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Interrelation between various branches of stable solitons in dissipative systems—conjecture for stability criterion

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Abstract

We show that the complex cubic-quintic Ginzburg–Landau equation has a multiplicity of soliton solutions for the same set of equation parameters. They can either be stable or unstable. We show that the branches of stable solitons can be interrelated, i.e. stable solitons of one branch can be transformed into stable solitons of another branch when the parameters of the system are changed. This connection occurs via some branches of unstable solutions. The transformation occurs at the points of bifurcation. Based on these results, we propose a conjecture for a stability criterion for solitons in dissipative systems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Coherent structures in dissipative systems play an important role in their dynamics. In optics, dissipative systems can be realized in nonlinear devices with gain and loss. Particular examples include wide-aperture lasers, all-optical transmission lines and passively mode-locked lasers. The generation of spatio-temporal dissipative structures in wide-aperture lasers [1–8] can be modeled by the $(2+1)$-dimensional complex Ginzburg–Landau equation (CGLE). This equation admits solutions in the form of various patterns, as well as vortex solitons [9]. Coherent structures, in the form of $(1+1)$-dimensional temporal solitons, appear in optical fiber systems with linear and nonlinear gain and spectral filtering (such as communication links with lumped fast saturable absorbers [10–14]. Of particular interest for us are passively mode-locked lasers such as fiber lasers with additive-pulse mode-locking or nonlinear polarization rotation [15–24]). Stable operation of these systems generating ultra-short pulses is crucial for practical purposes. Their stability is closely related to the issue of soliton stability in the theory of dissipative systems.

For the $(1+1)$-dimensional cubic-quintic CGLE, rather complete information can be obtained on the existence and competition of several types of coherent structures, such as fronts, pulses,

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sinks and sources [25]. The existence of soliton-like solutions of the quintic CGLE in the case of subcritical bifurcations ($\epsilon > 0$) has been numerically determined [26,27]. More recently, the regions in the parameter space where stable pulse-like solutions exist were found for the cases of anomalous [28] and normal dispersion [29]. A qualitative analysis of the transformation of the regions of existence of the pulse-like solutions, when the dissipative terms change from zero to infinity, was presented in Ref. [30].

Analytic soliton solutions to this equation in the form of stationary pulses are known (see Ref. [31] and references therein). However, analytic solutions exist only when there is a certain relation between the parameters, and most of them are unstable [28]. Numerical studies provide more branches of solitons, but require a large number of simulations. Knowledge of the complete set of solutions, even for some restricted range of parameters, is very important. The reason is that each stationary solution, even if it is unstable, is a singular point in the infinite-dimensional phase space of the system and it may play a decisive role in the complicated and rich pulse dynamics.

We already know [32] that, even in one dimension, solitons in dissipative systems appear in some multiplicity. A similar phenomenon is known for solitons in Hamiltonian systems [33]. However, there is a significant difference between solitons in Hamiltonian and dissipative systems. In Hamiltonian systems, soliton solutions appear as a result of a balance between diffraction (dispersion) and nonlinearity. Diffraction spreads the beam while nonlinearity focuses it and makes it narrower. The balance between the two opposed effects results in stationary solutions, which are usually a one-parameter family. In systems with gain and loss, in order to have stationary solutions, gain and loss must also be balanced. This situation is illustrated qualitatively in Fig. 1, which shows that this additional balance imposes a second constraint, so that, as a result we get solutions which are fixed. The shape, amplitude and the width are all fixed and depend on the parameters of the equation. There can be exceptions to this rule, but the solutions are usually fixed (i.e. isolated from each other).

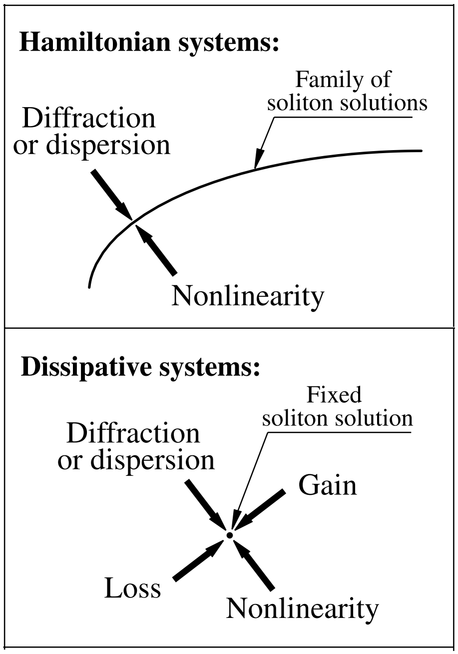

Fig. 1. Qualitative difference between the soliton solutions in Hamiltonian and dissipative systems. In Hamiltonian systems, soliton solutions are the result of a single balance, and comprise one- or few-parameter families, whereas, in dissipative systems, the soliton solutions are the result of a double balance and, in general, are isolated. On the other hand, it is quite possible for several isolated soliton solutions to exist for the same equation parameters.

