Non-ideal stability: variational method for the determination of the outer-region matching data

By A. PLETZER AND R. L. DEWAR

Department of Theoretical Physics and Plasma Research Laboratory, Research School of Physical Sciences and Engineering, The Australian National University, G.P.O. Box 4, Canberra, A.C.T. 2601, Australia

(Received 11 January 1991)

Within the framework of studies of the stability of magneto-plasmas to non-ideal modes, such as resistive modes, the problem of determining the asymptotic matching data arising from the outer (ideal) region is considered. Modes possessing both tearing and interchange (ballooning) parity are considered in finite-pressure plasmas. The matching data, which form a matrix whose elements represent the small solution response to forcing by a big solution, are shown to derive from a variational (energy) principle. The variational principle, as presented, applies to both cylindrical and two-dimensional (toroidal) geometries. Allowing for the presence of multiple rational surfaces, a reciprocity relation between off-diagonal elements of the matching data matrix is obtained. The variational principle is suitable for numerical approximation, and, in particular, for the finite-element method, for which convergence rates are estimated. By packing nodes near the rational surface, maximum convergence, proportional to the inverse square of the number of mesh nodes for tent functions, is achieved.

1. Introduction

We are interested in calculating the stability of magnetically confined plasmas to non-ideal modes such as resistive modes, which are thought to play an important role in observed disruptions and sawtooth oscillations. We consider here weakly non-ideal plasmas, so that resistivity, viscosity and other non-ideal effects can be neglected in the outer region, that is, everywhere except in thin layers around the surfaces where the safety factor is rational.

The thin-layer assumption makes the use of an asymptotic matching method (see e.g. Kevorkian & Cole 1981) appropriate. The ideal magnetohydrodynamics (MHD) equations (Dewar & Pletzer 1990) are solved in the outer region, and non-ideal equations, such as the linear resistive MHD equations (Coppi, Greene & Johnson 1966) or possibly even weakly nonlinear equations, are assumed to be solved in the inner layer, each of the domains giving rise to asymptotic solutions to be matched in their regions of overlap. This paper focuses on the matching data arising from the outer region. In order to allow maximum flexibility in the choice of physics in the inner region, a general formulation making no assumptions as to the symmetry of the inner layer will be developed. (For example, Dobrott, Prager & Taylor (1977) showed that
inclusion of equilibrium flow breaks the symmetry of the inner layer to lowest order.

In the outer region, the equations reduce to the vanishing of the force operator applied to the displacement field. The outer-region equation possesses regular singularities at the rational surfaces, with two independent solutions, called ‘big’ (dominant) and ‘small’ (subdominant), exhibiting different fractional power-like asymptotic behaviours. In a cylindrical geometry, the solutions to the left and right of a rational surface decouple, so that the matching data can be expressed only in terms of $\Delta_L$ and $\Delta_R$, the ratios of the leading Frobenius coefficients of the small to those of big solution (Glasser, Jardin & Tesauro 1984), on the left- and right-hand sides respectively. As there are two distinct modes in the finite-pressure-gradient case (Coppi et al. 1966), the tearing and interchange modes, which have definite (and opposite) parities in the inner region for a symmetric inner layer, we also formulate the outer matching data in terms of the sum $\Delta' \equiv \Delta_R + \Delta_L$ and difference $\Gamma' \equiv \Delta_R - \Delta_L$ parameters. In the two-dimensional case, however, $\Delta_L$ and $\Delta_R$ do not constitute a complete set of matching data. Toroidal coupling introduces a regular solution at the rational surface, which destroys the disjointness of the left- and right-sided solutions, leading to, at first sight, four independent matching data $\Delta'$, $\Gamma'$, $\Delta'$ and $B'$, corresponding to all combinations of odd and even parities of the small and big solutions. We shall, however, find that there remains a symmetry relation reducing the number of truly independent parameters to three.

Furth, Rutherford & Selberg (1973) gave a variational principle for the jump $\Delta'$ in the logarithmic derivative of the perturbed radial magnetic field, which determines the stability of tearing modes in a pressureless ($\beta = 0$) plasmas, but a similar variational principle has not previously been available for the finite-$\beta$ case. There are several reasons why variational expressions are desirable – numerical robustness, ease in theoretically demonstrating symmetries, and potential for physical interpretation of stability conditions in terms of energy considerations. In this paper, we present a new variational formulation for calculating the outer region matching data, a slightly modified version of the 'generalized Green function' algorithm first developed by Miller & Dewar (1986), which has been applied to resistive stability calculations of cylindrical plasmas by Pletzer & Dewar (1990). A similar formulation has recently been applied to a model equation by Chu et al. (1990). We show that this method is able to solve the accuracy and convergence problems due to the singular behaviour of the small solution in the finite-$\beta$ case, as encountered by Manickam, Grimm & Dewar (1983). It also allows easy demonstration of symmetry (reciprocity) relations between matching coefficients of opposite parity, and those relating different rational surfaces. The method applies to two-dimensional geometry, but for ease of exposition we develop it for a one-dimensional cylindrical geometry in the body of the paper. By assuming multiple rational surfaces and the possibility of hollow pressure profiles, most of the problems of the general two-dimensional case can be addressed in a cylindrical geometry. The extension to two-dimensional geometries is indicated in the appendix.

In §2 we review the Frobenius expansion of the linearized displacement in the ideal marginal case, and in §3 we show how this is related to the matching data.
The calculation of the matching data is formulated as a linear response in §4, the big solution acting as a forcing term, with coefficients of the small solutions forming the response. The variational formulation is developed in §5. After separating out the big solution around the rational surface, the matching data are shown in §5 to derive from an equation reminiscent of Green's theorem, involving the concomitant of a big forcing term and the small-solution response. The matching data appear as symmetrized matrix elements, stationary under variation of the small component. This leads to a reciprocity relation expounded in §7, particularly useful in the situation where several rational surfaces are present in the plasma. Uniqueness and related questions are discussed in §6.

In §8, we develop the finite-element method as a numerical method based on the variational formulation. Our numerical error analysis shows that a convergence of the matching data proportional to the square of the inverse number of mesh nodes is guaranteed for 'tent functions', provided the mesh is appropriately packed around the rational surface. In §9, numerical tests validating the method are conducted for the simple case of a plasma in a cylindrical geometry with a single rational surface present.

2. Frobenius expansion

In the outer region, where resistivity can be neglected, the ideal MHD equations are valid. With respect to the ideal Alfven time scale $\tau_A$, resistive modes appear quasi-static – the characteristic time scale of resistive modes being $\tau_R \approx \nu \tau_A$, $\nu \approx 10^7 \gg 1$ – and thus satisfy the marginal stability equation

$$L \xi(\psi) = 0,$$

with vanishing right-hand side, known as the Newcomb equation (Newcomb 1960). The boundary condition at the edge of the plasma, $\psi = \psi_a$,

$$\frac{d\xi(\psi_a)}{d\psi} = -\sigma_a \xi(\psi_a),$$

where $\sigma_a$ is a constant, is the most general boundary condition compatible with the Hermiticity of $L$ (see (27)). In the case of an infinitely conducting wall at $\psi_a$, we take $|\sigma_a| \rightarrow \infty$, so that $\xi$ satisfies $\xi(\psi_a) = 0$. However, (2) can also represent the boundary condition at the interface of plasma and vacuum. For a good exposition of the Newcomb formulation, including the vacuum contribution, see Freidberg (1987).

In (1), $L$ represents the component of the linearized force operator normal to the flux surfaces $\psi$, which reduces (Newcomb 1960; Dewar & Pletzer 1990) in a cylindrical geometry to a Sturm–Liouville operator

$$L = \frac{d}{d\psi} f(\psi) \frac{d}{d\psi} - g(\psi),$$

where $d/d\psi$ acts on everything to its right. In (1) and (2), $\xi(\psi)$ is a single Fourier mode amplitude such that

$$\xi(\psi) \exp \left[i (m\theta - n\zeta)\right] = \xi(\psi, \theta, \zeta) \cdot \nabla \psi,$$

where $\xi$ is the plasma displacement.
We follow the notation used in Dewar & Pletzer (1990), where the magnetic field

\[ \mathbf{B} = [\nabla \zeta - q(\psi) \nabla \theta] \times \nabla \psi \]

is expressed in the straight-field-line system of co-ordinates \((\psi, \theta, \zeta)\), \(2\pi\psi\) being the poloidal flux, \(\theta\) the poloidal angle and \(\zeta\) the angle that increases by \(2\pi\) along the axial periodicity length \(2\pi h^{-1}\). In (3), \(f(\psi)\) and \(g(\psi)\) are given by

\[ f(\psi) = \mathcal{G} \left[ m - nq(\psi) \right]^2 \mathcal{G}, \]

\[ g(\psi) = -2\mathcal{N} \mathcal{F} + \frac{d}{d\psi}(i[m - nq(\psi)]2\mathcal{G}). \]

It was shown in Dewar & Pletzer (1990) that these agree with the \(f\) and \(g\) functions of Newcomb (1960) when our use of \(\psi\) rather than radius \(r\) as independent variable, and \(\xi\) rather than the radial component of \(\xi, \xi_r = \xi/|\nabla \psi|\), is taken into account.

