series than that which we have been able to obtain analytically.

In contrast to the primitive theory, the renormalized perturbation theory in either its Lagrangian or canonical formulations uses $\Omega$ as the expansion variable. The resonant denominators in the renormalized theory will never be zero provided we choose an irrational value for $\Omega / 2\pi$. The perturbation theory will converge for small enough $\epsilon$ on a surface characterized by one such value. As $\epsilon$ is increased a point will be reached where the perturbation series will cease to converge. This point marks the destruction of the particular KAM surface corresponding to the chosen value of $\Omega$. It is sufficient to consider the perturbation series for $W$, since if $W$ diverges, so will all other quantities.

The perturbation theories can provide information about the local properties of the mapping close to a resonance. By expanding every term in the $\epsilon$ series for $W$ in partial fractions, we obtain a series whose general form is

$$W = \sum_{\mu} W_\mu,$$

with

$$W_\mu = \sum_{m} \sum_{n} \epsilon^{m(\mu)} \frac{a_{nmuv}}{\gamma^m (\gamma - \nu)^n},$$

where $\gamma = \exp(i \Omega)$, $\nu_{\text{max}}$ is an integer which depends on $n, m$ and $\mu$, $a_{nmuv}$ is some constant, and $Y_{\mu} = \exp(2i\pi \mu q/p)$ for $\mu = q/p$, where $q$ and $p$ are any two mutually prime integers such that $0 < \mu < 1$. $Y_{\mu}$ is termed a primitive $p$th root of unity. 20 We approximate $W_\mu$ by retaining only the most divergent terms, $\nu = \nu_{\text{max}}$, and get an approximation which is good near the $Y = Y_{\mu}$ resonance

$$W_\mu \approx \sum_{n} \frac{a_{nmuv}}{(\gamma - Y_{\mu})^{\nu_{\text{max}}}}.$$

To examine a simple case, we restrict ourselves to the Semistandard map, since in this case $a_{nmuv} = a_{n+\mu}, \delta_{v}$, we thus get:
TABLE I. Local approximations to the generating function $W$ for the Semistandard mapping at selected resonances. Notation corresponds to Eq. (72).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_\mu - \frac{1}{\nu_{\text{max}}}$</th>
<th>$Y_\mu = -\frac{1}{\nu_{\text{max}}}$</th>
<th>$Y_\mu = e^{2\nu_{\text{max}}/3}$</th>
<th>$Y_\mu = i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>144</td>
<td>1538</td>
</tr>
<tr>
<td>3</td>
<td>$rac{1}{8}$</td>
<td>$\frac{1}{32}$</td>
<td>2592</td>
<td>24576</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{128}$</td>
<td>$\frac{1}{2048}$</td>
<td>7776</td>
<td>49152</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{512}$</td>
<td>$\frac{1}{16384}$</td>
<td>41472</td>
<td>131072</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{1}{2048}$</td>
<td>$\frac{1}{131072}$</td>
<td>1942992</td>
<td>4718592</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{1}{8192}$</td>
<td>$\frac{1}{1048076}$</td>
<td>25</td>
<td>1143936</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{1}{32768}$</td>
<td>$\frac{1}{8388608}$</td>
<td>$-\frac{1}{25}$</td>
<td>50331648</td>
</tr>
</tbody>
</table>

\[ W_\mu \approx \sum_n \alpha_{\mu n} e^{\nu_{\text{max}} \exp(i\theta)} / (Y - Y_\mu)^{\nu_{\text{max}}}. \quad (72) \]

A table of $\alpha_{\mu n}$ and $\nu_{\text{max}}$ for the Semistandard map is shown in Table I, while Table II shows the values for the coefficients $\alpha_{\mu n_{\text{max}}}$ for a particular resonance, $\mu = 0$. A more refined approximation than Eq. (71) can be obtained by considering more terms in Eq. (70), since the order in $\nu_{\text{max}}$ at which the resonances at $Y_\mu$ first appears. This means that the size of the resonance at $Y = 1$ scales at $\sqrt{\nu_{\text{max}}}$, and the ordering at $\nu_{\text{max}}$ will be expected from direct mode-coupling between the primary resonances.

