PHYS3002 — Classical Mechanics

Based on my PHYS3001 1st Semester course 2002 and Matthew Hole's 2nd Semester course 2005. Builds on Craig Savage's PHYS3001 1st Semester Least Action segment.

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Textbook: My formal lecture notes, the current version being at wwwrsphysse.anu.edu.au/∼rld105/C01_ClassMech/C01_1.51.pdf

Announcements and lecture presentations will be posted on WebCT soon.
Read appropriate sections before each class. Print only part you need for present . . . will be revised en-route.

Assessment: 50% exam 50% assignments unless notified otherwise.
Tutorials: Best time to be discussed.

Topics covered last semester

- Principal of least action/Hamilton's principle
- Lagrangian Dynamics
- Noether's Theorem
- Hamilton's equations
- Poisson brackets

This semester

- Revision of above in depth with more examples
- Canonical transformation theory
- Hamilton–Jacobi theory
- Action-Angle theory
- Perturbation theory
- Nonlinear dynamics and chaos

Why study Classical Mechanics?

- Unified description of all classical (non-quantum) physics, chemistry and engineering. E.g.
  - celestial mechanics (motion of stars, planets and satellites)
  - plasma physics - particle orbits in complicated magnetic geometry (eg fusion plasmas)
  - molecular dynamics
  - mechanical (& electrical) engineering
- Provides formal infrastructure for the development of quantum mechanics. NB QED runs at Street Theatre till 22 July!
- Beautiful in its own right: *Again and again [I have] experienced the extraordinary elation of mind which accompanies a preoccupation with the basic principles and methods of analytical mechanics.* — Cornelius Lanczos 1949

History

- 16, 17th C: Particle Kinematics, Force and Momentum Vectors, Gravity, . . . : *Galileo, Newton, . . .*
- 18th C: Configuration Space Description, Energy, Variational Principles, . . . : *Euler, Lagrange . . .*
- 19th C: Phase Space Description, Electrodynamics, Statistical Mechanics, . . . : *Hamilton, Maxwell, Boltzmann, Gibbs . . .*
- 20th C: Integrability, Symmetry, Dynamical Systems Theory, Chaos, . . . : *Poincaré, Einstein, Kovalevskaya, Noether, Kolmorogorov, Arnol'd, Moser . . .*
- 21st C: Simulation, Visualization, Complexity, Biodynamics, . . . ?: *Your turn!*

Euler

- Leonhard Euler was born 15 April 1707 in Basel, Switzerland. He died 18 September 1783 in St Petersburg, Russia.
- His book *Mechanica* (1736-37), extensively presented Newtonian dynamics in the form of mathematical analysis for the first time, and started Euler on the way to major mathematical work.
- He studied continuum mechanics, lunar theory with Clairaut, the three body problem, elasticity, acoustics, the wave theory of light, hydraulics, and music. He laid the foundation of **analytical mechanics**, especially in his *Theory of the Motions of Rigid Bodies* (1765).
- We owe to Euler the notation $f(x)$ for a function (1734), e for the base of natural logs (1727), i for the square root of -1 (1777), π for pi, $\sum$ for summation (1755), the notation for finite differences Δy and $\Delta^2 y$ and many others.

Leonhard Euler

He produced half his works after he became totally blind.

Lagrange

- Joseph-Louis Lagrange (Lagrangia) was born 25 Jan 1736 in Turin, Sardinia-Piedmont (now Italy). He died 10 April 1813 in Paris, France.

- Lagrange based his early development on the principle of least action and on kinetic energy. He corresponded with Euler, who finally persuaded him to move to Berlin, where he worked for 20 years, producing a steady stream of top quality papers and regularly winning prizes from the Académie des Sciences of Paris.

- His *Méchanique analytique* (1788) summarised all the work done in the field of mechanics since the time of Newton and is notable for its use of the theory of differential equations. With this work Lagrange transformed mechanics into a branch of mathematical analysis.