It is not necessary for our purpose to enter into the details of the expressions for \(\mathcal{G}, \mathcal{F}\) and \(\mathcal{N}\), which have been defined by Dewar & Pletzer (1990). Let us just point out that \(\mathcal{G}\) vanishes on the magnetic axis for modes \(m \neq 0\), so that

\[ f(\psi) \sim \mathcal{G} \left[ m - nq(\psi) \right]^2 \frac{2h\psi}{m^2 + 4h^2n^2\psi/J(0)} \]

as \(\psi \to 0\). Also, for \(m \neq 0\),

\[ g(\psi) \sim \mathcal{G} \left[ m - nq(\psi) \right]^2 \frac{h}{2\psi} \]

as \(\psi \to 0\), while, for \(m = 0\),

\[ g(\psi) = O(\psi^0) \]

as \(\psi \to 0\), leading to the asymptotic behaviour of \(\xi \sim \psi^{\pm 1}\) for \(m \neq 0\) and \(\xi\) regular for \(m = 0\). Regularity of \(\xi\) selects only the solutions \(\sim \psi^{\pm 1}\) \((m \neq 0)\) and \(\sim \psi\) \((m = 0)\) as physically admissible. In (8), \(J_0(0)\) is the axial component of the current density on the magnetic axis \(\psi = 0\).

There is a second class of regular singularities, those at the rational surfaces \(\psi_i\), for which the safety factor \(q(\psi_i)\) is a rational number, \(m/n\). In §4 we shall assume the existence of \(N (i = 1, 2, \ldots, N)\) such surfaces, but for the moment we focus on a single \(\psi_i\). To investigate the behaviour of \(\xi\) near the singularity, we expand

\[ f(\psi) = \sum_{k=0}^{\infty} f_k(\psi - \psi_i)^{k+1}, \]

\[ g(\psi) = \sum_{k=0}^{\infty} g_k(\psi - \psi_i)^{k}, \]

so that \(f \sim f_0(\psi - \psi_i)^2\) and \(g \sim g_0\) as \(\psi \to \psi_i\), in agreement with (6) and (7). There is an important special case, \(g_0 = 0\) and \(g_1 \neq 0\), due to a zero pressure gradient at \(\psi_i\) (Dewar & Pletzer 1990), which we shall consider further in the following, though we are mainly concerned with the general, finite-pressure-gradient case. We restrict ourselves to cases where \(q'(\psi_i) \neq 0\), so that \(f_0\) does not vanish.
Ignoring boundary conditions for the moment, we seek solutions of (1) by assuming that the solutions can be expanded in Frobenius series around $\psi_1$,

$$\xi^{(a)} (\psi) = (\psi - \psi_1)^z \sum_{k=0}^{\infty} a_k (\psi - \psi_1)^k,$$

with $a$ a real number. Because $a$ is not in general an integer, the function $x^a$ is not uniquely defined. In this paper we shall take it to mean one of four functions, a 'left-sided' function $x^a_L$, equal to $|x|^a$ on $x < 0$ and zero elsewhere, a 'right-sided' function, $x^a_R$, with support on $x > 0$, an even-parity function $x^a \equiv x^a_R + x^a_L$, and an odd-parity function $x^a \equiv x^a_R - x^a_L$. The corresponding Frobenius solutions will be denoted by $\xi^{(a)}_L$, $\xi^{(a)}_R$, $\xi^{(a)}$ and $\xi^{(a)}_s$ respectively. (Note that $\xi^{(a)}_s$ have definite parity only at leading order.)

Because of the intervening inner layer, whose physics we regard as arbitrary for the purpose of this paper, it is physically meaningful to regard the left and right solutions as independent. As can be seen from the above definitions, the odd- and even-parity solutions are not a further two independent solutions, but do form an alternative solution set obtained from the left and right solutions by linear transformation. Both choices have their advantages - in this paper we shall use both.

It is perhaps worthwhile to remark here that a mathematical way of seeing that the odd and even (or left and right) solutions with the same $a$ are independent is to seek weak solutions of (1), that is, solutions for which the inner product of $L \xi(\psi)$ with any function taken from an appropriate test-function space vanishes. One can then fall back on the machinery of generalized function theory (Gel'fand & Shilov 1964) to show that both even and odd solutions satisfy (1) in the weak sense. The use of weak solutions is very natural in the context of the present paper because we are seeking a variational method for computing the matching data, and because the numerical approach we adopt is the Galerkin method. Note, however, that the necessity we shall encounter of taking inner products with singular functions takes us outside the standard generalized function theory, so that we shall have to develop our formalism from first principles.

At the lowest order, $k = 0$, (11) and (12) lead to the indicial equation $a(x + 1) f_0 - g_0 = 0$ whose two roots are

$$\begin{align*}
\alpha^{(a)} &\equiv -\frac{1}{2} + \mu, \\
\alpha^{(b)} &\equiv -\frac{1}{2} - \mu.
\end{align*}$$

(13)

where (Dewar & Pletzer 1990)

$$\mu \equiv \left( \frac{1}{4} + \frac{g_0}{f_0} \right) \equiv (-D_1)^{\frac{1}{2}}.$$

(14)

Here $D_1$ is the Mercier function of Glasser, Greene & Johnson (1975). The plasma is assumed to be stable to ideal modes, that is $D_1 \leq 0$, so that $\mu$ is positive and real. Although usually $\mu < \frac{1}{2}$ for a cylindrical plasma, we shall, with the toroidal case in mind, consider the case $\mu > \frac{1}{2}$ as well. This can occur in a cylindrical plasma with a hollow pressure profile. The superscript $(b)$ refers to the 'big', or dominant, solution in the limit $\psi \rightarrow \psi_1$, while the superscript $(s)$ refers to the 'small', or recessive, solution in this limit, the corresponding solutions (12) being denoted by $\xi^{(s)}$ and $\xi^{(a)}$. 

---

Non-ideal stability: variational method

---
We adopt the normalization $a_0^{(b)} = a_0^{(s)} = 1$. Higher-order Frobenius coefficients

$$a_k^{(s)} = \frac{\sum_{k=0}^{k-1} g_{k-k'} (z^{(s)} + k + 1) (z^{(s)} + k') f_{k-k'} a_{k'}^{(s)}}{f_0 k (k+2\mu)},$$  \hspace{1cm} (15)$$

$k = 1, 2, \ldots$, for the small-solution expansion, and

$$a_k^{(b)} = \frac{\sum_{k=0}^{k-1} g_{k-k'} (z^{(b)} + k + 1) (z^{(b)} + k') f_{k-k'} a_{k'}^{(b)}}{f_0 k (k-2\mu)},$$  \hspace{1cm} (16)$$

$k = 1, 2, \ldots$, for the big-solution expansion, are obtained by inserting (11) and (12) into (1) and equating coefficients at each order to zero.

The denominator in (16) vanishes when $k = 2\mu$, which will occur for some $k$ whenever $\mu$ is a half-integer. This situation is very important, since it includes the case of zero pressure gradient ($p'(\psi) = 0$) because $g_0 \rightarrow 0$ and $\mu \rightarrow \frac{1}{2}$ in this limit. In this case all the $a_k^{(b)}$ for $k > 0$ will diverge as $(1-2\mu)^{-1}$. Since the residues of these poles are proportional to $a_k^{(s)}$, the big and small solutions become increasingly linearly dependent in this limit. The divergences can be avoided (Miller & Dewar 1986) by subtracting off the part proportional to the small solution to give a redefined big solution

$$\xi_p^{(b)} = \xi_p^{(b)} - (s_\pm(\mu) + s_{\pm}(\mu) \text{sgn}(\psi - \psi_1)) \xi_p^{(s)}$$

where $p = +, -$ (or $L, R$), and $s_{\pm}$ are defined at half-integer and integer $\mu$ so as to subtract off the divergent part of the big solution, with the interpolation between these values of $\mu$ being somewhat arbitrary (Miller & Dewar 1986). Note that $s_+$ vanishes for $\mu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, while $s_-$ vanishes for $\mu = 1, 2, 3, \ldots$, since in the former case the divergent part has opposite parity to the big solution, while in the latter case it has the same parity.

A similar redefinition of the big solution has recently been invoked for calculating $A'$ for ballooning modes in tokamaks with asymmetric cross-sections (Wilson 1990). If we are not interested in values of $\mu$ near half-integers or integers then we can simply take $s_\pm \equiv 0$, as is the usual practice (Coppi et al. 1966). If we do wish to use the modified definition of the big solution then the definition used in the inner layer must also be modified to be consistent.