On the other hand, more than one fixed solution can exist for a given set of parameters of the system. In some cases, we can observe up to five stable solutions in a given system [32]. Each type of solution may be stable in a certain region of the space of parameters. The regions for different solutions can overlap in a narrow region, resulting in multistability. Now, a question arises as to whether these known branches are interconnected or completely independent. In other words, is it possible to change the parameters of the system continuously in such a way that we could start from one stationary solution and transform it continuously to a completely different solution? If we restrict ourselves to stable solitons, this is certainly not the case. However, leaving aside questions of stability, we find an indication that we can recover all known soliton solutions by continuously changing the parameters. In this work, we have at least uncover the connection between plain and composite solitons [32,34]. This may mean that there is a connection between other soliton solutions as well.

The existence of stable and unstable solitons for the CGLE raises another important issue: when we change the parameters of the equation, is it possible that there are regions of stable and unstable solitons? If the answer is positive, when does such a transition occur? In other words, what is the stability criterion for solitons in dissipative systems? Up till now, we are not aware of any work which answers the above questions. There are works dealing with the stability criterion for ground state solitons in Hamiltonian systems [35–37]. Some approaches for higher order solitons have also been developed [38,39]. However, as we discussed above, solitons in dissipative systems are qualitatively different from solitons in Hamiltonian systems. As a result, the stability criterion for Hamiltonian systems cannot easily be generalized to the case of dissipative systems.

Recent works by Kapitula and Sanstede consider the stability of CGLE (dissipative) solitons when they are perturbations of nonlinear Schrödinger equation (NLSE) solitons [40]. This is important for optical transmission lines which are governed by the perturbed NLSE. However, in lasers, e.g., dissipative terms are strong and the approach developed in Ref. [40] is not sufficiently general. In this work, we present a numerical example which might encourage further studies on stability. In particular, we find that the points where the stability changes abruptly coincide with the turning points of the branches representing the different families of solitons. These branches are represented by curves which show any soliton parameter (usually its peak amplitude or propagation constant) versus one equation parameter (in this work ϵ). Wherever this curve folds around itself, the corresponding soliton solutions change their stability properties.

Solving the whole propagation equation [32,34] allows us to obtain only stable structures. However unstable solitons may play an essential role in the overall dynamics when the system starts with an arbitrary initial condition. Therefore, it is important to know all soliton solutions when moving in the parameter space from one point to another. The conclusion is that both types of stationary solitons, stable and unstable, are important and deserve careful study.

In the situation where a single transverse (or temporal) coordinate is retained in the analysis, the cubic-quintic CGLE reads as [15]

$$\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi = \mathrm{i}\delta\psi + \mathrm{i}\epsilon|\psi^2|\psi + \mathrm{i}\beta\psi_{tt} + \mathrm{i}\mu|\psi|^4\psi - \nu|\psi|^4\psi, \qquad (1)$$

where D, δ, β, ϵ, μ, and ν are real constants (we do not require them to be small). The CGLE applies, as we mentioned, to the problem of ultra-short pulse generation in passively mode-locked lasers. In this case, t is a retarded time, z is the number of round trips, ψ is the complex envelope of the optical field, D is the dispersion (diffraction) coefficient, δ gives account of the linear gain, β describes spectral filtering or parabolic gain ($\beta > 0$), ϵ accounts for nonlinear gain/absorption processes, μ represents a higher order correction to the nonlinear amplification/absorption, and ν is a possible higher order correction term to the intensity-dependent refractive index. By a proper rescaling and without loss of generality D can be restricted to have the values $D = \pm 1$. We use $D = +1$ throughout this work.

In the present work, we are interested in the whole set of soliton solutions. These include both stable and unstable solitons existing for a given set of values of the equation parameters. We find that different types of solutions, with particular characteristics, which were previously viewed as belonging to different branches of solutions, in fact belong to the same branch, and can be obtained by continuously changing one parameter of the CGLE. We also numerically study their stability properties over their range of existence and calculate the growth rates for the unstable branches. From this stability analysis, we shed some light on a stability criterion for these solutions.

