**Classical Mechanics Questions for C01 exam, June 2001**

Q1. [20 marks total]

Consider a system with two degrees of freedom described by the generalized coordinates \( q_1 \) and \( q_2 \). Suppose the kinetic and potential energies are given by

\[
T = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2), \quad V = f(q_1 - q_2)
\]

where \( f \) is a given function.

(a) Write down a Lagrangian and Hamiltonian for the system.

(b) Find the configuration-space and phase-space equations of motion and determine the general solutions for \( q_1 \) and \( q_2 \) as functions of \( t \) if \( f(x) = x^2 \).

(c) Is the system autonomous? Is there a corresponding constant of the motion (and what is it)?

(d) Find a continuous symmetry of the system (other than time-translation invariance) and use Noether’s theorem to find the corresponding conserved quantity.

(e) By choosing suitable new coordinates, \( q_1' \) and \( q_2' \), find a new Lagrangian that is decoupled into a sum of Lagrangians for two independent one-degree-of-freedom subsystems.

Q2. [20 marks total]

(a) Write down the Lagrangian in cylindrical polar coordinates \((r, \varphi, z)\), \( r \equiv (x^2 + y^2)^{1/2} \), for a particle of charge \( e \) and mass \( m \) moving in an arbitrary scalar potential \( \Phi \) and a vector potential \( A = e_z A_z \), where \( e_z \) is the unit vector in the \( z \)-direction.

(b) Hence or otherwise show that the corresponding Hamiltonian is

\[
H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + \frac{(p_z - eA_z)^2}{2m} + e\Phi.
\]

(c) Given that the potentials are independent of \( \varphi \) and \( z \), write down two constants of the motion. Show that the particle motion in \( r \) can be described using a one-dimensional Hamiltonian with an effective potential \( V_{\text{eff}}(r) \). Given that the potentials are time-independent, write down another integral of the motion. Write down the one-dimensional Hamiltonian equations of motion and discuss the \( r \)-motion of the particle qualitatively assuming \( V_{\text{eff}}(r) \) goes to positive infinity as \( r \) approaches 0, and considering the two cases \( V_{\text{eff}}(r) \to \pm \infty \) as \( r \to \infty \). Sketch a phase-space diagram for the motions in the two cases.