For $\mu$ around $\frac{1}{4}$, we may take $s_+ \equiv 0$ and

$$s_- (\mu) = \frac{f_1 + g_1}{f_0 (1 - 2\mu)}.$$  \hspace{1cm} (18)$$

In the limit $\mu \rightarrow \frac{1}{2}$, the new big solution becomes

$$\xi_{\psi}^{(b)} = (\psi - \psi_1)^{\frac{1}{2}} + O(\psi - \psi_1) + \frac{f_1 + g_1}{f_0} \ln x |\psi - \psi_1|$$

as $\psi \rightarrow \psi_1$, where $\ln x$ denotes the even function $\ln |x|$, whereas $\ln x$ denotes the odd function $\text{sgn} x \ln |x|$. These logarithmic terms come from expanding the identity $x = \exp (e \ln x)$ to first order in $e$, where here $e$ is $\pm (\mu - \frac{1}{2})$. We thus recover the usual zero-$\beta$ solution without having to treat it as a special case necessitating a different ansatz, at the expense of using a different definition for the big solution. In the following, we shall take the superscript $(b)$ to denote the redefined big solution unless otherwise indicated.
3. Matching data

The two roots of the indicial equation correspond to two distinct asymptotic behaviours of $\xi$, with the big solution being dominant near the rational surface. With the left and right, or odd- and even-parity solutions for each of the two $\pm$, we see that the general solution of (1) has four arbitrary constants, despite the fact that it is a second-order differential equation (one can see $\Delta_L$ and $\Delta_R$ as boundary conditions at $\psi_-$ and $\psi_+$ respectively). Thus we can write the general solution as

$$
\xi = c_L(\xi_L^{(b)} + \Delta_L \xi_L^{(s)}) + c_R(\xi_R^{(b)} + \Delta_R \xi_R^{(s)}),
$$

where $c_L$, $c_R$, $\Delta_L$ and $\Delta_R$ are arbitrary until the two boundary conditions, (2) and regularity at $\psi = 0$, are taken into account, when $\Delta_L$ and $\Delta_R$ become geometrically determined quantities. As will be seen in §4, these may be thought of as linear response coefficients corresponding to forcing by the two independent big solutions. The two coefficients $\Delta_L$ and $\Delta_R$ should form a sufficient set of data for matching to the inside-layer solutions, no matter what the physics of this layer. Note that, from (20),

$$
\Delta_R = \lim_{\psi \to \psi_+} \left( \frac{\xi - c_R \xi_R^{(b)}}{\xi_R^{(b)}} \right) \left( \frac{\psi - \psi_+^{(2)}}{2} \right),
$$

$$
\Delta_L = \lim_{\psi \to \psi_-} \left( \frac{\xi - c_L \xi_L^{(b)}}{\xi_L^{(b)}} \right) \left( \frac{\psi - \psi_-^{(2)}}{2} \right).
$$

Using (21) for the computation of $\Delta_{R,L}$ is not very satisfactory from a numerical-analysis point of view, since it relies on pointwise-accurate representation of the solution as $\psi \to \psi_i$. It is much better (Miller & Dewar 1986) to use an integral definition that is less sensitive to local inaccuracy. This is one of our prime motivations for seeking a variational definition of the matching data.

The decoupling of the left and right solutions in one-dimensional geometries makes the formulation above rather simple. As mentioned previously, the need to take the regular solutions into account in the two-dimensional toroidal case will complicate the formulation by giving rise to a left small solution response to forcing by a right big solution, and vice versa (see the appendix). A similar complication arises already in the present cylindrical case when we write the general solution in terms of the odd- and even-parity solutions

$$
\xi = c_A[\xi_L^{(a)} + \frac{1}{2}(A'\xi_L^{(a)} + B'\xi_L^{(s)})] + c_B[\xi_R^{(a)} + \frac{1}{2}(\Gamma'\xi_R^{(a)} + \Delta'\xi_R^{(a)})],
$$

since, although we specify that the big components of the two fundamental solutions are of definite parity, the small components are of mixed parity. Clearly, not all of the four linear response coefficients $\Lambda'$, $B'$, $\Gamma'$ and $\Delta'$ are independent: in fact, writing $\xi_L^{(a)} = \frac{1}{2}(\xi_L^{(a)} - \xi_L^{(s)})$ and $\xi_R^{(a)} = \frac{1}{2}(\xi_R^{(a)} + \xi_R^{(s)})$ in (20), we see that

$$
\Lambda' = \Delta' = \Delta_R + \Delta_L,
$$

$$
B' = \Gamma' = \Delta_R - \Delta_L.
$$

We shall find that the symmetry $B' = \Gamma'$ is an example of a set of reciprocity relations following from the variational principle that we introduce in §5.
As an example of the use of the outer-matching data that we have defined to match to an inner-layer theory, we shall use the notation of Glasser et al. (1984) for a symmetric, linear, finite-$\beta$ inner layer. In the inner layer of width $\varepsilon \to 0$, their asymptotic solutions to be matched are

$$\xi \sim C_+ [X_+^{(b)} + \Delta_+(Q) X_+^{(e)}] + C_- [X_-^{(b)} + \Delta_-(Q) X_-^{(e)}]$$

as $X \equiv (\psi - \psi_*)/\varepsilon \to \infty$ and $\psi \to \psi_*$. In (24), $\Delta_+(Q)$ and $\Delta_-(Q)$ represent the inner-matching data for the resistive interchange mode and the tearing mode respectively as functions of the growth rate $Q$. As Glasser et al. (1984) use the simple unmodified definition of $\xi^{(b)}$, we shall do the same (i.e. we take $s(\mu) \equiv 0$ in (17)). Matching the coefficients of the big solutions in (24) with those in (22) gives $C_\pm = C_\pm/\varepsilon^{\alpha_0}$. Matching the coefficients of the even and odd small solutions gives two linear equations in $C_+$ and $C_-$. Requiring that they have a non-trivial solution yields the following determinantal dispersion relation:

$$\begin{vmatrix} e^{2\alpha \Delta_+} - 2\Delta_+(Q) & e^{2\alpha \Gamma_+} \\ e^{2\alpha \Gamma_-} & e^{2\alpha \Delta_-} - 2\Delta_-(Q) \end{vmatrix} = 0.$$  \hspace{1cm} (25)

This shows that either $\Delta_+(Q)$ or $\Delta_-(Q)$ or both must vanish in the limit $\varepsilon \to 0$. For example, $\Delta_+(Q) \sim e^{2\alpha}$ and $\Delta_-(Q) \sim 1$ corresponds to a mode that is essentially of odd (tearing) parity, while $\Delta_-(Q) \sim e^{2\alpha}$ and $\Delta_+(Q) \sim 1$ describes a resistive interchange mode.

4. Hilbert-space response formalism

Define the inner product $(\cdot, \cdot)$ by

$$(u, v) \equiv \int_0^\infty d\psi \, u^* v,$$ \hspace{1cm} (26)

where $u^*$ is the complex conjugate of $u$. If $u$ and $v$ are sufficiently well behaved to allow integration by parts, and they obey the boundary condition (2) at $\psi = \psi_*$ and the regularity condition at $\psi = 0$, then

$$(u, L^2 u) = (v, L^2 u)^* = -W(u, v),$$ \hspace{1cm} (27)

where the bilinear form $W(\cdot, \cdot)$ is defined by

$$W(u, v) \equiv \int_0^\infty d\psi \left( f \frac{du^*}{d\psi} \frac{dv}{d\psi} + gu^* v \right) + \sigma_a f(\psi_*) u(\psi_*)^* v(\psi_*).$$ \hspace{1cm} (28)

The quadratic form $W(u, u)$ is immediately recognized as being proportional to the ideal plasma energy $\delta W$ (Newcomb 1960; Freidberg 1987) for a perturbation $\xi = u$, and the result (27) as a proof of the well-known Hermiticity of $L$ (i.e. $L^* = L$).

Since we assume the plasma to be ideally stable, $W(u, u)$ must be positive for all non-zero $u$ satisfying the physically admissible boundary conditions, and so we can adopt

$$\|u\| \equiv ||W(u, u)|| > 0$$ \hspace{1cm} (29)

as norm in a Hilbert space $\mathcal{H}$ of functions satisfying the physical boundary conditions (2) with $W(u, v)$ as inner product (Miller & Dewar 1986). We shall show in $\S$5 that a sufficient condition for $u$ and $v$ to be ‘sufficiently well behaved’, used in deriving (27), is in fact the assumption that $u, v \in \mathcal{H}$. 

434 A. Pletzer and R. L. Dewar
If we could find a solution \( \xi \) to (1) such that \( \xi^{\text{et}} \) then (27) shows that \( W(\xi, \xi) \) would be zero. That is, the plasma would be only marginally stable, violating the assumption in (29). Thus we are led to conclude that the \( \xi \) we seek cannot lie in \( \mathcal{H} \). Indeed, it is easy to see that \( \| \xi^{(0)} \| = \infty \) — the big solutions have infinite energy (at least when the standard definition of energy is used).

Nevertheless, we seek a variational principle for \( \xi \) and the matching data. The critical step is (Dewar & Grimm 1984; Miller & Dewar 1986) to split \( \xi \) into a prescribed infinite-energy part \( \xi \), which captures the asymptotic behaviour of the big solution to sufficient accuracy so as not to pollute the small solution, and a complementary part \( \xi \equiv \xi - \xi \in \mathcal{H} \), which we can vary, using integration by parts in the usual way.

