EXACT FIRST-ORDER SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

N. N. Akhmediev, V. M. Eleonskii, and
N. E. Kulagin

A method is proposed for finding exact solutions of the nonlinear Schrödinger equation. It uses an ansatz in which the real and imaginary parts of the unknown function are connected by a linear relation with coefficients that depend only on the time. The method consists of constructing a system of ordinary differential equations whose solutions determine solutions of the nonlinear Schrödinger equation. The obtained solutions form a three-parameter family that can be expressed in terms of elliptic Jacobi functions and the incomplete elliptic integral of the third kind. In the general case, the obtained solutions are periodic with respect to the spatial variable and doubly periodic with respect to the time. Special cases for which the solutions can be expressed in terms of elliptic Jacobi functions and elementary functions are considered in detail. Possible fields of practical applications of the solutions are mentioned.

1. Introduction

The nonlinear Schrödinger equation (NSE) is one of the representatives of the class of completely integrable partial differential equations that has great applied importance. The classical method of its solution is the inverse scattering method [1] and its generalizations to the case of periodic solutions [2-4]. Among the other effective methods of constructing solutions of the NSE we mention the method of Darboux transformations [5]. To construct and classify solutions in the present paper, we use an approach based on the direct association of finite-dimensional completely integrable dynamical systems with definite classes of solutions of the NSE. In this approach, the problem of constructing and classifying the NSE solutions reduces to constructing and analyzing qualitatively the phase portrait of a finite-dimensional completely integrable dynamical system. The basis of this approach was some observations [6,7] about known NSE solutions and also the view put forward in [8,9] about solutions of soliton type on both homoclinic and heteroclinic trajectories belonging to finite-dimensional separatrix manifolds in the corresponding infinite-dimensional phase space.

It is shown below that the finite-dimensional completely integrable system generated by the linear manifold is the support of a large and, from the applied point of view, important class of NSE solutions, including both the classical envelope soliton [1] as well as the modulation soliton [7], which determines the evolution of an unstable wave with constant amplitude. Numerical calculations as well as examination of known exact solutions of the NSE show that to the class of rational solitons, and also the class of spatially periodic solutions doubly asymptotic in the limits $t \to \pm\infty$ to a wave with constant amplitude correspond to finite-dimensional completely integrable systems generated by algebraic manifolds in the functional phase space of the NSE. In such an approach, it is natural to ask whether or not this possibility is characteristic of only the particular completely integrable model we consider, namely, the NSE. Although at the present time it is difficult for us to estimate the degree of generality and effectiveness of the proposed approach for all known integrable models, we may draw attention to the following observation relating to the completely integrable sine–Gordon equation

$$u_{tt}-u_{xx}+\sin u=0.$$

Using the substitution $u = 4 \tan^{-1} \psi$, which reduces the sine–Gordon equation to the form of an equation with algebraic nonlinearities, one can show that the basic solutions of soliton type (kinks and breathers) can be associated with algebraic manifolds $P(\psi, \psi_t; x) = 0$ or $P(\psi, \psi_x; t) = 0$. The latter may be associated with dynamical systems generated by the

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 72, No. 2, pp. 183-196, August, 1987. Original article submitted March 12, 1986.

0040-5779/87/7202-0809$12.50

class of solutions that decrease rapidly with respect to x or the class of solutions periodic in time.

The investigation and classification that follow below of the NSE solutions associated with identification of a linear manifold in the phase space of the NSE have as outcome not only the finding of exact solutions of the NSE previously unknown in explicit form (something that can be achieved by various methods [2-5]) but also illustrate the fact that relatively simple algebraic manifolds can be the carrier of a large class of solutions. We hope that our observations will facilitate investigations of different versions of the problem of direct association of completely integrable field equations or continuum equations with finite-dimensional completely integrable dynamical systems.

