the last equation (11) assuring that $\zeta\mathfrak{B}$ is a perfect curl. But outside the plasma $\nabla\times\mathfrak{H}=\nabla\times\mathfrak{B}=0$ (assuming there are no currents in the vacuum) so that ζ must vanish everywhere. Hence, the determination of equilibrium again reduces to the solution of Eqs. (4).

In toroidal geometry the situation is slightly more complicated. As before, we may choose any convenient boundary condition for the functions σ and τ and Eqs. (10) and (11) remain true. In general, however, even if the magnetic field lines close on themselves after a small number of turns around the system, there is no guarantee that σ and τ will reflect this periodicity—for *any* choice of boundary conditions. But $\mathfrak{M}$ and $\mathfrak{H}$ are not directly observable, and our only real requirement is that $\mathfrak{B}$ be continuous. Hence, it is permissible to choose any convenient surface $s=0$ cutting the torus, requiring $\mathfrak{H}$ and $\mathfrak{M}$ to be single-valued only in the cut volume, and $\mathfrak{B}$ to be continuous across the cut.

A second complication arises for toroidal systems in that ζ no longer vanishes identically, corresponding to the possibility that axial currents can flow the entire length of a field line. The specification of these currents, however, is subject to an auxiliary boundary condition, determined for example by the emf exerted by external induction windings.

Again, let us set $s=0$ upon a surface $\mathbf{n}\cdot\nabla\mathfrak{B}=0$, and note that for a plasma in equilibrium, the distribution function depends only on the constants of motion. These we may take as the total energy, the magnetic moment, and the longitudinal invariant. It then follows that both $\mathbf{n}\cdot\nabla p_{\|}$ and $\mathbf{n}\cdot\nabla p_{\perp}$ must vanish with $\mathbf{n}\cdot\nabla\mathfrak{B}$. Thus, $\nabla\times\chi=4\pi\mathfrak{B}^{-2}\nabla p_{\perp}\times\nabla\mathfrak{B}$ lies along $\mathbf{n}$ and $u_1=u_2=0$. Hence, if $\tau=0$ at $s=0$, then $\nabla\tau=0$ as well and $\partial\sigma/\partial s=1$, independently of σ. We distinguish the symmetric case in which $\mathbf{n}\cdot\nabla\mathfrak{B}=0$ everywhere within the plasma from the circumstance in which we may choose our surface $s=0$ such that $\mathbf{n}\cdot\nabla(\mathbf{n}\cdot\nabla\mathfrak{B})\neq 0$. In the former case if $\sigma(s=0)=0$, then $\sigma=s$ and, on the surface, $\nabla\sigma=\mathbf{n}$ and $\mathbf{n}\cdot\nabla\times(\sigma\chi)=0$. Hence, when $\mathbf{n}\cdot\nabla\mathfrak{B}$ vanishes throughout the plasma,

$$\zeta=4\pi\mathfrak{B}^{-2}(1+\eta)c^{-1}\mathbf{j}\cdot\mathfrak{B}\,|_{s=0}. \tag{12}$$

More generally, putting $\sigma=0$ at $s=0$ now means that $\nabla\sigma$ must lie along $\nabla(\mathbf{n}\cdot\nabla\mathfrak{B})$. But $\mathbf{n}\cdot\nabla\sigma=1$ so that

$$\nabla\sigma=[\mathbf{n}\cdot\nabla(\mathbf{n}\cdot\nabla\mathfrak{B})]^{-1}\nabla(\mathbf{n}\cdot\nabla\mathfrak{B}).$$

Hence when $\mathfrak{B}\cdot\nabla(\mathbf{n}\cdot\nabla\mathfrak{B})\neq 0$,

$$\zeta=\{4\pi\mathfrak{B}^{-2}(1+\eta)c^{-1}\mathbf{j}\cdot\mathfrak{B} +[\mathfrak{B}\cdot\nabla(\mathbf{n}\cdot\nabla\mathfrak{B})]^{-1}\mathbf{n}\times\chi\cdot\nabla(\mathbf{n}\cdot\nabla\mathfrak{B})\}|_{s=0}. \tag{13}$$

If we now set the total magnetic field $\mathfrak{H}_T=\mathfrak{B}-4\pi\mathfrak{M}=\mathfrak{H}+\mathfrak{H}_A$, defining $\mathfrak{H}_A$ as the field due to a distribution of axial currents $4\pi c^{-1}\mathbf{j}_A=\zeta\mathfrak{B}$, the problem of equilibrium reduces once more to the solution of Eqs. (4).