To be specific, suppose that there are \( N \) rational surfaces \( \psi_i, i = 1, 2, \ldots, N \), in the plasma; then (20) becomes

\[
\xi = \sum_{r=1}^{N} \sum_{p=1}^{p} c_{t} \xi_{t}^{(p)}.
\]

where \( p = +, - \) (or \( L, R \)). (In a cylindrical plasma \( N \) is likely to be at most 2, as in the case of the double tearing mode (Manickam et al. 1983; Connor et al. 1988), but we keep \( N \) general with the toroidal case in mind.) We form the fundamental set \( \{ \xi_{t}^{(p)} \} \) of weak solutions by considering a set \( \{ \xi_{t}^{(p)} \} \) of 2\( N \) infinite-energy parts, where

\[
\xi_{t}^{(p)} \equiv H_{i}(\psi) \xi_{t}^{(p)}.
\]

extracts the behaviour of the big solutions \( \xi_{t}^{(p)} \) around \( \psi_{i} \) only. This is achieved by taking the localization functions \( H_{i}(\psi) \) (figure 1) to vanish outside finite intervals \((\psi_{i} - \delta, \psi_{i} + \delta)\) bracketing only the rational surfaces \( \psi_{i} \) and not including \( \psi = 0 \) or \( \psi = \psi_{a} \). Within \((\psi_{i} - \delta, \psi_{i} + \delta)\), however, the shapes of the \( H_{i}(\psi) \) are rather arbitrary (the complementary solutions will adjust to compensate — see §6). For simplicity, we assume first that \( H_{i}(\psi) \) is flat in a finite neighbourhood of \( \psi_{i} \):

\[
H_{i}(\psi) \equiv 1 \quad (\psi_{i} - \delta < \psi < \psi_{i} + \delta).
\]
where $e_i$, $0 < e_i < \delta_i$, is smaller than the convergence radius of (12). We also require the continuity of $H_1(\psi)$ and $dH_1(\psi)/d\psi$, so that the $H_1(\psi)$ go smoothly to zero between $\psi = \psi_i - e_i$ and $\psi = \psi_i + \delta_i$, and between $\psi = \psi_i + e_i$ and $\psi = \psi_i + \delta_i$. As a consequence,

$$L\epsilon = 0 \quad (\psi_i - e_i < \psi < \psi_i + e_i)$$

and (trivially) outside the support of $H_1(\psi)$. Elsewhere, $\hat{\xi}_{ip}$ does not satisfy (1), but since this occurs where there are no singularities, we have $L\hat{\xi}_{ip} \in \mathcal{H}$. The infinite-energy part of $\hat{\xi}_{ip}$ is $\hat{\xi}_{(p)}$, and the corresponding complementary, finite-energy, part is

$$\hat{\xi}_{(p)} = \hat{\xi}_{(p)} - \hat{\xi}_{ip}$$

(the parentheses around $ip$ being a reminder that $\hat{\xi}_{(p)}$ does not in general have a well-defined asymptotic parity $p$, nor is it localized near $\psi_i$, as is indicated in figure 2). Substituting (34) into (1), we find the inhomogeneous equation

$$L\hat{\xi}_{(p)} = -L\hat{\xi}_{ip}.$$ 

By the assumption of ideal stability, $L$ is a non-singular Hermitian operator within $\mathcal{H}$. Since, by construction, $\hat{\xi}_{(p)}$ and $L\hat{\xi}_{ip}$ lie within $\mathcal{H}$, we can solve (35) to give $\hat{\xi}_{(p)}$ as the linear response $-L^{-1}(L\hat{\xi}_{ip})$ driven by an imposed big solution $\hat{\xi}_{ip}$. The finite-element method developed in \S 8 can be thought of as a constructive proof of the existence of $L^{-1}$.

From (35), it is seen that $L\hat{\xi}_{(p)}$ also vanishes around $\psi = \psi_i$, and near the other rational surfaces, $\psi = \psi_j$, say, it satisfies $L\hat{\xi}_{(p)} = 0$ since, by construction, $\hat{\xi}_{ip}$ vanishes near $\psi = \psi_j$, $i \neq j$, and thus $\hat{\xi}_{(p)} = \hat{\xi}_{ip}$ there. Hence, near all rational surfaces, $\hat{\xi}_{(p)}$ exhibits the behaviour of the small solutions.
The coefficients of the leading terms are just the matching data as defined by Grimm, Dewar & Manickam (1983) and Manickam et al. (1983):

\[
\begin{align*}
\tilde{\xi}(u, \psi_j + x) &= \frac{1}{2} \left( A_{ij} \xi_j^{(u)} + B_{ij} \xi_{j'}^{(u)} \right), \\
\tilde{\xi}(u, \psi_j + x) &= \frac{1}{2} \left( \Gamma_{ij} \xi_j^{(u)} + \Delta_{ij} \xi_{j'}^{(u)} \right),
\end{align*}
\]

for \(-\epsilon_j < x < \epsilon_j\). The diagonal coefficients are the \(A', B', \Gamma'\) and \(\Delta'\) coefficients introduced in \(\S 3\), while the off-diagonal coefficients \(A'_{ij}, B'_{ij}, \Gamma'_{ij}\) and \(\Delta'_{ij}\) are the asymptotic response coefficients representing the excitation of small-solution behaviour at rational surface \(\psi_j\) by a big solution at surface \(\psi_i\). The full asymptotic matching problem for multiple rational surfaces involves all these coefficients (Grimm et al. 1983; Manickam et al. 1983; Connor et al. 1988). The dispersion relation (25) then becomes

\[
\frac{e^{2\alpha_i\Delta'_{ij}} - 2\Delta^{(u)}_{ij}(Q) \delta_{ij}}{e^{2\alpha_i\Gamma'_{ij}}} = \frac{e^{2\alpha_j\Delta'_{ij}} - 2\Delta^{(u)}_{ij}(Q) \delta_{ij}}{e^{2\alpha_j\Gamma'_{ij}}},
\]

(37)

It is perhaps in the ease with which (36) generalizes the matching data to multiple rational surfaces that the power of the Hilbert-space response formalism is most apparent, since we do not have to exhibit an explicit construction for the solution (cf. Connor et al. 1988), given that \(L^{-1}\) exists on quite general grounds.

5. Variational principle

Multiplying (1) for the fundamental solution \(\xi_{ip}\) by \(\xi_{ip}^*\) and integrating over the plasma, we have

\[
\left( \xi_{ip}, L\xi_{ip} \right) = 0,
\]

(38)

where \(p, q \in \{+, -\}\). Because \(\xi_{ip}\) does not lie in the Hilbert space \(\mathcal{H}\), we must proceed cautiously when integrating by parts. Introducing the decomposition (34) into (38), we have an equation involving four terms:

\[
\left( \xi_{ip}, L\xi_{ip} \right) + \left( \xi_{ip}, L\tilde{\xi}_{ip} \right) + \left( \xi_{ip}, L\tilde{\xi}_{ip} \right) + \left( \xi_{ip}, L\tilde{\xi}_{ip} \right) = 0.
\]

(39)

Recalling that we wish to vary the \(\xi\) component, we see that it is only the third term that presents a problem. We effect its integration by parts by first noting that the identity

\[
u^*(Lv) - v(Lu)^* = \frac{dP[u, v | \psi]}{d\psi}, \quad \psi + \psi_i,
\]

(40)

is pointwise-true (except at the rational surfaces if \(u\) and \(v\) are 'badly behaved'), where we have introduced the bilinear concomitant \(P\) (Morse & Feshbach 1953) defined by

\[
P[u, v | \psi] \equiv \int \left( u^* \frac{dv}{d\psi} - v \frac{du^*}{d\psi} \right).
\]

(41)

The importance of \(P\) derives from the fact, immediately apparent from (40), that \(P(\psi)\) is constant in any region where \(u\) and \(v\) satisfy (1) (except at the surfaces \(\psi_i\)). If \(u = O(|x|^\alpha)\) and \(v = O(|x|^\beta)\) in the neighbourhood of \(\psi_i\), with \(x \equiv \psi - \psi_i\), then \(P = O(|x|^\alpha + \beta - 1)\). Thus if \(\alpha_i + \beta_i > -1\) then \(P \to 0\) as \(x \to 0^\pm\) and \(P\)
is continuous at $x = 0$, but if $\alpha_i + \beta_i = -1$ then $P = O(\text{const})$ on either side of $\psi_i$, which, as noted above, is consistent with $u$ and $v$ locally being solutions of the Newcomb equation (1). However, the constant is typically different on either side of $\psi_i$, so that $P$ is discontinuous at $\psi_i$, and $dP/d\psi = [P_i, \delta(\psi - \psi_i)]$ in the neighbourhood of $\psi_i$, where $[P_i] \equiv P(\psi_i + 0) - P(\psi_i - 0)$ is the jump in $P$ at $\psi_i$. On the other hand, the left-hand side of (40) has at most an integrable singularity at $\psi = \psi_i$, with no $\delta$ function. Thus, to make (40) true in a generalized-function sense on the whole interval $0 < \psi < \psi_n$, we must subtract off the $\delta$-function contributions at the rational surfaces, yielding an equation reminiscent of that defining a Green function:

$$u^*(Lv) - v(Lu)^* = \frac{dP(\psi)}{d\psi} - \sum_{i=1}^{N} [P_i, \delta(\psi - \psi_i)].$$  (42)

We now see that the condition for the integration by parts leading to the Hermiticity condition (27) for $u$ and $v$ satisfying the physical boundary conditions (so that $P = 0$ at $\psi = 0$ and $\psi_n$) to be valid is that $[P_i]$ vanish at all rational surfaces, since the right-hand side of (42) is then a complete derivative. If $u, v \in \mathcal{H}$ then $\alpha_i > -\frac{1}{2}$ and $\beta_i > -\frac{1}{2}$ for $\|u\|$ and $\|v\|$ to converge; hence $\alpha_i + \beta_i > -1$ and $P$ is continuous, as noted above, and the $[P_i]$ vanish. Thus a sufficient condition for Hermiticity is $u, v \in \mathcal{H}$. It is readily verified that $u, v \in \mathcal{H}$ is also sufficient for $W(u, v)$ to be finite.