2. Construction of the Dynamical System

We write the NSE in the standard notation [1-4]:

$$i\psi_t+\psi_{xx}+2|\psi|^2\psi=0, \tag{1}$$

and, following [6], we suppose that the solutions satisfy a linear relationship between the real and imaginary parts of the unknown function $\psi = u + iv$:

$$u(x, t)=a_0(t)v(x, t)+b_0(t). \tag{2}$$

As in [6], we assume further that the coefficients a_0 and b_0 in (2) depend only on the variable t; it is this that will ultimately enable us to find the solution. If we go over to new functions $\varphi(t)$, $\delta(t)$, and $Q(x, t)$ by means of $a_0=\operatorname{ctg}\varphi$, $b_0=-\delta/\sin\varphi$, and $u=Q\cos\varphi-\delta\sin\varphi$. then the solution $\psi(x, t)$ will have the representation

$$\psi(x, t)=[Q(x, t)+i\delta(t)]e^{i\varphi(t)}, \tag{3}$$

in which only the function Q depends on the variable z.

Making the ansatz (3) in (1) and separating the real and imaginary parts, we obtain a system of equations for the function Q:

$$Q_{xx}-\delta_t-\varphi_tQ+2\delta^2Q+2Q^3=0, \tag{4}$$

$$Q_t-\varphi_t\delta+2\delta Q^2+2\delta^3=0. \tag{5}$$

Equation (4) admits the first integral

$$Q_x^2+Q^4+(2\delta^2-\varphi_t)Q^2-2\delta_tQ=h, \tag{6}$$

where $h = h(t)$ depends only on t. If the system of equations (5) and (6) is to be compatible, the Frobenius relation $Q_{xt} = Q_{tx}$ must hold. Substituting (5) and (6) in this relation and then equating its coefficients of equal powers of Q, we obtain a system of three differential equations:

$$8\delta\delta_t+\varphi_{tt}=0, \tag{7}$$

$$h_t+2\delta\delta_t\varphi_t-4\delta^3\delta_t=0, \tag{8}$$

$$4\delta h+\delta_{tt}-4\delta^3\varphi_t+\delta\varphi_t^2+4\delta^5=0. \tag{9}$$

The coefficients of Q to the fourth and fifth powers vanish identically.

Equations (7)-(9) can be integrated successively. The first integral of Eq. (7) has the form

$$4\delta^2+\varphi_t=W, \tag{10}$$

where W is a constant of integration. Substituting φ_t from (10) in Eq. (8), we find the second integral of the system

$$h+W\delta^2-3\delta^4=H, \tag{11}$$

in which H is a further constant of integration. Finally, using (10) and (11), we find an integral of Eq. (9):

$$\delta_t^2+(4H+W^2)\delta^2-8W\delta^4+16\delta^6=D, \tag{12}$$

where D is a third constant of integration. By means of the substitution $z = \delta^2$, Eqs. (12) and (10) can be rewritten in the form

$$z_t^2=-64z^4+32Wz^3-4(4H+W^2)z^2+4Dz, \tag{13}$$

$$\varphi_t=W-4z, \tag{14}$$

and Eq. (6) takes with allowance for (11) the form

$$Q_x^2=-Q^4+(W-6z)Q^2+\frac{z_t}{\sqrt{z}}Q+(H-Wz+3z^2). \tag{15}$$

Thus, if we do not consider shifts with respect to both variables and rotation in the complex plane through a constant angle, the class of solutions determined by the representations (2) or (3) is a three-parameter family of NSE solutions, and it can be found by successive solution of Eqs. (13), (14), and (15).

In the phase space of pairs of real functions (u, v), the relation (2) defines a linear manifold. The evolution of the linear manifold in time is determined by the solutions of the finite-dimensional dynamical system (7)-(9). The condition of compatibility of Eqs. (5) and (6) determines the correspondence between the evolution of the parameters of the linear manifold in time and the dependence of the solution on the spatial variable x.

The solutions of Eqs. (13) and (15) belong to the class of elliptic functions [10]. Reduction of Eq. (15) to the Weierstrass form [10] shows that its invariants g_2 and g_3 and, therefore, the period with respect to the variable x do not depend on the time and are determined by the constants of the first integrals of the dynamical system (7)-(9): $g_2 = W^2/12 - H$, $g_3 = D/4 - (H + W^2/36)W/6$. The relations given in this section completely determine the classification of the NSE solutions of first order in the constants of the first integrals of the dynamical system.