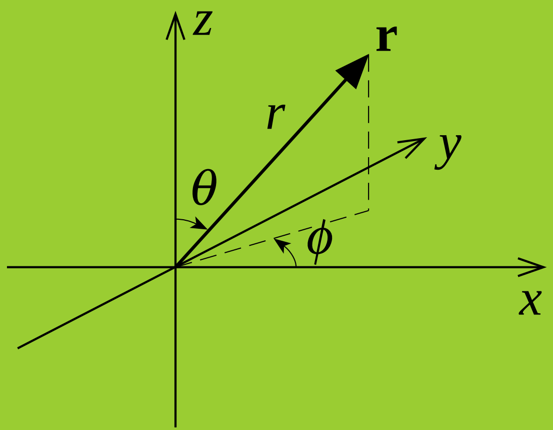
Joseph-Louis Lagrange

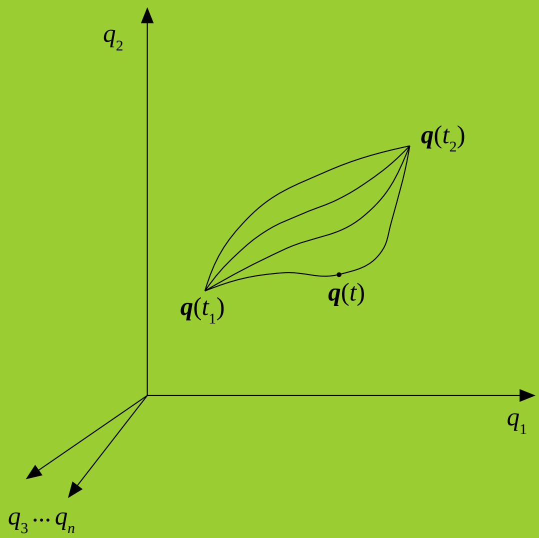
In 1787 he left Berlin to become a member of the Académie des Sciences in Paris, where he remained for the rest of his career. He was saved from arrest as an enemy alien during the Reign of Terror by Lavoisier (who wasn't so lucky himself—he was guillotined).

Generalized Coordinates

- Suppose force $F_i(\mathbf{r}, \dot{\mathbf{r}}, t)$, $i = 1, 2, 3$ acts on a particle of mass m. If coordinate system is Cartesian, then the *equations of motion* are the set of three second-order *differential equations* $m\ddot{x}_i - F_i = 0$.
- Consider a set of N Newtonian point masses interacting by various forces. There are then $3N$ equations of motion. Dynamics of a particle described in a $3N$ dimensional configuration space with *generalized coordinates* $q_1, ..., q_{3N}$. No particular metric is assumed.
- Whether the point masses are real particles like electrons, composite particles like nuclei or atoms, or mathematical idealizations like the infinitesimal volume elements in a continuum description, we shall refer to them generically as "particles".

Configuration Space





Big conceptual advance: Instead of thinking of a system as being made up of *many* points in 3-space, think of it as *one* point in the n-dimensional *configuration space* of the *generalized coordinates*. As time t changes, the point sweeps out a *path* through confuration space.

Holonomic Constraints

In general, the dimensionality n of the configuration space for an N-particle system in 3-space is $3N - m$, where m is the number of holonomic constraints,

$$f_j(\boldsymbol{q}) = 0 \; , \quad j = 1, 2, \cdots, m < 3N \; . \tag{1}$$

Example 1: Two particles are connected by a rigid rod so they are constrained to move a fixed distance apart.

Let the position of particle 1 with respect to a stationary Cartesian frame be $\{x_1, y_1, z_1\}$ and that of particle 2 be $\{x_2, y_2, z_2\}$. The rigid rod constraint equation is then

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l^2 \; . \tag{2}$$

Eq. 2 is a *holonomic* constraint, which *reduces* the number of degrees of freedom from 6 to 5.
Degrees of freedom could be taken be position of particle 1, $\{x_1, y_1, z_1\}$, and the spherical polar angles θ and ϕ.

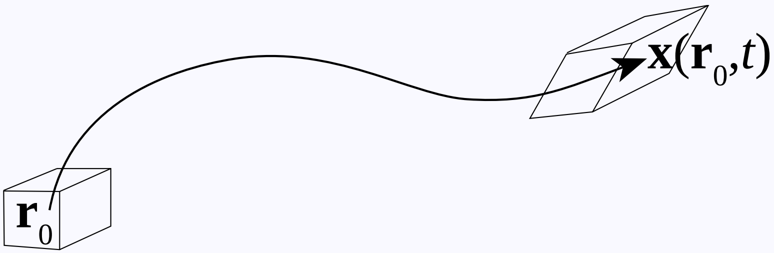
Example 2: The ideal fluid

Consider a fluid with density $\rho(\mathbf{r},t)$, presure $p(\mathbf{r},t)$ and velocity $\mathbf{v}(\mathbf{r},t)$ fields. Two different descriptions are,

Eulerian: fields indexed by actual positition, $\mathbf{r}$, and time t.