This is not the only circumstance in which $[P_i]$ vanishes, however. In particular, the concomitant between odd and even powers is continuous:

$$[P[x^4, x^4]]_i = 0,$$  (43)

even if $\alpha_i + \beta_i = -1$. Also, if $\alpha = \alpha_i^{(p)}$ then we can have $\alpha_i + \beta_i > -1$ simply by requiring $\beta_i > \alpha_i^{(m)}$. That is, if $u = O(\psi^{(p)})$ then we can still freely integrate by parts if we require $v$ to be $O(\psi^{(m)})$.

Using (42) in the third term of (39), we find an integral expression for the jump in the concomitant:

$$[P[\hat{\xi}_i, \hat{\xi}(p)]_i] = (\xi_j, L\xi(p)),$$  (44)

where we have defined the symmetrized matrix element

$$(\xi_j, L\xi(p))' = (\xi_j, L\xi(p)) + (\xi_j, L\xi(p)) + (\xi_j, L\xi(p)) + (\xi_j, L\xi(p)).$$  (45)

differing from that corresponding to (39) only in the third term on the right-hand side. Equation (44) is the fundamental result on which this paper is based.

From (31) to (36), we see that the jump is, to within a factor, the matching data we seek, so that (44) gives

$$\gamma_{ij} = \frac{(\xi_j, L\xi_i)}{2f_0^{(p)} \mu_j}, \quad \delta_{ij} = \frac{(\xi_j, L\xi_i)}{2f_0^{(p)} \mu_j}\left(\frac{\xi_j, L\xi_i}{\xi_j, L\xi_i}ight)^*.$$  (46)$$

Varying the $\xi$ component, we have

$$\delta(\xi_j, L\xi(p)) = (\delta\xi_{(j)}, L(\xi(p) + \xi_{(p)})) + (\delta\xi_{(p)}, L(\xi_{(j)} + \xi_{(j)}))^*.$$  (47)
which vanishes for arbitrary $\xi$ variations if and only if (35) is satisfied. Thus we have a variational principle: $A_{ij}', B_{ij}', \Gamma_j'$ and $\Delta_j'$ are stationary with respect to variations in the $\xi$ component if and only if the response equation (35) is satisfied.

6. Uniqueness and related questions

In this section, we consider the question of the behaviour of the matching data, as given now by the symmetrized matrix-element expressions (46), under transformations of the form

$$
\begin{align*}
\xi_{ip}' &\rightarrow \xi_{ip}' + \eta_{ip}' \\
\xi_{ip} &\rightarrow \xi_{ip} - \eta_{ip}'
\end{align*}
$$

where $\eta$ is an arbitrary function in $\mathcal{H}$. These transformations leave the fundamental solution $\xi_{ip}$ unchanged. Substituting (48) into (45), we find that all but two of the terms containing $\eta$ immediately cancel. Using (42) gives

$$
\xi_{ip} - 2\eta_{ip}' + [P(\xi_{ip} - \eta_{ip}'],
$$

First consider the case $\eta_{ip} = 0(\xi)$. From the discussion after (43), we know that the jumps $[P(\xi_{ip} - \eta_{ip}]]$ vanish in this case. Thus we have the result that the symmetrized matrix element is invariant under arbitrary redefinitions of the infinite-energy part, provided that the difference between the old and new definitions vanishes faster than the small solutions near the rational surfaces. This is consistent with (46) in that a change that was of the same order as the small solutions would 'pollute' the small-solution component of $\xi$ and thus affect the matching data, which are response coefficients measuring this component.

A useful corollary of this invariance is the fact that we can truncate the series (12) defining the big solution at a finite number of terms, provided that the truncation error $\eta_{ip}$ is of lower order than the leading term of the small solution. This is essential for practical purposes, since high-order coefficients $a_k^{(2)}$ and $a_k^{(3)}$ in the series (12) defining the big solution, as used in (31), involve high-order derivatives of equilibrium quantities, which are difficult to compute accurately. Summing from $k = 0$ to $k = k_t$ such that the first term omitted from (31), which is $O(|\psi'_{ij} - \psi'_{ji}|^{2+1+2})$, be of higher order than $O(|\psi'_{ij} - \psi'_{ji}|^{2+1+2})$, the leading-order term of the small solution (figure 3), we find the condition

$$
k_t > 2\rho_i - 1.
$$

As noted by Chu et al. (1990), we can do even better than this for $\frac{1}{2} < \rho_i < 1$ if we seek only to evaluate the matching data $A_{ij}'$ and $\Delta_j'$, which, by (48), involve matrix elements of $L$ between fundamental solutions with the same dominant parity. In these cases, the second term in the Frobenius expansion of the big solution contained in $\xi_{ip}$ is of opposite parity to the leading term of the big solution contained in $\xi_{ip}'$, so that if it is regarded as the error $\eta_{ip}$ then its contribution to the jump $[P(\xi_{ip} - \eta_{ip}']]$ vanishes identically by (43). Thus we can truncate at $k_t = 0$, keeping only one term in the Frobenius expansion in this case.

A similar argument allows us to extend the class of the $H_i(\psi)$, by taking

$\star$ This variational principle has been found independently by Chu et al. (1990).
Figure 3. Powers of $x$ occurring in the Frobenius expansion of the big (top) and small (bottom) solutions marked on the real line. Note that $\xi_{10}$ must be truncated at order $|x|^n + k > n$, i.e., $k > 2n - 1$, in order that $\xi_{10}$ extract the small-solution behaviour at leading order. In this particular case, $k > 1$.

$$H_i(\psi) = 1 + O((\psi - \psi_i)^{k+1})$$ as $|\psi| \to \infty$ instead of (32). With this choice of $\xi_{10}$, $H_i(\psi)$, $L\xi_{10} \in \mathcal{H}$ but vanishes only asymptotically, and (36) remains true at lowest order.

The result (49) can also be used to investigate the consequences of making different choices for the functions $s_{1,2}(\mu)$ used in the modified big solution $\xi^{(p)}_{10}$ defined by (17). In this case, we take

$$s_{1,2} = [\tilde{s}_{1,2}(\mu) + \text{sgn}(\psi - \psi_i) \tilde{s}_{2,1}(\mu)] H_i(\psi) \xi_{10}(\psi),$$

where $\tilde{s}_{1,2}$ represent the changes in the choices of $s_{1,2}$. Then (43) and (49) give

$$(\xi_{10}, L\xi_{10}) \to (\xi_{10}, L\xi_{10}) + 4\delta_{1,1} \delta_{\alpha,\beta} f^0_{\alpha} \mu_j \tilde{s}_{1,1}(\mu) + 4\delta_{1,1} \delta_{\alpha,\beta} f^0_{\alpha} \mu_j \tilde{s}_{2,1}(\mu).$$

Since $\tilde{s}_{1,2}(\mu) \equiv \tilde{s}_{2,1}(\mu) \equiv 0$, it follows that $A_i'$ and $A_i''$ are unaffected by the redefinition (17) in the low-$\beta$ limit $\mu \to \frac{1}{2}$.

7. Reciprocity relations and other symmetries

Note that, although we chose in (27) to use the Hermitian inner product because this will prove more appropriate in the toroidal case (see the appendix), $\xi_{10}$ is manifestly real in the present cylindrical case and the complex conjugate can be dispensed with.

As well as providing the basis for efficient finite-element computation of the matching data, as will be shown in §8, the explicit integral expressions for the matching data provided by (46) allow easy demonstration of fundamental symmetries between the coefficients.

The first type of symmetry, which, maintaining the linear response viewpoint, we call reciprocity relations, follows from the (Hermitian) symmetry of $(\xi_{10}, L\xi_{10})$ under interchange of $(j, q)$ and $(i, p)$. This is apparent in the first term of (45) from Hermiticity of $L$ within $\mathcal{H}$, manifestly for the sum of the next two terms and, by explicit consideration of the form of $\xi_{10}$, given by (31), for the last term. In the case $i = j + 1$, it is simplest for purpose of proof to assume that the supports of the shape functions $H_i(\psi)$ and $H_j(\psi)$ do not overlap, so that the last term of (45) vanishes identically, though invariance under (48) shows that it is not necessary to assume this.
The reciprocity relations are
\[ \begin{align*} 
\mu_j f^{(0)}_0 A_{ij} &= \mu_i f^{(0)}_0 A_{ji}, \\
\mu_j f^{(0)}_0 B_{ij} &= \mu_i f^{(0)}_0 B_{ji}, \\
\mu_j f^{(0)}_0 \Gamma_{ij} &= \mu_i f^{(0)}_0 \Gamma_{ji}, \\
\mu_j f^{(0)}_0 \Delta_{ij} &= \mu_i f^{(0)}_0 \Delta_{ji}. 
\end{align*} \] (53)

The relation \( B' = \Gamma' \), derived in \( \S 3 \), is an instance of a reciprocity relation: it expresses the fact that the even response (the amplitude of the even small solution) to an odd imposed big solution is equal to the odd response to an imposed even big solution. Since, from the appendix, (46) generalizes straightforwardly to the toroidal case, these reciprocity relations remain valid in that case also.