3. Finding of Solutions of the Dynamical System

To simplify the further calculations, we go over from the parameters D, H, and W, which determine the NSE solutions, to three other equivalent parameters α_1, α_2, and α_3, which are roots of the polynomial of fourth degree on the right-hand side of Eq. (13). The fourth root of this polynomial is obviously zero. These triplets of parameters are related to each other by Viète's relations for a cubic polynomial:

$$W=2(\alpha_1+\alpha_2+\alpha_3), \tag{16a}$$

$$H=2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)-\alpha_1^2-\alpha_2^2-\alpha_3^2, \tag{16b}$$

$$D=16\alpha_1\alpha_2\alpha_3. \tag{16c}$$

In the region of allowed parameters, determined below, each point in the space of the parameters (α_1, α_2, α_3) corresponds to a definite solution of the NSE. Equation (13) can be expressed in terms of the new parameters as follows:

$$z_t^2=-64z(z-\alpha_1)(z-\alpha_2)(z-\alpha_3). \tag{17}$$

We shall be interested in only real positive solutions of Eq. (17), since by definition $z = \delta^2$ and δ are real. Therefore, at least one of the roots must be positive. Let this be α_3. The two other roots may be real or complex conjugates. If real, we order them and will assume $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$. In the case of complex roots, we go over from α_1 and α_2 to two other parameters ρ and η: $\alpha_1 = \alpha_2^* = \rho + i\eta$.

Specifying the roots of the right-hand side of Eq. (17) and obtaining its solutions, we must then find the roots of the polynomial of fourth degree on the right-hand side of Eq. (15). With allowance for Eq. (17), we rewrite this polynomial in the form

$$Q^4-(W-6z)Q^2-8\sqrt{(\alpha_1-z)(\alpha_2-z)(\alpha_3-z)}Q-(H-Wz+3z^2)=0. \tag{18}$$

By means of Ferrara's formulas [11], we can decompose the polynomial (18) into the following two quadratic trinomials:

$$Q^2\pm2\sqrt{\alpha_3-z}Q+\alpha_3-\alpha_1-\alpha_2+z\pm2\sqrt{(\alpha_1-z)(\alpha_2-z)}=0, \tag{19}$$

whose discriminants are

$$D(\pm)=\alpha_1+\alpha_2-2z\mp\sqrt{(\alpha_1-z)(\alpha_2-z)}. \tag{20}$$

Further, it is necessary to analyze separately the cases of real and complex roots α_1 and

α_2.

We consider first the case of real α_i. In this case, formula (20) can be written in the form

$$D(\pm)=\begin{cases} -(\sqrt{z-\alpha_1}\pm\sqrt{z-\alpha_2})^2 & \text{for } z\geqslant\alpha_2, \\ (\sqrt{\alpha_1-z}\mp\sqrt{\alpha_2-z})^2 & \text{for } z\leqslant\alpha_1. \end{cases} \tag{21}$$

It can seen from formulas (21) that the polynomial (18) has four real roots only for solutions of Eq. (17) in the interval $0 \leqq z \leqq \alpha_1$, when all the parameters α_i are positive, and it has one multiple real root in the case of coincident negative $\alpha_1 = \alpha_2 \leqq 0$ and $0 \leqq z \leqq \alpha_3$. In the remaining cases, the roots of the polynomial (18) are complex, and Eq. (15) does not have real solutions. Let all the roots α_i be positive. Then the solution of Eq. (17) on the interval $0 \leqq z \leqq \alpha_1$ can be expressed in terms of elliptic Jacobi functions:

$$z(t)=\frac{\alpha_1\alpha_3\,\mathrm{sn}^2(\mu t,k)}{\alpha_3-\alpha_1\,\mathrm{cn}^2(\mu t,k)}, \tag{22}$$

where $\mu=4\sqrt{\alpha_2(\alpha_3-\alpha_1)}$, $k^2=\alpha_1(\alpha_3-\alpha_2)/\alpha_2(\alpha_3-\alpha_1)$ is the modulus of the elliptic functions. The roots of the polynomial (18) are real and have the form

