C01 Classical Mechanics

Examination: 3pm–5pm, 2 May 1998

Attempt all questions. The maximum marks for each question (totalling 50) are indicated in square brackets [...].

Question 1 [Total marks 16]

(a) [2] Show that the geodesic joining two points in the Euclidean plane is a straight line. [Hint: in Sec. 1.4.1 take $\tau = x$ so that the length element becomes $\mathrm{d}l = \sqrt{1+y'^2}\,\mathrm{d}x$, where $y'(x) \equiv \mathrm{d}y/\,\mathrm{d}x$.]

(b) [3] If f does not depend explicitly on x in the functional

$$I[y] = \int_{x_1}^{x_2} f(y, y')\ \mathrm{d}x\ ,$$

show that functions $y(x)$ for which $I[y]$ is stationary with respect to infinitesimal variations δy satisfy the identity

$$y' \frac{\partial f}{\partial y'} - f = \text{const}\ .$$

What is the corresponding result in dynamics?

Two point masses m_1 and m_2 are joined by a rigid massless rod of length l, and they can slide freely, under the influence of gravity, on a frictionless surface inclined at an angle ψ to the horizontal.

(c) [2] How many degrees of freedom has this system? Explain the reasoning behind your answer.

(d) [3] Choose generalised coordinates (draw a diagram) and write down a Lagrangian for the system.

(e) [2] Write down as many independent integrals of the motion as you can.

(f) [4] Initially the rod is spinning with an angular speed ω, and the centre of mass is at rest. Describe mathematically, in words and with a diagram the subsequent motion.

Question 2 [Total marks 17]

One-dimensional Newtonian dynamics is invariant under the transformation $x = x' + ut$, where x is the coordinate measured in the laboratory frame and

x' is the coordinate measured in a frame moving with uniform velocity u.

(a) [2] Draw a diagram to visualise the Galilean transformation described above and work out the transformation equations for velocity, $\dot{x}$, and for momentum, $p \equiv m\dot{x}$.

(b) [1] Verify that the Lagrangians in the laboratory and moving frames, $L = \frac{1}{2}m\dot{x}^2$ and $L' = \frac{1}{2}m\dot{x}'^2$ respectively, both give valid equations of motion for a free-particle (i.e. one not acted on by any forces).

(c) [4] Find the Lagrangian (denoted $\tilde{L}$, say) in the moving frame by the point transformation procedure in Section 2.6.1 of the notes. Show the Lagrangian equations of motion derived from $\tilde{L}$ are the same as those derived from L' but the Hamiltonian is different. Resolve this paradox.

(d) [2] Show that the generating function

$$F_2(x, p') = (x - ut)(p' + mu)$$

generates the Galilean transformation equations for position and momentum and that

$$F_2(x, J) = kxJ \ ,$$

where k is a constant, generates a stretching transformation to a new coordinate $\theta = kx$, and its canonically conjugate momentum J.

(e) [5] A particle of charge e and mass m moves in an electrostatic plasma wave with electric potential given by

$$\varphi(x, t) = \varphi_0 \cos(kx - \omega t) \ , \tag{1}$$

where the wavevector k, amplitude φ_0 and frequency ω are constants. By making a Galilean canonical transformation to a frame moving with the phase velocity, ω/k, combined with a stretching transformation, show that the Hamiltonian can be made the same, with appropriate identifications, as that of the physical pendulum.

(f) [3] Sketch typical particle orbits in phase space over two wavelengths in the x-direction and show that the wave separates phase space into three regions: particles that move slower than the wave, particles that move faster than the wave, and particles that move on average at the same speed as the wave. Estimate (to within a numerical factor) the average width, $\Delta v = \Delta p/m$, in velocity of the latter region.

Question 3 [Total marks 16]

A particle of charge e and mass m is in a constant magnetic field $B = m\omega_{\rm c}/e$ directed parallel to the z-axis and a constant electric field E directed parallel to the x-axis.

(a) [1] Verify that the electrostatic potential

$$\Phi = -Ex$$

and the vector potential

$$\mathbf{A} = \left(\frac{m\omega_{\rm c}}{e}\right) x\mathbf{e}_y$$

give the fields as specified and that the Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{(p_y - m\omega_{\rm c}x)^2}{2m} + \frac{p_z^2}{2m} - eEx \ .$$

(b) [2] Write down the Hamiltonian equations of motion and give three integrals of the motion.

(c) [4] Show that x obeys the equation for a harmonic oscillator subject to a forcing term,

$$\ddot{x} + \omega_{\rm c}^2 x = \frac{p_y\omega_{\rm c} + eE}{m} \ ,$$

and that this is satisfied by

$$x = r_{\rm L}\sin\omega_{\rm c}t + \frac{(p_y\omega_{\rm c} + eE)}{m\omega_{\rm c}^2} \ ,$$

where $r_{\rm L}$ is an arbitrary constant (the Larmor radius).

(d) [3] By calculating the time averages $\langle\dot{x}\rangle$ and $\langle\dot{y}\rangle$ show that the particle drifts perpendicular to $\mathbf{B}$ at a speed E/B in the $\mathbf{E}\times\mathbf{B}$ direction.

(e) [4] Construct the general solution for y and z and sketch the projection of the motion in the x, y plane.

(f) [2] Now suppose the magnetic field is slowly increased until it is twice its original value. Show that the Larmor radius decreases by a factor $1/\sqrt{2}$.