The second type of symmetry, which is peculiar to the cylindrical problem, is a consequence of the disjointness of the solution on either side of a rational surface. First note that, as a consequence of this, the support of \( \xi \) is limited to the region on either side of \( \psi = \psi_i \) bounded by \( \psi_{i+1} \) (or a boundary). Thus, in the unlikely event that there were more than two rational surfaces, there would be no overlap for \( |i-j| > 1 \), and the matching coefficients \( A_{ij} \) etc. would vanish for \( |i-j| > 1 \).

Also, since we can change the sign of \( \xi \) in the interval \( \psi_i < \psi < \psi_{i+1} \) and still have a valid solution of (1), there is a simple relation between the basis functions corresponding to big solutions of opposite parities:
\[ \xi_{i+p}(\psi_i + x) = \text{sgn } x \xi_{i-p}(\psi_i + x). \] (54)

Consider first the case \( i = j+1 \). Using (54) in (45), we find
\[ (\xi_{i-q}, L \xi_{i+1, p}) = -(\xi_{i-q}, L \xi_{i+1, p}), \] (55)
whence
\[ A'_{j+1, j} = B'_{j+1, j} = -\Gamma'_{j+1, j} = -\Delta'_{j+1, j}. \] (56)

Now consider the case \( i = j \). We obtain
\[ A'_{ii} = \Delta'_{ii}, \] (57)
as well as the reciprocity relation \( B'_{i} = \Gamma'_{i} \), which we had from (53). As mentioned above, (57) is not expected to remain valid in the toroidal case.

8. Finite-element solution

After extraction of the infinite-energy part \( \xi \), the remaining contribution \( \xi \) can be obtained by means of the finite-element method, applicable only to solutions of finite energy norm (29).

This leads us to solve the weak form of (38).
\[ W(u, \xi_{i(p)}) = (u, \xi_{i(p)}), \] (58)
obtained after multiplying (35) by \( u \in \mathcal{H} \), integrating over the plasma and using (27). We choose the trial functions \( u \) to lie within \( \mathcal{H}_M \), where \( \mathcal{H}_M \subset \mathcal{H} \) is spanned by the basis functions \( \psi_v(\psi) (v = 1, 2, \ldots, M) \). The Ritz approximation
\[ \tilde{\xi}_{i(p)}^{M} = \sum_{v=1}^{M} \Xi_{v}^{(i)} \psi_v(\psi) \] (59)
Vti 4. Basis functions $e_i$. Each element $e_i$ extends from node $\psi_{r-1}$ to node $\psi_{r+1}$. The distribution of nodes can be non-uniform about the singularity $\psi_r$. Note that no singular-shaped element is used.

is used to approximate $\hat{e}_{(ip)}$, where $\Xi_{ip}$ are coefficients to be determined. Replacing $\hat{e}_{(ip)}$ by $\hat{e}_{(ip)}^{M}$ in (58) yields the Galerkin equation

$$W(u, \hat{e}_{(ip)}^{M}) = (u, L\hat{e}_{(ip)}), \tag{60}$$

which leads, after substitution of (59) into (60), to the system of linear equations

$$\sum_{r=1}^{M} \Xi_{ip}^{(r)} W(e_r, e_r) = (e_r, L\hat{e}_{(ip)}) \tag{61}$$

for the $\Xi_{ip}^{(r)}$. Note that, by combining (58) and (60),

$$W(u, \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}) = 0 \tag{62}$$

$\forall u \in H_{M}$, from which it is seen that the error $\hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}$ is orthogonal to any function $u$ belonging to $H_{M}$.

The $e_r$ must be carefully chosen so that $W(e_r, e_r)$ of (61) be finite. We shall consider here the simplest choice of such basis functions $e_r$, which possess a finite support extending from nodes $\psi_{r-1}$ to $\psi_{r+1}$ (except for $e_1$ and $e_M$, which have supports extending from $\psi_1$ to $\psi_2$ and $\psi_{M-1}$ to $\psi_M$ respectively): the tent functions shown in figure 4.

We shall estimate, in the following, the error resulting from the Ritz approximation in the calculation of the matching data (46). Note that the first two terms of (45) cancel by virtue of (35), so that the error of the matching data reduces to

$$\frac{(\hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}, \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M})}{2 \mu_j f_0^{(0)}} = \frac{W(\hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}, \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M})}{2 \mu_j f_0^{(0)}}$$

$$= \frac{W(\hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}, \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M})}{2 \mu_j f_0^{(0)}} \tag{63}$$

since, from (62), $W(\hat{e}_{(ip)}^{M}, \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}) = 0$. Equation (63) can be further bounded by

$$|W(\hat{e}_{(ip)} - \hat{e}_{(ip)}^{M}, \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M})| \leq \| \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M} \| \| \hat{e}_{(ip)}^{M} - \hat{e}_{(ip)}^{M} \|$$

$$= O(M^r) \tag{64}$$

as $M \to \infty$, where the convergence rate $r$ is defined by

$$\| \hat{e}_{(ip)} - \hat{e}_{(ip)}^{M} \| = O(M^r) \tag{65}$$
as $M \to \infty$. Note that $r$ is the convergence rate for the solution $\xi$. Equation (64) shows that the convergence rate $2r$ of the matching data is twice as fast as that of the solution itself, a characteristic property of a variational principle.

From (65), it is seen that $\ell^{(M)}_{s(i)}$ can be made arbitrarily close to $\ell^{(M)}_{s(i)}$ by increasing $M$ if $r < 0$. To estimate $r$, we first show that, using (62),

$$
\|\ell_{s(i)}^{(p)} - u\| = \|\ell_{s(i)}^{(p)} - \ell^{(M)}_{s(i)} + \ell^{(M)}_{s(i)} - u\|
$$

$$
= (\|\ell_{s(i)}^{(p)} - \ell^{(M)}_{s(i)}\| + \|\ell^{(M)}_{s(i)} - u\|)
$$

(66)

$\forall u \in \mathcal{X}_c$. Hence $\|\ell_{s(i)}^{(p)} - u\| \geq \|\ell^{(M)}_{s(i)} - \ell^{(M)}_{s(i)}\|$; that is, the finite-element method provides the best approximation of $\ell_{s(i)}^{(p)}$ by minimizing $\|\ell_{s(i)}^{(p)} - \ell^{(M)}_{s(i)}\| = \inf_{u \in \mathcal{X}_c} \|\ell_{s(i)}^{(p)} - u\|$ (recalling that we assume $\|\cdot\|$ to be real and positive by virtue of the ideal stability hypothesis of (29)). In particular, we have

$$
\|\ell_{s(i)}^{(p)} - \ell^{(M)}_{s(i)}\| \leq \|\ell_{s(i)}^{(p)} - \ell^{(M)}_{s(i)}\|,
$$

(67)

where $\ell^{(M)}_{s(i)} \in \mathcal{X}_c$ linearly interpolates $\ell_{s(i)}^{(p)}$ (for tent functions),

$$
\ell^{(M)}_{s(i)}(\psi) = \ell^{(M)}_{s(i)}(\psi).
$$

(68)

between the nodes $\nu = 1, 2, \ldots, M$. To evaluate the right-hand side of (67), we introduce $e, (\psi) \equiv \ell^{(p)}_{s(i)}(\psi) - \ell^{(p)}_{s(i)}(\psi)$ and $e, \equiv de, (\psi)/d\psi$ defined for $\psi, \psi \leq \psi_{p+1}$, which can be shown (Morton 1987) to be bounded by

$$
e_, \leq \ell^{(p)}_{s(i)}(\psi_{p+1} - \psi),
$$

$$
e_+ \leq \ell^{(p)}_{s(i)}(\psi_{p+1} - \psi),
$$

(69)

where $\ell^{(p)}_{s(i)}$ represents the maximum value of $d^2\ell^{(p)}_{s(i)}/d\psi^2$ within $(\psi, \psi_{p+1})$. From (67) and (69), we obtain

$$
\|\ell^{(p)}_{s(i)} - \ell^{(M)}_{s(i)}\| \leq \left\{ \sum_{\nu=1}^{M-1} \ell^{(p)}_{s(i)}(\psi_{p+1} - \psi,)^2 \int_{\psi,}^{\psi_{p+1}} d\psi [f + \ell^{(p)}_{s(i)}(\psi_{p+1} - \psi,)^2] \right\}^{1/2}.
$$

(70)

Assuming $\ell^{(p)}_{s(i)}$ sufficiently regular in $(0, \psi_x)$ so that $|\ell^{(p)}_{s(i)}| < \infty$, (70) reduces to

$$
\|\ell^{(p)}_{s(i)} - \ell^{(M)}_{s(i)}\| \leq \text{const} \ell^{(p)}_{s(i)},
$$

(71)

where $h \equiv \max_{\nu=1,2,\ldots,M-1} (\psi_{p+1} - \psi,)$ and $\ell^{(p)}_{s(i)} \equiv \max_{\nu=1,2,\ldots,M-1} \ell^{(p)}_{s(i)}$. Thus a linear mesh gives a convergence rate $r = -1$, and often provides the best accuracy since $h \propto M^{-1}$ is minimal.