The rest of the paper is organized as follows. In Section 2 we present the method for finding stationary localized solutions. Section 3 presents the results for continuous wave (CW) and soliton solutions which follow from our analysis. Section 4 presents the results for stability of solitons and presents the conjecture for a stability criterion. Finally Section 5 summarizes our main conclusions.

2. Analytical approach for finding stationary localized solutions

Exact analytical solutions can be found only for certain combinations of the values of the parameters [31]. On the other hand, we know that CGLE has soliton solutions which cannot be expressed analytically. Hence, we need to develop some technique to find stationary solutions. One way to do it is by reducing Eq. (1) to a set of ordinary differential equations (ODE). We do that by seeking solutions in the form:

$$\begin{aligned}\psi(t,z) &= \psi_0(\tau)\exp(-\mathrm{i}\omega z)\\ &= a(\tau)\exp[\mathrm{i}\phi(\tau)-\mathrm{i}\omega z],\end{aligned} \tag{2}$$

where a and ϕ are real functions of $\tau = t - vz$, v is the pulse velocity and ω is the nonlinear shift of the propagation constant. Eq. (2) is general representation of a complex function and covers all possible solutions. Substituting Eq. (2) into Eq. (1), we obtain an equation for two coupled functions, a and ϕ. Separating real and imaginary parts, we get the following set of two ODEs:

$$\begin{aligned}&\left[\omega - \tfrac{1}{2}D\phi'^2 + \beta\phi'' + v\phi'\right]a + 2\beta\phi' a'\\ &\quad + \tfrac{1}{2}Da'' + a^3 + \nu a^5 = 0,\\ &\Big(-\delta + \beta\phi'^2 + \tfrac{1}{2}D\phi''\Big)a + (D\phi' - v)a'\\ &\quad - \beta a'' - \epsilon a^3 - \mu a^5 = 0,\end{aligned} \tag{3}$$

where each prime denotes a derivative with respect to τ.

It can be transformed into:

$$\begin{aligned}&\omega a + v\frac{M}{a} - \frac{DM^2}{2a^3} + \frac{\beta M'}{a} + \frac{D}{2}a''\\ &\quad + a^3 + \nu a^5 = 0,\\ &-\delta a - va' + \frac{\beta M^2}{a^3} + \frac{DM'}{2a} - \beta a''\\ &\quad - \epsilon a^3 - \mu a^5 = 0,\end{aligned} \tag{4}$$

where $M = a^2\phi'$.

Separating derivatives, we obtain a dynamical system in standard form [42] with the first order derivatives on the left hand side:

$$\begin{aligned}M' &= \frac{2(D\delta - 2\beta\omega)}{1+4\beta^2}a^2 + \frac{2(D\epsilon - 2\beta)}{1+4\beta^2}a^4\\ &\quad + \frac{2(D\mu - 2\beta\nu)}{1+4\beta^2}a^6 - \frac{4\beta v}{1+4\beta^2}M\\ &\quad + \frac{2Dv}{1+4\beta^2}ay,\\ y' &= \frac{M^2}{a^3} - \frac{2(D\omega + 2\beta\delta)}{1+4\beta^2}a - \frac{2(D + 2\beta\epsilon)}{1+4\beta^2}a^3\\ &\quad - \frac{2(D\nu + 2\beta\mu)}{1+4\beta^2}a^5 - \frac{4\beta v}{1+4\beta^2}y\\ &\quad + \frac{2Dv}{1+4\beta^2}\frac{M}{a},\\ a' &= y.\end{aligned} \tag{5}$$

This set contains all stationary and uniformly translating solutions. The parameters v and ω are the eigenvalues of Eq. (5). In the (M,a) plane, the solutions corresponding to pulses are closed loops starting and ending at the origin. The latter happens only at certain values of v and ω. If v and ω differ from these fixed values, the trajectory cannot comprise a closed loop.

If we are only interested in zero-velocity ($v = 0$) solutions, Eq. (5) can be further simplified:

$$\begin{aligned}M' &= \frac{2(D\delta - 2\beta\omega)}{1+4\beta^2}a^2 + \frac{2(D\epsilon - 2\beta)}{1+4\beta^2}a^4\\ &\quad + \frac{2(D\mu - 2\beta\nu)}{1+4\beta^2}a^6,\\ y' &= \frac{M^2}{a^3} - \frac{2(D\omega + 2\beta\delta)}{1+4\beta^2}a - \frac{2(D + 2\beta\epsilon)}{1+4\beta^2}a^3\\ &\quad - \frac{2(D\nu + 2\beta\mu)}{1+4\beta^2}a^5,\\ a' &= y.\end{aligned} \tag{6}$$

This set of three coupled first order ODEs can be solved numerically. For localized solutions with correctly chosen ω, the amplitude a should go exponentially to zero outside the region of localization. The exponential part of the solution is unique as it can be found using the linear approximation. This way we leave out fronts, sources and sinks. Only the value of ω plays a role of an eigenvalue in this nonlinear problem. It is fixed for a given solution but other solutions have different values of ω unless there is a degeneracy.