$$\begin{aligned} &Q_1=\sqrt{\alpha_1-z}+\sqrt{\alpha_2-z}+\sqrt{\alpha_3-z}, \qquad Q_2=-\sqrt{\alpha_1-z}-\sqrt{\alpha_2-z}+\sqrt{\alpha_3-z},\\ &Q_3=-\sqrt{\alpha_1-z}+\sqrt{\alpha_2-z}-\sqrt{\alpha_3-z}, \qquad Q_4=\sqrt{\alpha_1-z}-\sqrt{\alpha_2-z}-\sqrt{\alpha_3-z}, \end{aligned} \tag{23}$$

where the signs in front of the roots are chosen such that $Q_4 \leqq Q_3 \leqq Q_2 \leqq Q_1$. Equation (15) has solutions in two intervals,

$$\begin{aligned} &Q=\frac{Q_1(Q_2-Q_4)-Q_4(Q_1-Q_2)\,\mathrm{sn}^2(px,m)}{(Q_2-Q_4)-(Q_1-Q_2)\,\mathrm{sn}^2(px,m)}, \qquad Q_2\leqslant Q\leqslant Q_1,\\ &Q=\frac{Q_4(Q_1-Q_3)-Q_1(Q_3-Q_4)\,\mathrm{sn}^2(px,m)}{(Q_1-Q_3)-(Q_3-Q_4)\,\mathrm{sn}^2(px,m)}, \qquad Q_4\leqslant Q\leqslant Q_3, \end{aligned} \tag{24}$$

where $p=\sqrt{\alpha_3-\alpha_1}$, $m^2=(\alpha_2-\alpha_1)/(\alpha_3-\alpha_1)$. To each solution (24) there corresponds a solution of the NSE. However, these solutions can be transformed into each other by shifts with respect to the variables x and t. Formulas (22) and (24) are not the only form of expression of the solutions; we shall see that in a number of special cases they can be expressed more simply. We find the function φ by integrating (14). Taking into account (16b), we obtain

$$\varphi=Wt-4\int_0^t z\,dz=2(\alpha_1+\alpha_2-\alpha_3)t+\frac{4\alpha_3}{\mu}\Pi(n;\mu t,k), \tag{25}$$

where $\Pi(n;\mu t,k)=\int_0^{\mu t}\frac{d\tau}{1-n\,\mathrm{sn}^2(\tau,k)}$ is the incomplete elliptic integral of the third kind [12], $n = \alpha_1/(\alpha_1 - \alpha_3)$. Since $\varphi(t)$ occurs in the solution (3) as an argument of trigonometric functions, the solutions of the NSE of first order are in the general case conditionally periodic with respect to the time (with two-frequency basis). For brevity, we have omitted in Eqs. (22), (24), and (25) the obvious constants of integration t_0, x_0, and φ_0, although in them and everywhere below we shall understand the possibility of the substitutions $t \to t - t_0$, $x \to x - x_0$, and $\varphi\to\varphi-\varphi_0$. At the same time, the possible dependence of t_0 and φ_0 on x that exists in the theory is eliminated, since such a dependence would contradict the original assumptions of x-independence of the coefficients in (2), and the assumption of a t-dependence of x_0 contradicts Eq. (5).

The case of negative $\alpha_1 = \alpha_2$ can be regarded as a special case of complex conjugate roots with vanishing imaginary part. Since Q in this case does not depend on x, the NSE solution is also independent of x. It is easy to show that it has the form $\psi=-\sqrt{\alpha_3}\exp(2i\alpha_3 t)$.

We analyze the cases of complex roots in general form. Let $\alpha_1 = \alpha_2^* = \rho + i\eta$. The root α_3 in this case can be positive or zero. The solution of Eq. (17) can be written in the form

$$z(t)=\frac{(1-\nu)(1-\mathrm{cn}(2\mu t,k))}{4(1-\nu\,\mathrm{cn}(2\mu t,k))} \tag{26}$$

where

$$\nu=\frac{f-g}{f+g},\quad \mu=4\sqrt{fg},\quad k^2=\frac{1}{2}\left[1-\frac{\eta^2+\rho(\rho-\alpha_3)}{fg}\right],\quad f=\sqrt{(\alpha_3-\rho)^2+\eta^2},\quad g=\sqrt{\rho^2+\eta^2}.$$