The situation is quite different for $|\ell^{(p)}_{s(i)}| \to \infty$ in $(0, \psi_x)$. We expect in this case that increasing the density of nodes around the singularity improves accuracy, and is even necessary to recover the convergence rate $r = -1$. We can pack the nodes by taking a mesh-generation function $F : \psi, = F(t_x)$, where $t_x \equiv (\nu-1)/(M-1)$ ($\nu = 1, 2, \ldots, M$), such that

$$
\psi, = \psi_x - \text{const} (t_x - t_x)^{p_x}
$$

(72)

for $t_x < t_x$, with $t_x = F^{-1}(\psi_x)$, and

$$
\psi, = \psi_x + \text{const} (t_x - t_x)^{p_x}
$$

(73)

for $t_x > t_x$ near $\psi_x$ and by taking $p_x > 1$ (figure 5).

We are confronted with two singular points located at the rational surface $\psi_x = \psi, (6)$, and on the magnetic axis $\psi_x = 0, (8)$. Each singularity is characterized by the exponent $\beta$ of the leading-order behaviour of $f$ near $\psi_x$. 

Non-ideal stability : variational method
Figure 5. Mesh distribution function. Note the three scaled sections located on the right-hand side of \( \psi = \psi_s = 0 \) and on the left- and right-hand sides of \( \psi = \psi_i \). The dashed horizontal line represents the position of the rational surface \( \psi_i \). In the limit \( M \to \infty \), when \( t = \nu/M \) becomes a continuous variable, \( \psi_i(t) \) and \( d\psi_i/dt \) are continuous.

\[ f \sim f_0(\psi - \psi_s)^\mu \quad \text{as} \quad \psi \to \psi_s: \quad \beta = 1 \quad (m \neq 0) \quad \text{for} \quad \psi_s = 0 \quad \text{and} \quad \beta = 2 \quad \text{for} \quad \psi_s = \psi_i. \]

Similarly, we have \( \xi_{\alpha}(t) \sim a_0(\psi - \psi_s)^\xi \) as \( \psi \to \psi_s: \quad \alpha = \frac{1}{2}m \quad (m \neq 0) \quad \text{for} \quad \psi_s = 0 \quad \text{and} \quad \alpha = -\frac{1}{2} + \mu_i \) for \( \psi_s = \psi_i \) respectively. Taking the limits \( M \to \infty \), \( t, \to t = \nu/M \).

\[ 1/M \to dt, \quad \psi_r = \psi_i - d\psi_i = 0 \quad \text{and} \quad d\psi_i/dt \to \psi_i \to \psi_i \to \psi_i \]

This gives the following minimal scaling of the mesh: \( \gamma \geq \frac{2}{2m + \beta + 1} \)

\[ \alpha = \frac{1}{2}m, \quad \text{or} \quad \beta = 2 \quad \text{for} \quad \psi_s = 0 \quad \text{and} \quad \beta = 1 \quad \text{for} \quad \psi_s = \psi_i, \]

or the particular case of \( \beta = 0 \) or \( \mu_i = \frac{1}{2} \), the small solution being regular about the singular point (\( \alpha = 0 \)), maximum convergence is achieved with linear distributed nodes (\( \gamma_i = 1 \)).

9. Numerical results

In order to access the regime \( \mu_i > \frac{1}{2} \), we are led to consider hollow pressure profiles

\[ p(\psi) = p_0(1 + \rho) \frac{1 - (\psi/\psi_s)^2}{1 + \rho(1 - (\psi/\psi_s)^2)} \]

in a cylindrical geometry. For \( 0 \leq \rho \leq 1 \), the pressure decreases monotonically from \( \psi = 0 \) to \( \psi = \psi_s \). However, for \( \rho > 1 \), the pressure profile is hollow with positive pressure gradient at the rational surface.

We take the safety factor to be a monotonically growing function of the radius \( r, q(r) = 1 + 3(r/r_a)^2, r_a \equiv r(\psi_s) \), so that there always exists one and only
one rational surface between \( r = 0 \) and \( r = r_a \) for \( m = 2 \) and \( n = 1 \). The following profiles, denoted by \( a \), \( b \) and \( c \), and illustrated in figure 6, are considered:

\[
\begin{align*}
\text{profile} & \quad p_0 & \quad \rho & \quad \mu_i \\
\text{a} & \quad 1 & \quad 0 & \quad 0.41673 \\
b & \quad 0 & \quad 0 & \quad 0.5 \\
c & \quad 1 & \quad 20 & \quad 0.60305 \\
\end{align*}
\]  

\hspace{1cm} (77)

Because in a cylindrical geometry the left- and right-hand-side problems decouple, it is advantageous to use the left-sided and right-sided functions for calculation of the matching data:

\[
\begin{align*}
\mu_i f_0^{(L)} \Delta_L &= W(\xi(L), \xi(L)) + (\xi(L), \xi(L)) \xi(L) \\
\mu_i f_0^{(R)} \Delta_R &= W(\xi(R), \xi(R)) + (\xi(R), \xi(R)) \xi(R) \\
\end{align*}
\]  

\hspace{1cm} (78)

We set \( s(\mu_i) = 0 \) in (31), except for case \( b \), where \( s_i \) is given by (18).

The convergence of the algorithm (46) and (78) is tested by varying the number of nodes (10–200), as well as \( \gamma \). Each mesh is characterized by \( M_L \), the number of mesh nodes on the left-hand side of \( \psi_i \), and \( M_R \), the number of mesh nodes on the right-hand side of \( \psi_i \), with \( M = M_L + M_R \). The mesh is divided in sections of linear distribution of nodes,

\[
\psi_i = bt_i + \frac{bw(1-\gamma_0)}{\gamma_0} \\
\]  

\hspace{1cm} (79)

for \( w \geq t_i = (v-1)/(M-1) \) \( < t_i - w \) and

\[
\psi_i = bt_i + 1 - b \\
\]  

\hspace{1cm} (80)

for \( t_i + w \geq t_i \leq 1 \), as well as three sections of relative width \( w < \frac{1}{2} \) where mesh nodes \( \psi_i \), \( v = 0, 1, ..., M_i \), are packed about \( \psi_x \) with exponent \( \gamma_x \geq 1 \), \( x \in (0, i) \),

\[
\psi_i = \frac{bw(t_x)^{\gamma_x}}{\gamma_x(w)} \\
\]  

\hspace{1cm} (81)
for $t_s < w$,

$$\psi_i = -\frac{bw}{\gamma_i} \left( \frac{t_i - t_s}{w} \right)^{\gamma_i} + \psi_i$$

(82)

for $t_i - w \leq t_s < t_i$ and

$$\psi_i = \frac{bw}{\gamma_i} \left( \frac{t_i - t_s}{w} \right)^{\gamma_i} + \psi_i$$

(83)

for $t_i \leq t_s < t_i + w$. In (79)-(83), the parameters

$$b^{-1} = 1 + w \left( \frac{1 - \gamma_o}{\gamma_o} + 2 \frac{1 - \gamma_i}{\gamma_i} \right)$$

$$t_i = \frac{\psi_i}{b} - w \left( \frac{1 - \gamma_o}{\gamma_o} + \frac{1 - \gamma_i}{\gamma_i} \right)$$

(84)

are adjusted so as to generate a mesh that satisfies the requirement of continuous $\psi(t)$ and $d\psi(t)/dt$ in the limit $M \to \infty$ (figure 5). The different meshes are summarized as follows:

<table>
<thead>
<tr>
<th>p profile</th>
<th>mesh</th>
<th>$w$</th>
<th>$\gamma_o$</th>
<th>$\gamma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a1</td>
<td>0.2</td>
<td>1.1</td>
<td>1.0</td>
</tr>
<tr>
<td>a</td>
<td>a2</td>
<td>0.2</td>
<td>1.1</td>
<td>2.4</td>
</tr>
<tr>
<td>a</td>
<td>a3</td>
<td>0.2</td>
<td>1.1</td>
<td>4.0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0.2</td>
<td>1.1</td>
<td>1.0</td>
</tr>
<tr>
<td>c</td>
<td>c1</td>
<td>0.2</td>
<td>1.1</td>
<td>1.0</td>
</tr>
<tr>
<td>c</td>
<td>c2</td>
<td>0.2</td>
<td>1.1</td>
<td>1.7</td>
</tr>
</tbody>
</table>

(85)

The pressure profile $a$ is the numerically most constraining case, since $\mu_i$ is small so that a high density of nodes is required according to (75). This is exhibited by figures 7 and 8. For a linear mesh, $\Delta_L$ and $\Delta_R$ converge (negative $r$), but, as expected, without reaching the maximal convergence rate of
Non-ideal stability: variational method

Figure 8. Convergence of $\Delta_2$ versus the inverse square of the number of mesh nodes $M_2$ on the right-hand side of $\psi_r$ for pressure profile $a$. Three meshes, $a_1$, $a_2$ and $a_3$, are considered, with mesh scaling exponents $\gamma_i = 1.0$, 2.4 and 4.0 respectively.