In summary, we have shown that it is possible, in general, to describe the equilibrium between a plasma and a magnetic field in terms of a magnetization $\mathfrak{M}$, Eq. (10), thus reducing the problem essentially to the level of classical magnetostatics, i.e., the solution of Eqs. (4). In general, however, $\mathfrak{M}$ is defined nonlocally through its relationship to two scalar functions σ and τ. These functions are expressible as line integrals along a line of force such that each component of Eq. (8) vanishes. The results are to pose the problem of equilibrium in soluble form, making techniques of solution available to the computation of general equilibria which previously could be applied only when the plasma pressures were functions of $|\mathfrak{B}|$ alone.

The author is grateful to Dr. William A. Newcomb for a number of critical comments.

This work was performed under the auspices of the U.S. Atomic Energy Commission.

[1] L. S. Solov'ev and V. D. Shafranov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1970), Vol. 5, p. 1.
[2] J. G. Cordey and C. J. H. Watson, in *Proceedings of the British Nuclear Energy Society Nuclear Fusion Reactor Conference* (British Nuclear Energy Society, Culham, 1970), p. 122.
[3] L. S. Hall, Phys. Fluids (to be published).

THE PHYSICS OF FLUIDS VOLUME 15, NUMBER 4 APRIL 1972

Frequency Shift Due to Trapped Particles

R. L. DEWAR
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540

(Received 14 June 1971; final manuscript received 29 November 1971)

The time asymptotic distribution functions corresponding to adiabatic and sudden excitation of an electrostatic wave are calculated. These distributions are compared and used to calculate the nonlinear response of the plasma, and Poisson's equation is used to find a nonlinear dispersion relation.

Recently, Manheimer and Flynn[1] have shown that trapped-particle effects lead to an $O(\phi^{1/2})$ correction to the linear dispersion relation (where ϕ is the electrostatic potential). This is to be contrasted with the $O(\phi^2)$ contribution from nonresonant particles,[2] which is, in general, very small. These authors, however, used the approximations of simple harmonic motion for the trapped particles and constant velocity for the untrapped particles, and also derived the dispersion relation in an only approximately correct fashion. In view of the importance of the frequency shift in the stability theory of large amplitude plasma waves,[3] we have attempted a more accurate analysis.

We have also felt it desirable to draw attention to the nonuniqueness of the distribution of trapped particles,[4] and to examine two cases which must, in some sense, be extremes of physically reasonable distributions. On the one hand, we have the distribution corresponding to the time asymptotic limit of the nonlinear wave studied by O'Neil,[5] which is the one used by Manheimer and Flynn.[1] The procedure in this case is to turn the wave on suddenly and to allow the distribution function to phase mix until it is constant along lines of constant wave-frame energy. This yields for the distribution function, as a function of wave-frame energy W,

$$F_{\text{sud}}(W)=\sum_{\pm}\left\langle\frac{\theta(W-e\phi)f_0[v_p\pm u(W,x)]}{u(W,x)}\right\rangle\Big/\left\langle\frac{\theta(W-e\phi)}{u(W,x)}\right\rangle, \tag{1}$$

where $\theta(W-e\phi)$ is the unit step function, $f_0(v)$ is the distribution function before the wave was turned on, $v_p\equiv\omega/k$, and

$$u(W,x)\equiv(2/m)^{1/2}(W-e\phi)^{1/2} \tag{2}$$

is the magnitude of the velocity in the wave frame. We take the potential $\phi(x-v_pt)$ to be periodic with wavelength $\lambda=2\pi/k$, and to be such that $\langle\phi\rangle=0$, where $\langle\ \rangle$ denotes the space average,

$$\lambda^{-1}\int_0^\lambda dx.$$

On the other hand, we may excite the wave by switching it on adiabatically from some time in the remote past, allowing only the trapped particles to phase mix. As shown by Best,[6] all but an exponentially small fraction of particles conserve their average velocity

$$\bar{u}(W)\equiv\langle\theta(W-e\phi)u(W,x)\rangle, \tag{3}$$

as an adiabatic invariant. Tracing this function back into the past we see that it is the initial velocity of any particle with present energy W. Hence,

$$F_{\text{ad}}(W)=\sum_{\pm}f_0[v_p\pm\bar{u}(W)]. \tag{4}$$

Finally, we need a reference distribution for determining which part of the response is essentially nonlinear. We shall call this reference distribution the linear distribution, and we require that the charge density derived from it be only the linear part of the response. It may be shown that the function[7]

$$F_{\text{lin}}(W)=\sum_{\pm}f_0[v_p\pm(2/m)^{1/2}W^{1/2}] \tag{5}$$

satisfies this criterion up to, and including, $O(\phi^{3/2})$ which is adequate for our purposes, as we shall calculate the response only up to $O(\phi^{3/2})$.