The asymptotic behavior of Eq. (6) at small a is given by

$$a = a_0 \exp(g\tau),$$

$$M = \frac{D\delta - 2\beta\omega}{g(1+4\beta^2)} a_0^2 \exp(2g\tau),$$

where a_0 is an arbitrary small amplitude and the soliton tail exponent g can be found from the biquadratic equation

$$g^4 + \frac{2(D\omega + 2\beta\delta)}{1+4\beta^2} g^2 - \frac{(D\delta - 2\beta\omega)}{(1+4\beta^2)^2} = 0. \qquad (7)$$

Thus

$$g^2 = \pm\sqrt{\frac{\omega^2+\delta^2}{1+4\beta^2}} - \frac{D\omega + 2\beta\delta}{1+4\beta^2}. \qquad (8)$$

Using this approximation for the tails and adjusting properly the eigenvalue ω, it is possible to find the rest of the pulse solution with a shooting method.

3. Solitons and continuous wave solutions

Using the above described technique we studied stationary soliton solutions in a certain range of parameters. It would be a confusing and complicated task to try to change several parameters at once. So we restricted our study and we fixed all parameters except ϵ which was variable. Other parameters were chosen close to a region of transition between soliton solutions and fronts where multiplicity of solutions appear [33]. This region is relatively wide [33] and allows some variations. The particular choice of parameters is not critical and we got qualitatively similar results for other sets of parameters inside of the transitional region. In addition, there are some restrictions dictated by the underlying physics of the problem under consideration.

In particular, δ must be negative in order for the background to be stable, μ must be also negative to limit the amplitudes from above and β must be positive in order to stabilize the soliton in the frequency domain. These considerations show clearly that ϵ can be the only gain term (> 0) in this model. In a way, it is responsible for existence of any structure in this dissipative system. Hence, we can consider ϵ as the most important parameter and study the properties of solitons when ϵ changes.

As the cubic gain term is the only one which is positive, there is a minimum threshold value of ϵ below which all solutions die away. Soliton solutions exist only above this threshold. The numerical results are presented in the following set of figures. The solid curve in Fig. 2 shows the peak amplitude of the single pulse (SP) soliton branch versus ϵ. This is the branch of solitons with the "plain" or bell-shape profile which resamples usual *sech*-profile solitons. The shape of the soliton continuously changes when we change ϵ. As a result we have a "branch" of solitons. It extends from the threshold both up and down in amplitude. For every value of ϵ above the threshold we have two SP soliton solutions.

Soliton solutions are tightly related to CW solutions which exist in the system. To show this, in

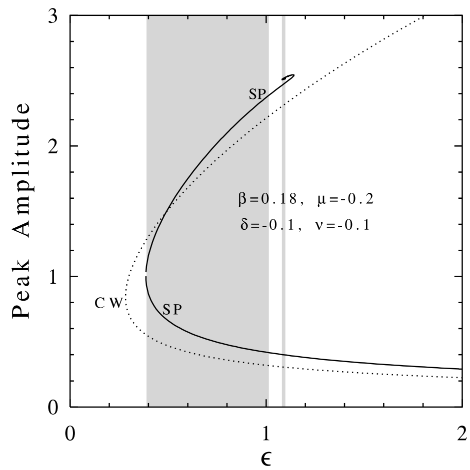

Fig. 2. Maximum SP soliton amplitude (solid curve) versus ϵ. The amplitude of the CW solutions (dotted curve) is shown for comparison. These two lines show the amplitudes a_+ and a_- of the two CW solutions with $M = 0$ as given by Eq. (9). The shadow represents the values of the parameter ϵ where SP solitons of the upper branch are stable.

dotted line we plotted the curve for the CW solution which is given by

$$a_{\pm} = \sqrt{\frac{-\epsilon \pm \sqrt{\epsilon^2 - 4\delta\mu}}{2\mu}}, \quad \omega = -\nu a_{\pm}^4 - a_{\pm}^2. \tag{9}$$

Just like for SP solitons, there are two types of CW solutions with low and high amplitudes respectively. They coincide at the lowest value of ϵ at which they exist, $\epsilon = (4\delta\mu)^{1/2}$. This is the threshold for existence of CW solutions. The low amplitude CW solutions are always unstable. However, the CW solutions of the upper branch have a chance to be stable. The stability of the upper branch CW relative to modulation instability can be studied analytically.