Figure 9. Convergence of $\Delta_2$ versus the inverse square of the number of mesh nodes $M_2$ on the left-hand side of $\psi_r$ for the $\beta = 0$ case $b$ and for a linear mesh ($\gamma_i = 1$).

$2r = -2$. Increasing $\gamma_i$ to $\gamma_i = 2.4 \approx 1/\mu_i$ is sufficient to recover $r = -1$, so that $\Delta_L$ and $\Delta_R$ converge as $M^{-2}$ for $M > 40$. The convergence is further improved at smaller $M$ for $\gamma_i = 4.0$ at the expense of a deteriorated accuracy. A characteristic feature of $\mu_i$ approaching $\frac{1}{2}$ when $s_i(\mu_i)$ is set equal to zero is $|\Delta'| \ll |\Gamma'|$, already noticeable at $\mu_i = 0.41673$ (case $a$) in that the sum of $\Delta_L$ and $\Delta_R$ is an order of magnitude smaller than the difference.

In the pressureless case $b$, the dominant behaviour of $L_{\theta}^2 = O(\psi - \bar{\psi}_r)$ is less localized, leading to a convergence in $M^{-2}$ at low $M \approx 10$ for a linear mesh, as illustrated in figures 9 and 10. For higher values of $\mu_i$ (see case $c$ in figures 11 and 12), the mesh scaling becomes less crucial, since linear meshes often turn out to be sufficiently accurate and convergent.
10. Conclusions

A numerically efficient and accurate method for the calculation of the outer-region asymptotic matching data has been presented. Numerical convergence is guaranteed to be proportional to the inverse square of the number of mesh nodes for 'tent functions', provided the mesh is carefully scaled near the singular (rational) surfaces.

In our approach, no particular assumption regarding the inner layer physics is made, so that the method applies to a large number of situations where non-ideal effects are small. It is not excluded, however, that under some
Non-ideal stability: variational method

Figure 12. Convergence of $\Delta_n$ versus the inverse square of the number of mesh nodes $M_n$ on the right-hand side of $\dot{\psi}$, for the hollow pressure profile $c$. Two meshes, $c_1$ and $c_2$, are considered, with mesh scaling exponents $\gamma_1 = 1.0$ and $1.7$ respectively.

circumstances, such as the tearing mode in $\beta = 0$ plasmas (Furth, Killeen & Rosenbluth 1963), the number of matching data can be reduced owing to a parity selection rule in the inner layer. (In the $\beta = 0$ case, $\Delta'$ entirely determines the stability properties.)

Since the matching data represent a generalization of $\Delta'$ to non-ideal modes in plasmas possessing finite pressure gradients at the rational surfaces, it is interesting to speculate whether the tearing-stability criterion $\Delta' < 0$ can be generalized so as to involve the complete set of matching data. This question remains open at the present time, but the demonstration in this paper that the matching data are related to a well-defined symmetrized matrix element $(\cdot, \cdot)'$ does suggest that the sign of the energy-like quantity $W' \equiv -\langle \xi, L\xi \rangle'$ may be related to non-ideal stability, with the jumps $[P]$ being related to energy fluxes into the inner layers. The non-uniqueness of the symmetrized inner product under the redefinition of the big solution (17), as shown by (52), does suggest, however, that energy arguments cannot be made purely on the basis of an analysis of the outer region.

We are grateful to Dr J. M. Greene, whose quest for an energy principle for finite-$\beta$ resistive stability has been a major factor in leading us to the present formulation of the outer-region matching-data problem. We also thank Dr A. D. Miller for making available to us the finite-element code used in Miller & Dewar (1986). The numerical calculations in this work were carried out using the Fujitsu VP100 of the Australian National University Supercomputer Facility.

Appendix. Matching data in two-dimensional geometry

To take into account toroidal (or helical) effects in the calculation of the matching data, we follow Dewar & Pletzer (1990), where (1) becomes $L\xi(\psi, \theta) = 0$, with

$$L \equiv (\partial_{\theta} D_{\theta} + L') \mathcal{A}(2 + D_{\theta} \partial_{\psi}) + \mathcal{X}$$

(86)
and $\xi$ satisfying (2) (with $\sigma_\theta$ a function of $\theta$) at $\psi = \psi_t$. In (86), we assume continuous symmetry along the $\zeta$ co-ordinate, so that $n$ remains a good quantum number. All operators

$$\mathcal{D}_\theta \equiv \partial_\theta - i n q(\psi),$$

(87)

$\mathcal{G}$ and $\mathcal{K}$, involve only $\partial_\theta \equiv \partial/\partial \theta$ derivatives (but no $\partial_\psi \equiv \partial/\partial \psi$). The expressions for $\mathcal{G}$, $\mathcal{I}$ and $\mathcal{K}$ are given in Dewar & Pletzer (1990). For the purpose of this appendix, we need only know that $\mathcal{G}$ and $\mathcal{K}$ are Hermitian; that is,

$$\langle \psi, \mathcal{G} \psi \rangle = \langle \mathcal{G} \psi, \psi \rangle,$$

$$\langle \psi, \mathcal{K} \psi \rangle = \langle \mathcal{K} \psi, \psi \rangle$$

(88)

with respect to the surface inner product

$$\langle \psi, \psi \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \psi \ast \psi,$$

(89)

where $u = u(\theta)$ and $v = v(\theta)$ are $2\pi$-periodic functions.

Because in the two-dimensional situation the $\theta$ dependence of $\xi(\psi, \theta)$ is not in general that of a single Fourier mode exp $(im\theta)$, we find, for given $n$, an extended class of rational surfaces located at every $\psi_t$ where (87) has a vanishing eigenvalue $i(m_t - n q(\psi_t))$, with $m_t$ possibly different at each $\psi_t$.

The second complication arising from toroidal coupling is due to the existence of a third type of asymptotic solution near $\psi_t$. In addition to the small and big solutions of §2, there is a set of regular solutions $\xi^{(t)}$, which are analytic everywhere except at $\psi = 0$. The $\xi^{(t)}$ couple the solutions across the rational surfaces, so that it is not possible to solve each region delimited by rational surfaces (or boundary conditions) independently as in the cylindrical case.

The formalism presented in §4 is, however, sufficiently general, in allowing for the presence of several rational surfaces and treating both parities $\xi_+$ and $\xi_-$ simultaneously, that the results obtained in §§5 and 7 can be extended to the toroidal case, with $\xi_{\#}$ extracting the big solution near $\psi_t$, and $\xi^{(t)}$ containing both the small and regular solutions.

The concomitant (40) becomes

$$P[u, v, \psi] \equiv \langle u, \mathcal{D}_\theta \mathcal{G}(2 + \mathcal{D}_\theta \partial_\psi) v \rangle + \langle \mathcal{D}_\theta \mathcal{G}(2 + \mathcal{D}_\theta \partial_\psi) u, v \rangle.$$  

(90)

We focus on $P[u, v, \psi]$ in (44) to obtain the matching data (46). It can easily be shown that the contributions from big and small solutions yield (40), with

$$f_0^{(t)} = \frac{\pi a_i^2 \mu_i}{\langle \xi^{(t)} \rangle},$$

(91)

where $\langle \cdot \rangle_j \equiv \langle \exp (im_j \theta), \cdot \exp (im_j \theta) \rangle$ evaluated at $\psi_j$, provided that the regular solutions do not contribute to the matching data.

We can easily show, from a surface-averaged version of (40), that $dP[u, v, \psi]/d\psi$ is zero where $Lu = Lv = 0$ (except at $\psi_t$). Thus $P[u, v, \psi]$ must be constant in neighbourhoods immediately to the right and left of the $\psi_t$ for any choice of Frobenius solution, including the big and regular solutions (cf. equation (68) of Bineau 1966). However, setting $u = \xi^{(b)}_t$ and $v = \xi^{(r)}_t$, where superscripts (b) and (r) denote big and regular solutions respectively, we see
from the discussion in §5 that $P(\xi_{0}^{(0)}, \xi_{1}^{(0)} | \psi) = x_{1}^{(0)}\phi(x)$ for $\psi = \psi_{0} + x$, with $\alpha_{1}^{(0)} + 1 = \frac{1}{2} - \mu$, and $\phi(x)$ a Taylor series in $x$. For $\mu_{q} = \frac{1}{2}$, this is incompatible with the constancy of $P(\xi_{0}^{(0)}, \xi_{1}^{(0)} | \psi)$ unless all the coefficients in $\phi$ vanish. It can be verified by explicit calculation that this is in fact the case. Thus the regular solutions do not affect the jump in $P(\xi_{0}^{(0)}, \xi_{1}^{(0)} | \psi)$ at the rational surface and remains true. This must be true in the $\beta = 0$ case also, since (see §2) we handle this case by taking the limit $\mu \to \frac{1}{2}$.

REFERENCES

Bennett, M. 1966 Nucl. Fusion, 2, 130.