The discriminants of Eqs. (19) are

$$D(\pm)=2(\rho-z)\mp\sqrt{(\rho-z)^2+\eta^2}. \tag{27}$$

For all values of ρ and η, two roots of Eq. (18) are real, and two complex:

$$Q_{1,2}=-b\pm d,\quad Q_{3,4}=b\pm ic, \tag{28}$$

where $b=\sqrt{\alpha_3-z}$; $d,c=\sqrt{2[\sqrt{(\rho-z)^2+\eta^2}\pm(\rho-z)]}$. A solution of Eq. (15) exists in the interval $Q_2 \leqq Q \leqq Q_1$:

$$Q=-b-d\frac{r-\mathrm{cn}(2px,m)}{1-r\,\mathrm{cn}(2px,m)}, \tag{29}$$

where

$$p=\sqrt[4]{(\alpha_3-\rho)^2+\eta^2},\quad m^2=\frac{1}{2}\left[1-\frac{\alpha_3-\rho}{p^2}\right],\quad r=\frac{\sqrt{p^2+b-bd}-\sqrt{p^2+b+bd}}{\sqrt{p^2+b-bd}+\sqrt{p^2+b+bd}}.$$

For the function φ in this case we obtain by integration of (14) with allowance for (26) the expression

$$\varphi=\left(2\rho+\alpha_3-\frac{2g}{f}\right)t+\frac{2g^2}{\alpha_3\mu f}((n_1-1)\Pi(n_1;\mu t,k)+(1-n_2)\Pi(n_2;\mu t,k)], \tag{30}$$

where $n_1=2fk^2/(f-g+\alpha_3)$, $n_2=2fk^2/(f-g-\alpha_3)$.

Formulas (3), (22), (24)-(26), (29), and (30) determine the solutions of the NSE of first order for all α_i in the region of allowed values. From these formulas we can select some special cases of the greatest practical interest. We first make an important observation. Division of the roots α_i by some positive number q is equivalent to transition from the solution $\psi(x, t)$ corresponding to the roots α_i to a different solution $\psi'(x, t)$ corresponding to the roots $a_i = \alpha_i/q$. These two solutions are connected by the transformation

$$\psi(x,t)=q\psi'(qx,q^2t). \tag{31}$$

As q we can choose, for example, the value of one of the roots and seek a two-parameter family of solutions $\psi'(x, t)$, specifying directly the a_i. The third parameter is introduced into the solution by means of the transformation (31). Below, when considering special cases, we shall proceed in precisely this manner and restrict ourselves to finding the solutions $\psi'(x, t)$, regarding the transformation (31) as trivial. For convenience of comparison with the previously obtained results in the cases when $\alpha_3 \neq 0$, we set $q = 2\alpha_3$, so that the two parameters a_1 and a_2 of the family of solutions $\psi'(x, t)$ are related to the parameters of the required family of solutions $\psi(x, t)$ by $a_1=\alpha_1/2\alpha_3$, $a_2=\alpha_2/2\alpha_3$, and the third parameter is $a_3 = \frac{1}{2}$.

4. Special Cases

We consider first cases when all roots are real.

1. Suppose $0\leqslant a_1=a_2\leqslant a_3={}^1/_2$. In this case, we obtain from (22)

$$z=\frac{a_1\,\mathrm{sh}^2\beta t}{\mathrm{ch}^2\beta t-2a_1}, \tag{32}$$

where $\beta^2 = 8a_1(1 - 2a_1)$, and for the function φ from (25) we obtain

$$\varphi=t+\mathrm{arctg}\,\gamma, \tag{33}$$

where $\gamma=\sqrt{2a_1/(1-2a_1)}\,\mathrm{th}\,\beta t$. The expressions for the roots (23) take the form

$$Q_{1,2}=\frac{\sqrt{1-2a_1}(\mathrm{ch}\,\beta t\pm\sqrt{2a_1})}{\sqrt{2}\sqrt{\mathrm{ch}^2\beta t-2a_1}},\quad Q_{3,4}=-\frac{\sqrt{1-2a_1}\,\mathrm{ch}\,\beta t}{\sqrt{2}\sqrt{\mathrm{ch}^2\beta t-2a_1}}. \tag{34}$$