Lagrangian: fields indexed by initial positition, $\mathbf{r}_0$, of particle passing through point $\mathbf{r} = \mathbf{x}(\mathbf{r}_0, t)$.

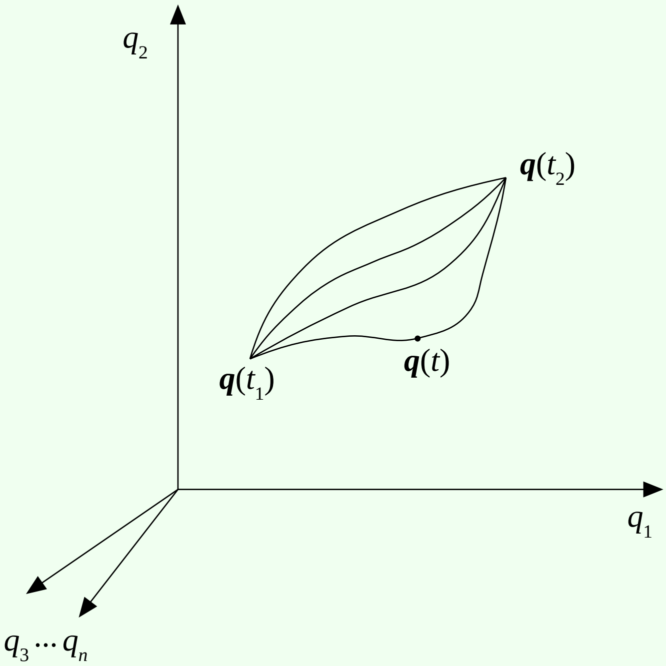
- $\mathbf{x}(\mathbf{r}_0, t)$ can be regarded as an infinite set of generalized coordinates.
- volume elements related by Jacobian $J(\mathbf{r}_0, t) : dV = J(\mathbf{r}_0, t)dV_0$.
- mass conservation, equation of state act as holonomic constraints $\rightarrow \rho, p$ are not additional generalized coordinates.



A fluid element advected from point $\mathbf{r} = \mathbf{r}_0$ at time $t = 0$ to $\mathbf{r} = \mathbf{x}(\mathbf{r}_0, t)$ at time t.

Variation of paths in configuration space

- Assume that the dimensionality n of the configuration space of generalized coordinates $\boldsymbol{q} = \{q_1, q_2, \ldots, q_n\}$ has been reduced to a minimim by taking into account all holonomic constraints.
- Consider arbitrary variations of the path between two fixed initial and final points.



Variational Calculus

Consider an *objective functional* $I[\boldsymbol{q}]$, defined on the space of all differentiable paths between two points in configuration space, $\boldsymbol{q}(t_1)$ and $\boldsymbol{q}(t_2)$

$$I[\boldsymbol{q}] \equiv \int_{t_1}^{t_2} \mathrm{d}t\, f(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) \; . \tag{3}$$

Making arbitrary variations $\delta\boldsymbol{q}(t)$ in the path and integrating by parts gives

$$\begin{aligned} \delta I[\boldsymbol{q}] &\equiv \int_{t_1}^{t_2} \mathrm{d}t \left[\delta\boldsymbol{q}(t)\boldsymbol{\cdot}\frac{\partial f}{\partial \boldsymbol{q}} + \delta\dot{\boldsymbol{q}}(t)\boldsymbol{\cdot}\frac{\partial f}{\partial \dot{\boldsymbol{q}}}\right] \\ &= \left[\delta\boldsymbol{q}\boldsymbol{\cdot}\frac{\partial f}{\partial \dot{\boldsymbol{q}}}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathrm{d}t\, \delta\boldsymbol{q}(t)\boldsymbol{\cdot}\frac{\delta f}{\delta \boldsymbol{q}} \; , \end{aligned} \tag{4}$$

where

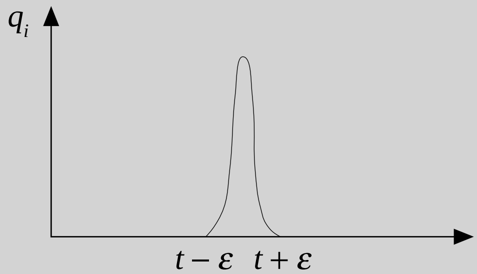
$$\frac{\delta f}{\delta \boldsymbol{q}} \equiv \frac{\partial f}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{\boldsymbol{q}}} \; . \tag{5}$$

