Classical Mechanics Questions for C01 exam, June 2001

Q1. [20 marks total]

Consider a system with two degrees of freedom described by the generalized coordinates q_1 and q_2. Suppose the kinetic and potential energies are given by

$$T = \tfrac{1}{2}(\dot{q}_1^{\,2} + \dot{q}_2^{\,2}) \,, \qquad V = f(q_1 - q_2)$$

where f is a given function.

(a) Write down a Lagrangian and Hamiltonian for the system.

(b) Find the configuration-space and phase-space equations of motion and determine the general solutions for q_1 and q_2 as functions of t if $f(x) = x^2$.

(c) Is the system autonomous? Is there a corresponding constant of the motion (and what is it)?

(d) Find a continuous symmetry of the system (other than time-translation invariance) and use Noether's theorem to find the corresponding conserved quantity.

(e) By choosing suitable new coordinates, q_1' and q_2', find a new Lagrangian that is decoupled into a sum of Lagrangians for two independent one-degree-of-freedom subsystems.

Q2. [20 marks total]

(a) Write down the Lagrangian in cylindrical polar coordinates (r, φ, z), $r \equiv (x^2 + y^2)^{1/2}$, for a particle of charge e and mass m moving in an arbitrary scalar potential Φ and a vector potential $\mathbf{A} = \mathbf{e}_z A_z$, where $\mathbf{e}_z$ is the unit vector in the z-direction.

(b) Hence or otherwise show that the corresponding Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + \frac{(p_z - eA_z)^2}{2m} + e\Phi \,.$$

(c) Given that the potentials are independent of φ and z, write down two constants of the motion. Show that the particle motion in r can be described using a one-dimensional Hamiltonian with an effective potential $V_{\text{eff}}(r)$. Given that the potentials are time-independent, write down another integral of the motion. Write down the one-dimensional Hamiltonian equations of motion and discuss the r -motion of the particle qualitatively assuming $V_{\text{eff}}(r)$ goes to positive infinity as r approaches 0, and considering the two cases $V_{\text{eff}}(r) \to \pm\infty$ as $r \to \infty$. Sketch a phase-space diagram for the motions in the two cases.