Similar to CW solutions, the lower branch of SP solitons is always unstable and the solitons of the upper branch have a chance to be stable. These features are due to the nonlinearity which in our case is quintic. Namely, the CGLE has at least three solutions including the trivial zero solution for CW as well as for solitons. Due to our choice of parameters, the trivial solution is stable, and the stability of the upper branch solitons is the main problem in this work. We stress again that SP soliton solutions of either branch can be found only numerically in contrast to CW solutions. Consequently, the stability of the SP solitons can reliably be studied only numerically.

The main qualitative difference of the curve for SP soliton solutions from the curve for CW solutions is that their amplitude is limited from above. In fact, we can see from Fig. 2 that the solid curve does not extend above the value approximately of $\epsilon = 1.15$. Instead, the curve starts to spiral at the upper end at around $\epsilon \approx 1.1$. The spiraling happens if we plot any other parameter of the solutions rather than the amplitude. For example, the propagation constant ω versus ϵ for solitons and CW solutions for the same data as in Fig. 2 is shown in Fig. 3. The spiraling of the soliton curve in Fig. 3 is clearly seen.

To show the spiraling of the curve for SP solitons in detail we calculated this curve with higher precision, i.e. increasing considerably the sampling in ϵ values. The result is shown in Fig. 4. The spiraling seems to happen indefinitely. Fig. 4(b) shows a magnification of the part of the curve shown in (a) which in turn has this spiraling feature. We do not have enough accuracy in our calculations to study this issue deeper. Fig. 5

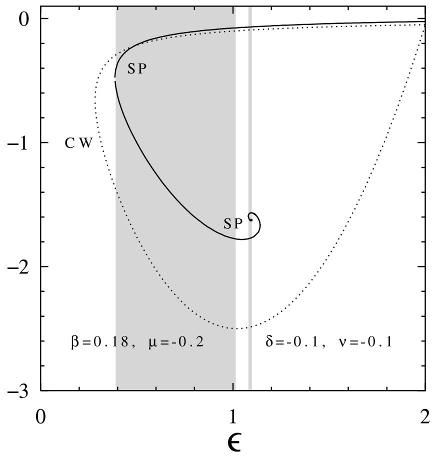

Fig. 3. Propagation constant ω versus ϵ for the same data as in Fig. 2.

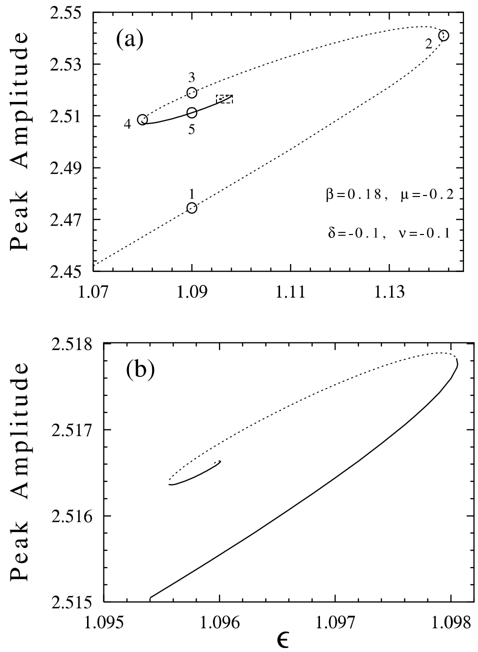

Fig. 4. (a) Maximum soliton amplitude versus ϵ for SP solitons at the top of the branch shown in Fig. 2. (b) Magnification of the curve shown in (a) inside the small rectangle.

