Classical Mechanics Questions for C01 exam

Q1. [20 marks total]

Express your own insights into/interpretations of the relative merits of the Newtonian, Lagrangian and Hamiltonian formulations of mechanics, from the material covered in this course. Include in your discussion

- the role played by Hamilton's principle in the Lagrangian and Hamiltonian approaches and give a major advantage of this principle, as a basis of mechanics, over Newton's laws;
- symmetries and their relation to conservation laws;
- mechanics as a dynamical system in the Lagrangian and Hamiltonian approaches.

Q2. [20 marks total]

(a) [2] Write down the Lagrangian in cylindrical polar coordinates (r,φ,z), $r \equiv (x^2+y^2)^{1/2}$, for a particle of charge e and mass m moving in an arbitrary scalar potential Φ and a vector potential $\mathbf{A} = \mathbf{e}_z A_z$, where $\mathbf{e}_z$ is the unit vector in the z-direction.

(b) [5] Hence or otherwise show that the corresponding Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + \frac{(p_z - eA_z)^2}{2m} + e\Phi .$$

(c) [9] Given that the potentials are independent of φ and z, write down two constants of the motion. Show that the particle motion in r can be described using a one-dimensional Hamiltonian with an effective potential $V_{\text{eff}}(r)$. Given that the potentials are time-independent, write down another integral of the motion. Write down the one-dimensional Hamiltonian equations of motion and discuss the r-motion of the particle qualitatively assuming $V_{\text{eff}}(r)$ goes to positive infinity as r approaches 0, and considering the two cases $V_{\text{eff}}(r) \to \pm\infty$ as $r \to \infty$. Sketch a phase-space diagram for the motions in the two cases.

(d) [4] The scalar and vector potentials for the electric and magnetic fields produced by a long straight filament carrying current I in the z-direction and charged uniformly with a line density λ are $\Phi(r) = \Phi_0 \ln\frac{a}{r}$ and $\mathbf{A}(r) = \mathbf{e}_z \Psi_0 \ln\frac{a}{r}$, respectively, where r is the distance from the filament (the z-axis), a is an arbitrary reference distance, and the constants Φ_0 and Ψ_0 are defined in terms of I and λ by $\Phi_0 = \frac{\lambda}{2\pi\varepsilon_0}$, $\Psi_0 = \frac{\mu_0 I}{2\pi}$.

Show that a charged particle in an evacuated vessel containing such a filament is confined by the magnetic field (i.e. the motion is bounded in r) even when the charge on the particle and on the filament are of the same sign. (Consider the initial conditions $r = a, \varphi = z = 0, \dot{r} = v_0 > 0, \dot{\varphi} = \omega_0 > 0, \dot{z} = 0$.)