1 Constrained particle motion under gravity

Recall that, in a uniform gravitational field producing downwards acceleration g, the potential energy of a particle of mass m at height z above some reference level is $V = mgz$, while the kinetic energy is $T = \frac{1}{2}m\mathbf{v}^2$. Thus the total energy is $E = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$.

Now suppose the particle is *constrained* to move on the surface $z = h(x, y)$. Then

$$E = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \left(\frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial y}\dot{y}\right)^2\right] + mgh(x, y)\ . \tag{1}$$

For frictionless motion, E is conserved: $E = E_0$, where E_0 is the initial value of the total energy. In 1-D, $y = 0$, $z = h(x)$, eq. (1) may be solved for $\dot{x}$:

$$\dot{x} = \pm\sqrt{2g}\left[\frac{h(x_0) - h(x)}{1 + h'(x)^2}\right]^{1/2}\ . \tag{2}$$

2 Statics ⇔ minimum energy states

For dissipative motion, E decreases with time — $E \le E_0$ for $t \ge t_0$. Assuming $h(x, y)$ is bounded from below (a convex function) E obtains its time-asymptotic minimum value when the K.E. and P.E. are *simultaneously* minimized: $\dot{x} = \dot{y} = 0$ *and*

$$h(x, y) = \mathrm{Min}[h(x, y)]\ . \tag{3}$$

Note that there may be several local minima — which one the particle approaches depends on initial conditions (the basin of attraction in which we start).

A state for which $\dot{x} = \dot{y} = 0$ for all time is known as an *equilibrium* state, and the branch of mechanics concerned with finding such states is known as *statics.* We have illustrated the fact that *statics problems can be solved by finding the minimum potential energy states of a system.*

To find a minimum we want $\delta h = 0$. Assuming h differentiable, $\delta h = (\delta x, \delta y) \cdot \nabla h$, thus a necessary condition for h to be a minimum is that its 2-D gradient vanish:

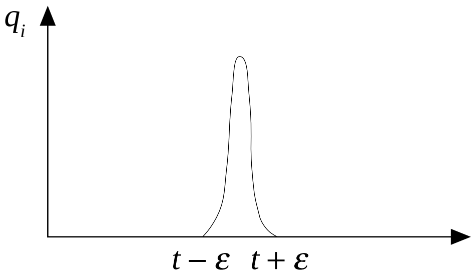
$$\nabla h \equiv (\partial_x h, \partial_y h) = 0\ . \tag{4}$$

3 Variational principle - recap

Consider variational principle $\delta I = 0$, where objective functional $I \equiv I[\boldsymbol{q}, \dot{\boldsymbol{q}}, t]$ and endpoint values of $\boldsymbol{q}$ are fixed:

$$\int_{t_1}^{t_2} \mathrm{d}t\, \delta\boldsymbol{q}(t)\boldsymbol{\cdot}\frac{\delta f}{\delta \boldsymbol{q}} = 0 \tag{5}$$

for $\delta \boldsymbol{q}(t)$ arbitrarily localized in t:



Equation 5 can be satisfied for such variations iff

$$\frac{\delta f}{\delta \boldsymbol{q}} \equiv \frac{\partial f}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{\boldsymbol{q}}} = 0\ . \tag{6}$$

at each value of t. This represents n equations — the *Euler–Lagrange* equations.

4 A geometric variational principle: geodesics

Define the *metric tensor* $g_{i,j}$ such that a length element $\mathrm{d}l$ is given by

$$(\,\mathrm{d}l)^2 = \sum_{i,j=1}^{n} \mathrm{d}q_i\, g_{i,j}\, \mathrm{d}q_j\ . \tag{7}$$

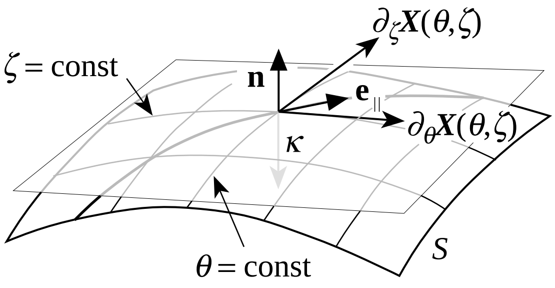
Then the total length l of a curve, paramtrized by τ,

$$l = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \left[\sum_{i,j=1}^{n} \dot{q}_i g_{i,j}(\boldsymbol{q},\tau)\dot{q}_j\right]^{1/2} \tag{8}$$

is the objective functional for defining a *geodesic*: a curve between two points whose length (calculated using the given metric) is stationary against infinitesimal variations about that path.

5 Example: geodesics on arbitrary surface

Prob: Show that any geodesic $\mathbf{r} = \mathbf{x}(\tau)$ on a two-dimensional manifold $S :\ \mathbf{r} = \mathbf{X}(\theta,\zeta)$ embedded in ordinary Euclidean 3-space, where θ and ζ are arbitrary curvilinear coordinates on S, has only normal curvature [i.e. is such that the curvature vector $\boldsymbol{\kappa}(\tau)$ is everywhere **normal** to S (or zero)].



The curvature vector is defined by $\boldsymbol{\kappa} \equiv \mathrm{d}\mathbf{e}_\parallel / \mathrm{d}l$, where $\mathbf{e}_\parallel(\tau) \equiv \mathrm{d}\mathbf{x}/\mathrm{d}l$ is the unit tangent vector at each point along the path $\mathbf{r} = \mathbf{x}(\tau)$.

See Answers in Chapter 5 of the course notes. The result can also be derived physically by analogy with an elastic string, and has physical application in magnetically confined plasmas.

6 The brachistochrone problem

Writing $\dot{x} = \mathrm{d}x/\mathrm{d}t$, we use eq. (2) to find the time $\mathcal{T}_{ab}$ taken for a frictionless particle released from the initial point $x = a$, $z = h(a)$ to roll to another point $x = b$, $z = h(b)$

$$\begin{aligned} \mathcal{T}_{ab} &= \frac{1}{\sqrt{2g}} \int_a^b \left[\frac{1 + h'(x)^2}{h(a) - h(x)}\right]^{1/2} \mathrm{d}x \\ &\equiv \frac{1}{\sqrt{2g}} I[h, h'] \; . \end{aligned} \tag{9}$$

The brachistochrone problem is to minimize the objective functional I, which is of the form $I = \int_a^b f(h, h', x)\,\mathrm{d}x$, with

$$f \equiv \left[\frac{1 + h'(x)^2}{h(a) - h(x)}\right]^{1/2} \; . \tag{10}$$

7 Variational approximation: trial function method

Variational principles can be used to derive "the best" approximation: use a *trial function*

$$\boldsymbol{q}(t) = \boldsymbol{q}_K(t, a_1, a_2, \ldots, a_K)$$

where $\boldsymbol{q}_K$ is (hopefully) an approximating function involving a finite number of parameters a_k, $k = 1, \ldots, K$ to be determined variationally:

$$\delta I = \sum_{k=1}^{K} \frac{\partial I}{\partial a_k} \delta a_k = 0 \; . \tag{11}$$

The condition for a stationary point is thus

$$\frac{\partial I}{\partial a_k} = 0, \quad k = 1, \ldots, K \; , \tag{12}$$

that is, that the K-dimensional gradient of I vanish. This method is applied to the anharmonic oscillator in the notes and accompanying Maple workbook