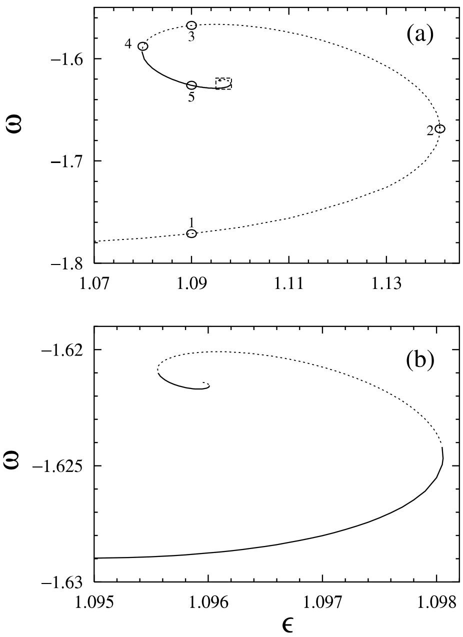

Fig. 5. (a) Propagation constant ω versus ϵ for the branch named SP. The parameters of the equation (except ϵ) are the same as in Fig. 2. (b) The part of (a) enclosed in a dashed line rectangle in a magnified scale.

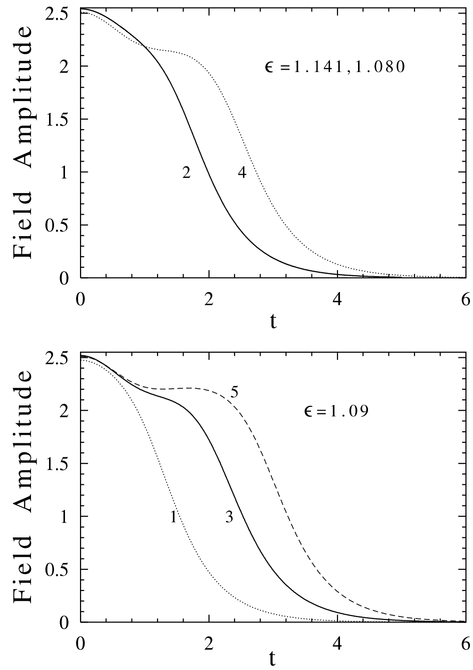

Fig. 6. Amplitude profiles of the stationary solutions marked in Figs. 4(a) and 5(a) by the open circles. The numbers below the curves correspond to the labels for the open circles in Figs. 4(a) and 5(a). The values of ϵ are shown in the figure.

shows the propagation constants for the curves presented in Fig. 4. These again have the same spiraling structure.

The amplitude profile of the SP soliton has plain bell-shape profile for almost any value of ϵ except for the spiraling parts. Examples of the amplitude profiles are shown in Fig. 6. Only half of the profiles is plotted as they are even functions of t. These examples correspond to the circles in Figs. 4 and 5 accordingly numbered. We can see clearly that before the curves start to spiral (the points 1 and 2), the amplitude profiles have simple bell shape. The amplitude profile resembles the profiles for composite solitons [34] inside the spirals (the points 3, 4 and 5). This fact may bring us to the conclusion that all soliton solutions (or at least those we know about) are interconnected, i.e. continuously changing parameters of CGLE we can transform one type of soliton into another. However, we do not have enough evidence for this far reaching idea.

4. Stability results

We have studied the stability of the SP solitons at every point of the solid curves in Figs. 2 and 3 with the main attention concentrated on the spiraling part of the curves. In numerics, we used the linear stability analysis, described in detail in Ref. [28] and verified its results through direct numerical solution of the CGLE when the initial input is one of the SP solitons. The stability analysis allows us to calculate the growth rates (g) for the unstable solitons. The zeros of the growth rate g versus ϵ curves give us the boundaries of instability. This way we were able to find which parts of the spiral correspond to stable solitons and which parts do not. In particular, we confirmed that the lower branch of the SP solitons is unstable up to the threshold point in Fig. 2. The SP solitons are stable above this point.

The growth rate curves calculated for SP solitons in the spiraling part of the curve in Fig. 2 are

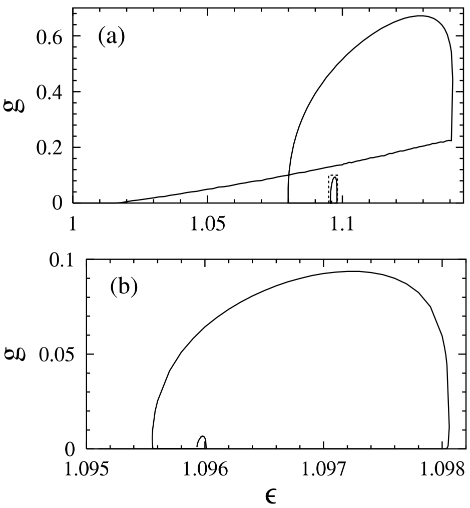

Fig. 7. (a) Perturbation growth rates versus ϵ for SP solitons around the spiral. (b) Magnification of the portion in (a) enclosed in a dashed rectangle.