The partial fraction decomposition provides a more rigorous version of the overlapping resonances picture of the breakup of KAM surfaces than does Chirikov's criterion. Consider an irrational value of $\Omega / 2\pi$. Define $\rho_n(\Omega)$ as the "radius of divergence" associated with the $n$th convergent of $\Omega / 2\pi$, $\mu_n = q_n / p_n$ [Ref. 21] (rational truncations of the continued fraction expansion). The invariant curve associated with $\Omega$ will be preserved only if $\rho_n(\Omega) \to 0$ faster than $q_n / p_n \to \Omega / 2\pi$ as $n \to \infty$. As $\epsilon$ is increased, $\rho_n(\Omega)$ ceases to be a rapidly decreasing function of $n$ as $n \to \infty$, and there-

TABLE II. Coefficients for the partial fraction expansion of $W$ near the primary resonance $|Y_\mu| = 1$ for the Semistandard map. Notation corresponds to Eq. (70).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\nu_{\text{max}}$</th>
<th>$\alpha_{\nu_{\text{max}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{32}$</td>
<td>$\frac{1}{16}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{4608}$</td>
<td>$\frac{1}{384}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{165888}$</td>
<td>$\frac{1}{55296}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{9830400}$</td>
<td>$\frac{1}{481}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{21233664000}$</td>
<td>$\frac{1}{9927143}$</td>
</tr>
</tbody>
</table>

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fore the invariant curve will be destroyed when $\epsilon$ is greater than $\epsilon_{\text{crit}}(\Omega)$, the value at which $\rho_\lambda(\Omega)$ and $\Omega - 2\pi\mu_\lambda$ approach 0 at the same asymptotic rate. Even if an invariant curve is destroyed on the real $\Omega$ axis, "surfaces" will still be preserved for complex $\Omega$'s far enough away from the axis, to avoid being overlapped by any of the circles of divergence (see Fig. 1). Because $\rho_\lambda(\Omega)$ decreases rapidly with $n$, the region of convergence will have "tendrils", touching the real line at the preserved primary KAM curves, if any of them are left. An alternative method for extrapolating the behavior of $\rho_\lambda(\Omega)$ at infinitely high order has been suggested by Escande and Dovell, through the use of renormalization group techniques.

VIII. CONVERGENCE OF THE $\epsilon$ POWER SERIES

In either the Lagrangian or the canonical perturbation theory, the functions of interest are expressed as a double series: a power series in $\epsilon$ and a Fourier series in $\Theta$. Greene and Percival examine the convergence of the Fourier series to determine analyticity of their perturbation theory. This assumes that the power series expansion for each Fourier mode converges. We shall examine the convergence of the Fourier series in Sec. IX, but it is necessary to examine the $\epsilon$ power series for convergence first.

For the semistandard mapping the double summation collapses into a single sum. For this case the convergence study is reduced to a determination of the convergence of the Fourier series (see Sec. IX). Regrettably, this cannot be done for the Standard mapping, where the assumption of convergence of the $\epsilon$ power series must be justified.

Figure 2(a) shows a plot of the magnitude of the coefficients $\theta_{nm}$ of the Lagrangian theory [see Eqs. (53)] for $m = 1$, $\Omega/2\pi = (\sqrt{5} - 1)/2$ as a function of the $\epsilon$ power index $n$. The general features of this plot are obtained for other harmonics as well. The magnitude of the coefficients has a ragged behavior, which will be explained in Sec. IX. However, the general trend is for these coefficients to decay rapidly for small $n$, and then more and more slowly as $n$ increases. In this case, the first few terms of the $\epsilon$-series give a good approximation to each Fourier coefficient, even when the series is not absolutely convergent.

In contrast, Fig. 2(b) shows the same plot for a surface that breaks very early, $\Omega/2\pi = \pi - 3$. Now the coefficients grow exponentially. For a sufficiently small value of $\epsilon$, however, the exponential growth can be overcome. For this value of $\Omega$, we estimate that the power series for each Fourier mode can be majorized by a convergent geometric series if $\epsilon < 0.465 \pm 0.02$. For this particular value of $\Omega/2\pi$, it is difficult to insure that the majorization will persist beyond the point where the recursive calculation was stopped. This is due to the peculiar continued-fraction expansion that the number $\pi - 3$ possesses. In general, for nonquadratic irrational values of $\Omega/2\pi$, we can safely establish values of $\epsilon$ beyond which the power series is guaranteed to diverge, but it appears hazardous to claim that the power series would converge for $\epsilon$ less than a specific number. Note that this qualification is unnecessary for quadratic irrational values of $\Omega/2\pi$.