By adding and subtracting F_{lin} from $F\equiv F_{\text{ad}}$ or F_{sud}, the charge response may be split into a linear and a nonlinear part

$$\rho=\Pi(v_p)\phi+\sum_s\frac{e}{m}\int_{(e\phi)_{\min}}^{\infty}(F-F_{\text{lin}})\frac{\theta(W-e\phi)\,dW}{u(W,x)}, \tag{6}$$

assuming initial charge neutrality, where $\sum_s$ sums over species, and the linear response function Π is given by

$$\Pi(v_p)=-\sum_s(e^2/m)P\int dv[kf_0'(v)/(\omega-kv)]. \tag{7}$$

The integral in Eq. (6) is $O(\phi^2)$ when $W=O(1)$, but is $O(\phi^{1/2})$ when $W=O(\phi)$, so that, to $O(\phi^{3/2})$, the nonlinear response comes only from the region $W=O(\phi)$. In this region we find, to $O(\phi)$,

$$F_{\text{sud}}(W)=2f_0(v_p)+f_0''(v_p)[\bar{u}(W)/m\bar{u}'(W)], \tag{8a}$$

$$F_{\text{ad}}(W)=2f_0(v_p)+f_0''(v_p)[\bar{u}(W)]^2, \tag{8b}$$

$$F_{\text{lin}}(W)=2f_0(v_p)+f_0''(v_p)[2W/m], \tag{8c}$$

where primes denote differentiation. These functions, in the case of sinusoidal ϕ, are shown in Fig. 1. As a simple check on these distributions, we should be able to show that charge is conserved, i.e., that $\langle\rho\rangle=0$. Using such identities as $\langle\theta(W-e\phi)/u(W,x)\rangle=m\bar{u}'(W)$, and the asymptotic expansion of $\bar{u}(W)$ at large W, we may indeed show that the integral in Eq. (6) vanishes when averaged, for both F_{ad} and F_{sud}, and for any ϕ such that $\langle\phi\rangle=0$.

We now require that ϕ satisfy Poisson's equation to make the problem self-consistent. Multiplying Poisson's equation by ϕ and averaging we find the nonlinear dispersion relation

$$1-\frac{4\pi\langle\rho\phi\rangle}{\langle(\partial\phi/\partial x)^2\rangle}=0. \tag{9}$$

From Eq. (6), we have

$$\langle\rho\phi\rangle=\Pi(v_p)\langle\phi^2\rangle+\sum_s\int_{(e\phi)_{\min}}^{\infty}(F-F_{\text{lin}})(W\bar{u}'-\tfrac{1}{2}\bar{u})\,dW. \tag{10}$$

The function $[W\bar{u}'(W)-\frac{1}{2}\bar{u}(W)]$ is shown in Fig. 2, assuming ϕ to be sinusoidal. It has an integrable, logarithmic infinity at $W=|e\phi_1|$, and a zero near

$$W=0.65\,|e\phi_1|,$$

where ϕ_1 is the amplitude of the potential. Inspection

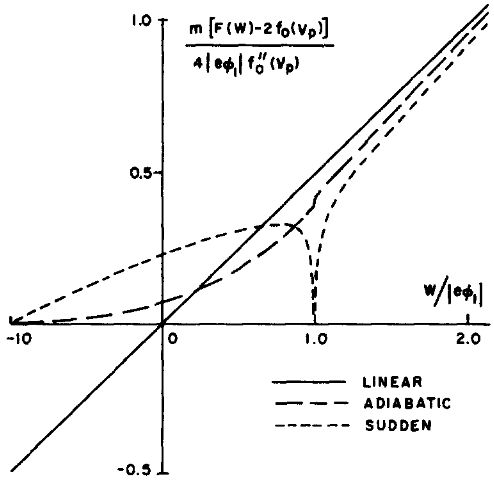

FIG. 1. The nonconstant parts of the distribution functions corresponding to adiabatic and sudden excitation of a sinusoidal wave, as a function of wave-frame energy. The distribution needed for linear response is also plotted.

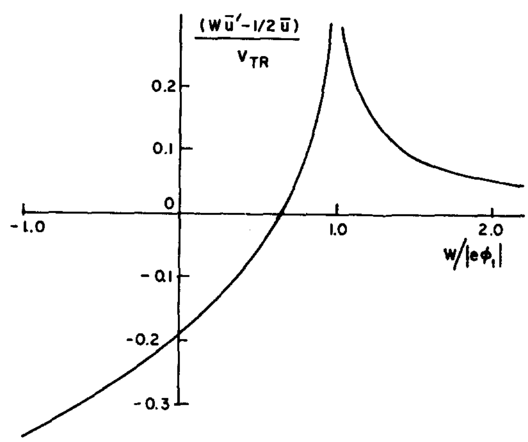

FIG. 2. The weight function used in determining the frequency shift, as a function of wave-frame energy.