shown in Fig. 7(a). The curve with the larger growth rate corresponds to the outer part of the spiral which is shown in magnified version in Fig. 4(a). The arc of the growth rate curve from $\epsilon = 1.08$ to 1.14 corresponds to the part of the spiral from point 4 to point 2. The lower part of the growth rate which gradually goes to zero at smaller ϵ corresponds to the spiral from point 2 to point 1 and further away from the spiral. Between the different portions of the growth rate curve with the points $g = 0$ SP solitons are stable. It is clear that we have eigenvalue transformations from being purely real to imaginary at the edges of soliton existence where g turns to zero. These are points of bifurcation as it follows from the general theory of dynamical systems [42]. The behavior of the solutions above and below the points of bifurcation are qualitatively different.

The two bifurcations in the stability of solitons located at points numbered 4 and 2 are qualitatively different. At point 4, the growth rate goes to zero and the next part of the spiral corresponds to stable solitons. On the other hand, at point 2 the mode of instability changes but the solitons remain unstable. Moving to the left along the branch for SP solitons, the value of the growth rate monotonically decreases and below $\epsilon = 1.02$ the SP solitons become stable up to $\epsilon = 0.4$ which is the absolute minimum point of the SP soliton existence. Turning down beyond this point, SP solitons, which now become low amplitude ones, are everywhere unstable. This turning point is qualitatively similar to the point 4.

The point 2 in the spiral is a special one. Although this is the point of bifurcation, the SP soliton is unstable at both sides of this point. What changes at this point is the mode of instability. Namely, using the standard stability analysis for solitons we can write the solution in the form

$$\Psi(t,z) = [\psi_0(t) + \alpha f(t)\exp(gz)]\exp(-\mathrm{i}\omega z), \qquad (10)$$

where $\psi_0(t)$ is a stationary soliton solution, α is a small parameter, $f(t)$ is the perturbation function and g is its growth rate. Substituting Eq. (10) into Eq. (1), assuming that α is small and linearizing around the soliton solution we will get an equation for eigenfunctions $f(t)$ and eigenvalues g. For details see e.g. Section 13.7 of Ref. [33]. This equation can be solved numerically. This way we can find the perturbation function $f(t)$. We can also call it "perturbation eigenfunction".

Fig. 8(a) shows the perturbation function just below the point 2 while Fig. 8(b) shows the perturbation function just above. It is clear that we

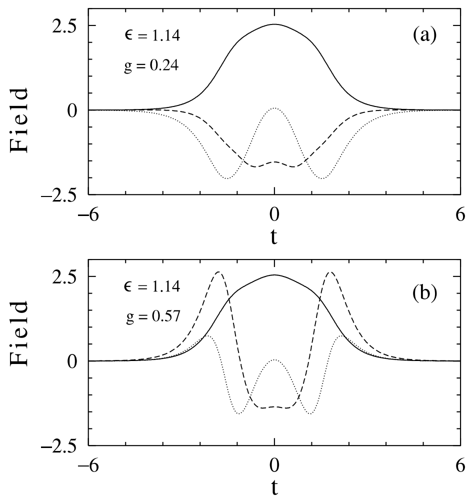

Fig. 8. Real (dashed line) and imaginary (dotted line) parts of the eigenfunction of perturbation for unstable SP soliton (a) below the point 2 and (b) above the point 2. Solid line shows the SP soliton profile for each case.

have a bifurcation at point 2 where the mode of perturbation changes. Moreover, the eigenvalue of the perturbation transforms from being complex with imaginary part different of zero below the point 2 to be purely real above the point 2. Its real part is the growth rate g, and it is written inside each figure. The instability has purely exponential growth above the point 2 and it is oscillatory below the point 2. The oscillatory instability is related to the birth of pulsating solitons [41] rather than to radiation phenomena. This is confirmed by the fact that the eigenfunction of perturbation in this region is concentrated in the soliton center.

It appears clearly from these data that, the points of local minima and maxima of the range of soliton existence in the spiral are the points of bifurcation. Depending on the nature of the modes of perturbation these points might be the turning points of stability. This conjecture is confirmed with further investigation. The small curve in the dashed rectangle in Fig. 7(a) corresponds to the inner part of the spiral which is shown in Fig. 4(b). This small portion surrounded in dashed line is conveniently magnified in Fig. 7(b). Note the two different x- and y-scales. The smaller loop in Fig. 7(b) corresponds to the innermost part of the spiral in Fig. 4(b). To establish the correspondence between the stationary solutions and its stability deduced from this analysis, the stable soliton branches are shown in Figs. 4 and 5 by solid lines. These curves confirm our conjecture that local maxima and minima of ϵ (i.e. the local edges in region of soliton existence) are the points where stability changes. We believe that further magnification could reveal more detailed structure, but we are close to the accuracy limit of our numerical method.