In the next section we will discuss the analyticity of the Fourier series. For this analysis, we will assume that the power series for each Fourier mode converges, at least up to $\epsilon_{\text{crit}}(\Omega)$, the critical value at which the Fourier series ceases to converge uniformly for real $\Theta$. We have examined the
perturbation theories for several different values of $\Omega / 2\pi$. For all quadratic irrational values of $\Omega / 2\pi$ examined the empirical evidence is that each $\epsilon$ series has a radius of convergence approximately equal to $\epsilon_{\text{crit}}(\Omega)$. The agreement is to within 2% or better with a computational horizon (i.e. the value of $n$ at which we stop our calculation) of $n = 80$. Thus to get very accurate values for the Fourier coefficients near $\epsilon = \epsilon_{\text{crit}}(\Omega)$ we would need a large number of terms in the $\epsilon$ series. The study of convergence properties of the power series defining one Fourier coefficient can be a more sensitive diagnostic of convergence than the study of the complete Fourier series for $\epsilon \approx \epsilon_{\text{crit}}(\Omega)$. Because of the finite computational horizon we may see no apparent divergence in the Fourier series up to this horizon while the power series convergence estimate indicates that the double series is indeed diverging. It becomes apparent that the analyses described herein are valid provided we choose a large enough horizon to be able to discern some self-similar behavior, thus justifying the extrapolations used. This can be examined by continuing the exact fraction decompositions of $\Omega / 2\pi$, which will also be done in the next section. Since nonquadratic irrational values of $\Omega / 2\pi$ do not exhibit self-similar behavior in their continued fraction expansions, one can safely infer only that $\epsilon < \epsilon_{\text{crit}}(\Omega)$ is a necessary but not sufficient condition for the convergence of the series.

The difficulties of principle in determining convergence do not appear when using superconvergent expansions. Indeed, for a rigorous proof of convergence one probably has to resort to these expansions at the expense of increased complexity of the perturbation series. For practical work a simple perturbation theory is sufficient to furnish reasonably high-precision quantitative results.

**IX. THE STRUCTURE OF THE FOURIER SERIES**

Assuming that the $\epsilon$ series for the Fourier coefficients converges, and summing over different orders in $\epsilon$, we can examine the existence of the transformation by studying the convergence of the Fourier series, without using the partial fraction decomposition. This is the approach used by Greene and Percival. Because of the superposition of all the different resonances, the Fourier coefficients have a very irregular dependence on $m$, the mode index, necessitating the calculation of the series to very high order. The Lagrangian formulation seems ideally suited to this task, because the algebraic nature of its recursion relations permits us to obtain extremely high order results $[O(e^{3000})$ for the Semistandard mapping] in a purely numerical fashion.

The coefficients of the Fourier series must decay exponentially if the series is to converge. We can estimate the decay rate of the Fourier coefficients by least-squares fits of exponentials or exponential-polynomial products to the values of the coefficients of the series for $\theta$ obtained from the perturbation theory. This yields the value for the critical $\epsilon$ needed to break the golden-mean surface $[\Omega / 2\pi = (\sqrt{5} - 1)/2] to within 1% of the value Greene obtained for the Standard map $[\epsilon_{\text{crit}} = 0.97]$, considering only the first 80 Fourier coefficients. Figure 3 shows a least squares fit of $am^b \exp m\gamma$ to the Fourier coefficients for the Standard mapping for the golden mean, where we adjust $\alpha, \beta$, and $\gamma$. We show two such fits, for different values of $\epsilon$; $\gamma$ changes sign for $\epsilon = 0.96 ...$, which marks the point where the Fourier series starts to diverge. Higher accuracy can be obtained simply by considering a larger number of Fourier coefficients.

The Fourier coefficients for both the Standard and the Semistandard maps show some very interesting structure for real $\Omega$. The structure is a direct consequence of the resonant denominators. If we construct the convergents $q_n/p_n$ of any irrational value of $\Omega$, we can see that the Fourier coefficients with mode index $m = p_n$ are the slowest decaying ones. Further subsequences can be picked by a simple algorithm. It is easy to see that the subsequence $m = p_n + p_{n-1}$ again decays slowly, though not as slow as the subsequence $m = p_n$. In general, coefficients with mode index corresponding to linear combinations of $p_n$'s with integer coefficients form slowly decaying subsequences, as can again be expected from mode beating arguments. Subsequences formed by modes where $m$ cannot be expressed as such one such linear combination will decay the fastest, and in general

**FIG. 3.** Fourier coefficients for the Standard Mapping obtained from the Lagrangian perturbation theory, summed up to order 78 in the perturbation theory. Least-squares fits of $am^b \exp m\gamma$ are shown for two different values of $\epsilon$, before $[\epsilon = 0.95$, dashed curve] and after breakage $[\epsilon = 0.95$, solid curve] of the KAM surface at the golden mean, $\Omega / 2\pi = (\sqrt{5} - 1)/2$.