Euler-Lagrange Equations

Require $\delta I = 0$ for all functions $\delta \boldsymbol{q}(t)$ such that $\delta \boldsymbol{q}(t_1) = \delta \boldsymbol{q}(t_2) = 0$. From eq. (4) we have

$$\int_{t_1}^{t_2} \mathrm{d}t\, \delta \boldsymbol{q}(t) \boldsymbol{\cdot} \frac{\delta f}{\delta \boldsymbol{q}} = 0 \tag{6}$$

for $\delta \boldsymbol{q}(t)$ arbitrarily localized in t:



Equation 6 can be satisfied for such variations iff

$$\frac{\delta f}{\delta \boldsymbol{q}} \equiv \frac{\partial f}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial \dot{\boldsymbol{q}}} = 0\ . \tag{7}$$

at each value of t. This represents n equations — the *Euler–Lagrange* equations.

Variational approx.: trial functions

Aim : Solve for q s.t.$\delta f/\delta q = 0$.
Approach : variational principles can be used to derive "the best" approximation: use a *trial function*

$$\boldsymbol{q}(t) = \boldsymbol{q}_K(t, a_1, a_2, \ldots, a_K)$$

where $\boldsymbol{q}_K$ is (hopefully) an approximating function involving a finite number of parameters a_k, $k = 1, \ldots, K$ to be determined variationally:

$$\delta I = \sum_{k=1}^{K} \frac{\partial I}{\partial a_k} \delta a_k = 0\ . \tag{8}$$

The condition for a stationary point is thus

$$\frac{\partial I}{\partial a_k} = 0, \quad k = 1, \ldots, K\ , \tag{9}$$

that is, that the K-dimensional gradient of I vanish.

Lagrange Multipliers

Aim : Generalize method to handle auxillary constraints, $\delta f^{(j)} = 0$.
General Idea : Find a transformation of $\delta f / \delta \boldsymbol{q}$ to get $\delta f / \delta \boldsymbol{q} = 0$.

- For holonomic problems, the set of differential forms

$$\sum_{i=1}^{n} \omega_i^{(j)}(\boldsymbol{q})\, \mathrm{d}q_i = 0 \;, \tag{10}$$

  can be integrated to give $f_j(\boldsymbol{q}) = 0, j = 1, 2, .., n$.

- Suppose $\exists m < n$ auxilliary constraints of form

$$\delta f^{(j)} \equiv \boldsymbol{\omega}^{(j)}(\boldsymbol{q}, t)\boldsymbol{\cdot}\delta \boldsymbol{q} = 0 \;. \tag{11}$$

  - vectors $\boldsymbol{\omega}^{(j)}$, $j = 1, \ldots, m$ are linearly independent, & span an m-dimensional subspace, $V_m(t)$, of n-dimensional vector space V_n occupied by the unconstrained variations.
  - Eqs. (11) constrain the variations $\delta \boldsymbol{q}$ to lie within an $(n - m)$-dimensional subspace, $V_{n-m}(\boldsymbol{q}, t)$, complementary to V_n.

- Constrained variational problem now reads

$$\frac{\delta f}{\delta \boldsymbol{q}} \cdot \delta \boldsymbol{q} = 0 \quad \forall\, \delta \boldsymbol{q} \in V_{n-m}(\boldsymbol{q}, t)\ . \tag{12}$$

- Solution by Lagrange: Eq.(12) says the *projection*, $(\delta f / \delta \boldsymbol{q})_{n-m}$, of $\delta f / \delta \boldsymbol{q}$ into $V_{n-m}(\boldsymbol{q}, t)$ is required to vanish.
- Rewrite

$$(\delta f / \delta \boldsymbol{q})_{n-m} \longrightarrow \delta f / \delta \boldsymbol{q} - (\delta f / \delta \boldsymbol{q})_m, \tag{13}$$

with $(\delta f / \delta \boldsymbol{q})_m = -\sum \lambda_j \boldsymbol{\omega}^{(j)}$.
- Variational formulation reads

$$\left[\frac{\delta f}{\delta \boldsymbol{q}} + \sum_{j=1}^{m} \lambda_j \boldsymbol{\omega}^{(j)} \right] \cdot \delta \boldsymbol{q} = 0 \quad \forall\, \delta \boldsymbol{q} \in V_n(\boldsymbol{q}, t)\ . \tag{14}$$

That is, *by using the Lagrange multipliers we have turned the constrained variational problem into an unconstrained one.*

Preparation for next lecture

Revise lecture notes from PHYS3001: Least Action in Physics