The general results of the stability analysis are illustrated in Figs. 2 and 3. The shadowed area in these figures show the values of ϵ where SP solitons of the higher amplitude are stable. The edges of stability are related to the local minima or maxima of ϵ on the curve for the SP solitons. An exception is the right hand side edge of the wide stripe around $\epsilon = 1$ where stability changes gradually. In between both shadowed areas, any input tends to give place to a pulsating solution in agreement with the above results on oscillatory instability.

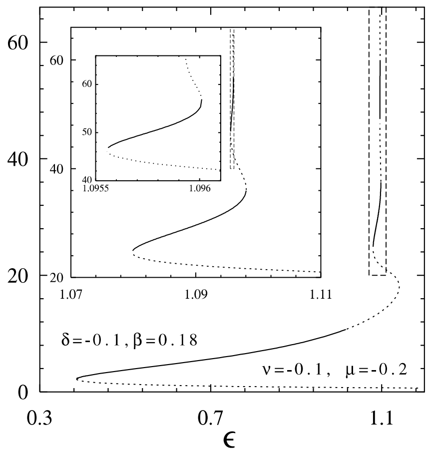

Fig. 9. Energy of the soliton Q versus ϵ for SP solitons. Stable solitons are shown by solid line and unstable solitons by dotted line. The parameters of the equation (except ϵ) are the same as in Fig. 2. Two consecutive magnifications of the small parts of the curve enclosed in dashed rectangles are shown in the insets.

The conjecture for soliton stability can be further refined if we plot as one of the parameters a measurable quantity, namely the soliton energy versus ϵ. The energy, $Q = \int_{-\infty}^{\infty} |\Psi_0(t)|^2 \, \mathrm{d}t$, has been calculated numerically for each point of the SP soliton branch. The value of Q versus ϵ for the SP solitons is shown in Fig. 9. Energy increases to infinity while we move in along the spiral in Fig. 2. The reason is that the width of the soliton increases indefinitely. The soliton becomes a composite structure consisting of the central peak and two fronts attached to it from both sides. There is an indication that the curve can be continued indefinitely in the direction of large energy Q. However, when the width of the soliton becomes large, the binding energy between the central peak and the two fronts is weak. Stability of these higher order solutions is also weak.

Comparison with the above stability results shows that the parts of the curve in Fig. 9 with negative slope are all unstable. More accurately, the solid parts of the curve in Fig. 9 correspond to stable solitons and the dotted parts to unstable solitons. As we can see, the stability change happens at the local minima and maxima of ϵ vs. Q, i.e. at the local edges of soliton existence in ϵ parameter. The point 2 is not an exception. It represents a point of bifurcation as discussed above but the modes of perturbation on each side of this maximum are different. As a result, the soliton has different type of instability on each side of this point.

Summarizing, we can formulate our conjecture for the stability criterion as follows. The local edges of soliton existence in ϵ-parameter are the points of bifurcation where stability of solitons changes. Between the points of bifurcation, the solitons are unstable if the slope of energy versus ϵ curve is negative. The opposite is not necessarily true, i.e. solitons might be unstable even if the slope is positive.

Clearly, numerical studies are restricted and cannot give the complete answer to the problem of soliton stability in dissipative systems. As it always happen, this stability criterion might work only for certain class of soliton solutions. Nevertheless, we are sure that our brief study will serve as a sparkle which at least will ignite the interest to this important problem. In this regard, we recall that even in the case of Hamiltonian systems the problem of soliton stability is not completely solved and research in this direction continues. We also think that an extension of this study to the $(2+1)$-dimensional case might give valuable information about spatio-temporal structures in wide-aperture lasers [1–8].

5. Conclusion

In conclusion, we studied stationary soliton solutions for the cubic-quintic CGLE which models a passively mode-locked laser system. This study revealed interconnections between several soliton branches which had been considered completely independent before. In particular, we found a branch of soliton solutions of the quintic CGLE which interconnects SP solitons and composite solitons. Moreover, this study allowed us to make a conjecture about the soliton stability. We found that the boundaries of soliton existence in ϵ-parameter are the points of bifurcation where the stability of solitons changes. We have also found that between the points of bifurcation, the solitons are unstable if the slope of energy versus ϵ curve is negative.

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