of Eqs. (8) shows that $(F_{\rm sud}-F_{\rm lin})$ has a zero at the same point, thus making the integrand negative definite if $f_0''(v_p)>0$. In the adiabatic case there is a small positive region, but its contribution to the integral is small. The second term in Eq. (10) provides an $O(\phi^{1/2})$ contribution to Eq. (9), while harmonics generated by the nonlinearity contribute, through the $\langle\phi^2\rangle$ and $\langle(\partial\phi/\partial x)^2\rangle$ terms, an $O(\phi)$ contribution. Thus, to $O(\phi^{1/2})$, the sinusoidal approximation is perfectly adequate. Within this approximation $\bar{u}$ and $\bar{u}'$ may be evaluated in terms of the complete elliptic integrals $E(m)$, $K(m)$ [8]:

$$\bar{u}(W)=(2\sqrt{2}/\pi)v_{\rm tr}w^{1/2}E(w^{-1}),$$
$$\bar{u}'(W)=(\sqrt{2}/\pi)(mv_{\rm tr})^{-1}w^{-1/2}K(w^{-1}), \qquad (11)$$

for $w\equiv(W+|e\phi_1|)/(2|e\phi_1|)>1$; and

$$\bar{u}(W)=(2\sqrt{2}/\pi)v_{\rm tr}[E(w)-(1-w)K(w)],$$
$$\bar{u}'(W)=(\sqrt{2}/\pi)(mv_{\rm tr})^{-1}K(w) \qquad (12)$$

for $0<w<1$. The trapping velocity $v_{\rm tr}$ is defined by $v_{\rm tr}\equiv(2|e\phi_1|/m)^{1/2}$.

Using Eqs. (8), (11), and (12) in Eqs. (9) and (10), and numerically evaluating the integrals, we find the nonlinear dispersion relation to be similar to that found by Manheimer and Flynn[1]

$$\epsilon(k,\omega)=-\sum_s(\alpha\omega_p^2v_{\rm tr}/n_0k^2)f_0''(v_p), \qquad (13)$$

where the correction coefficients α are found to be $\alpha_{\rm sud}=1.163$, $\alpha_{\rm ad}=0.770$, and $\epsilon(k,\omega)=1-4\pi k^{-2}\Pi(v_p)$ is the linear dielectric constant. The contribution of untrapped particles to $\alpha_{\rm sud}$ is about 14%, while that from the region $0.9<w<1.1$ is about 12%, so the approximations of Manheimer and Flynn were not unreasonable for an order of magnitude calculation. More important is the 50% difference between $\alpha_{\rm sud}$ and $\alpha_{\rm ad}$, although the sign of shift is the same in both cases. Which distribution should be used depends on the case in hand; the adiabatic being more appropriate to the final state of a weakly unstable wave, for instance, and the sudden being more appropriate to a plasma wave launched by a grid. The frequency shifts are much smaller than that calculated by Goldman and Berk[4] on the basis of a bunched-beam model, indicating that the frequency shift has a smaller effect on the sideband instability than found by these authors.

It is a pleasure to acknowledge useful discussions with Drs. W. M. Manheimer and W. L. Kruer.

This work was started with the aid of a Fellowship from the Center for Theoretical Physics, University of Maryland, and its completion was supported by U. S. Air Force Office of Scientific Research Contract F44620-70-C-0033.

[1] W. M. Manheimer and R. W. Flynn, Phys. Fluids **14**, 2393 (1971).

[2] R. L. Dewar and J. Lindl, Phys. Fluids (to be published).

[3] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957), have shown that the distribution of trapped particles is uniquely specified only if both the waveform and the untrapped distribution are known. They do have a small-amplitude expansion, but this is invalidated in our case by the logarithmic singularities of the untrapped distributions.

[4] M. V. Goldman and H. L. Berk, Phys. Fluids **14**, 801 (1971).

[5] T. M. O'Neil, Phys. Fluids **8**, 2255 (1965).

[6] R. W. B. Best, Physica **40**, 182 (1968).

[7] By expanding about $W=0$ it is seen that $F_{\rm lin}$ is real, even for W negative, because of the summation over positive and negative wave-frame velocities. Equation (5) may be verified by comparing it with the linearized Vlasov result $\Sigma_\pm[f_0(v_p\pm u)\pm P(e\phi/mu)f_0'(v_p\pm u)]$ at both small and large W.

[8] *Handbook of Mathematical Functions* (National Bureau of Standards Applied Mathematics Series, No. 55) (U.S. Government Printing Office, 1964), p. 590.