**FIG. 4.** Smoothly decaying subsequences for the Standard Mapping. We show a case for $\Omega / 2\pi = 2/(3 + \sqrt{5}) = [0.2, 1, 1, 1, ...]$. Compare the positions of the peaks with those in Fig. 3, where $\Omega / 2\pi = [0, 1, 1, 1, ...]$. The dashed curves pass through two "slowly decaying" subsequences of Fourier coefficients.
three-mode combinations decay faster than two-mode combinations. It is sufficient to fit exponential-polynomial decay curves to those coefficients decaying the slowest to determine convergence or divergence of the whole series. Figure 4 shows the absolute value of the Fourier coefficients of $\theta$ for the Semistandard mapping. The exponential-power fit to each subsequence is now much better than in Fig. 3.

When we analytically continue the mappings into the complex $\Omega$ plane, the spiky structure of the Fourier coefficients is preserved close to the real axis. As we get farther from the real axis and the resonances therein, the magnitude of the Fourier coefficients begins to drop dramatically, again showing the convergence of the Fourier expansion in the same domain where the $\epsilon$ expansion converges. Figure 5 shows a comparison of the Fourier coefficients on and off the real axis.

By scanning over complex values of $\Omega$, we determine numerically the regions of divergence of the Fourier series for fixed values of $\epsilon$ by looking at the exponential decay of the Fourier coefficients. Figure 6 shows such a scan for the Semistandard mapping, for $\Omega$ in the vicinity of the golden mean. The “tendril” structure is again obtained. In the case of the Semistandard map this is not surprising, since the Fourier series is equivalent to the perturbation series in $\epsilon$; this is not true in the Standard mapping case, where the convergence studies can be carried out independently of each other.

X. SUMMARY

The renormalized perturbation theory, both in its Lagrangian and in its canonical formulation appears to be a much more rigorous tool to study nonintegrable mappings than many others hitherto used. The theory is convergent on KAM surfaces and is remarkably useful both for understanding the analytical structure of resonance-resonance interactions as well as for obtaining numerical estimates (such as critical values for the breakup of the KAM surfaces).

The canonical formulation has a simpler recursive structure in general than the Lagrangian theory, but it involves differential operators that limit its use. Even though these differential operators are normal to the KAM surfaces, their presence is not of great concern because they can be understood as derivatives taken in the complex $\Omega$ plane; since a “tendril” of analyticity reaches down to a preserved KAM surface, the differential operators can be thought of as the analytic continuation to the real line of well defined operators in the complex plane.

The process of analytic continuation allows us to understand the process of the breakup of KAM surfaces. Since we can compute the size of the divergent regions in the complex angular frequency plane, we can see clearly that KAM surfaces are destroyed by an overlapping process reminiscent of that of Chirikov. In contrast to the Chirikov picture, however, the resonances that ultimately destroy a given KAM surfaces are not the primary resonances; they are the high-order resonances lying nearby. The scaling of the widths of these resonances with $\epsilon$ seems to be consistent with a picture where the primary resonances beat directly against each other, i.e., where we neglect the interaction between resonances that have themselves been generated by beating. This fact is substantiated by a comparison between the divergent regions predicted by the Fourier series convergence studies and the analytical estimations of the radii of divergence of the perturbation series.

The values of the perturbation parameters necessary to break any given KAM surface can be obtained with arbitrary accuracy given enough computer time, using the theories described in this paper to compute the regions of divergence associated with high-order resonances. This is a significant achievement for a perturbation theory, since, as discussed above, the existence of KAM surfaces is determined by the interaction of resonances of arbitrarily high order.

A study of the libratory motion of the physical pendulum, to be reported elsewhere, suggests that secondary (is-
land-like) KAM surfaces correspond to certain complex values of $\Omega$. If this is true as well for nonintegrable systems, then the perturbative theories, coupled with the analytic continuation methods, may allow us to explore the nature of stochastic motion in a region where the quasilinear diffusion approximation fails.\textsuperscript{23}

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