One of the roots $Q_3 = Q_4$ is multiple, and the solution corresponding to this case does not depend on x: $Q = Q_3$. The solution of Eq. (15) in the interval $Q_2 \leqq Q \leqq Q_1$ has the form

$$Q=\frac{\sqrt{1-2a_1}}{\sqrt{\operatorname{ch}^2\beta t-2a_1}}\,\frac{\operatorname{ch}^2\beta t-4a_1+\sqrt{2a_1}\cos 2px\operatorname{ch}\beta t}{\sqrt{2}\,(\operatorname{ch}\beta t-\sqrt{2a_1}\cos 2px)} \tag{35}$$

where $p=\sqrt{2(1-2a_1)}$. The solution of the NSE for this case can be simplified by means of the formula

$$\psi'(x,t)=\frac{(Q-\gamma\sqrt{z})+i(\gamma Q+\sqrt{z})}{\sqrt{1+\gamma^2}}e^{it} \tag{36}$$

For Q independent of x, we obtain the stationary solution

$$\psi'=-(1/\sqrt{2})e^{it}, \tag{37}$$

and for Q expressed by (35) we obtain the solution

$$\psi'(x,t)=\frac{(1-4a_1)\operatorname{ch}\beta t+\sqrt{2a_1}\cos 2px+i\beta\operatorname{sh}\beta t}{\sqrt{2}(\operatorname{ch}\beta t-\sqrt{2a_1}\cos 2px)}e^{it}. \tag{38}$$

Thus, the case $a_1 = a_2$ corresponds to two solutions. One of them, (37), describes a wave with constant amplitude; the other, (38), describes exponential growth of perturbations periodic in x superimposed on a constant amplitude and a subsequent return after attainment of the maximal modulus to the original state (37). For $\alpha_1 = 1/4$, the growth rate of the perturbations is maximal, $\beta = 1$, and the solution (38) simplifies to

$$\psi'(x,t)=\frac{\cos x+i\sqrt{2}\operatorname{sh}t}{\sqrt{2}\operatorname{ch}t-\cos x}e^{it}. \tag{39}$$

In the limit $a_1 \to \frac{1}{2}$, we obtain from (38) a rational solution with power-law growth of the localized perturbation:

$$\psi'(x,t)=-\left[1-4\frac{1+2it}{1+2x^2+4t^2}\right]\frac{e^{it}}{\sqrt{2}} \tag{40}$$

while in the case $a_1 = 0$ the solution (38) degenerates into (37).

If $a_1 = a_2 < 0$, then only the multiple root $Q_3 = Q_4$ is real, and in this region there exists only the stationary solution (37). The initial stage of growth of the periodic perturbations, the so-called modulation instability, has been investigated several times in a number of physical applications [13,14]. The solution (38), which describes the further evolution of the modulation instability, was given earlier in our [7] (in other notation).

2. We set $a_1 + a_2 = a_3 = \frac{1}{2}$. Then the function (22) can be transformed by means of a lowering transformation of Landen's descending transformation [12] to the form

$$z(t)=\frac{\varkappa^2\operatorname{sn}^2(t,\varkappa)\operatorname{cn}^2(t,\varkappa)}{2[1-\varkappa^2\operatorname{sn}^4(t,\varkappa)]}, \tag{41}$$

where we have gone over from the modulus $k = \alpha_1/\alpha_2$ of the elliptic function to the new modulus $\varkappa=2\sqrt{k}/(1+k)$. In this case, the expression for the function φ can also be written in the form (33), where, however

$$\gamma=\frac{\operatorname{sn}(t,\varkappa)\operatorname{dn}(t,\varkappa)}{\operatorname{cn}(t,\varkappa)}. \tag{42}$$

The values of the roots (23) can be written in the form

$$Q_{1,2}=(1\pm v_0)f_0,\quad Q_{3,4}=-(1\mp w_0)f_0, \tag{43}$$

where

$$v_0=\frac{\sqrt{1+\varkappa}[1-\varkappa\operatorname{sn}^2(t,\varkappa)]}{\operatorname{dn}(t,\varkappa)},\quad w_0=\frac{\sqrt{1-\varkappa}[1+\varkappa\operatorname{sn}^2(t,\varkappa)]}{\operatorname{dn}(t,\varkappa)},\quad f_0=\frac{\operatorname{dn}(t,\varkappa)}{\sqrt{2}\sqrt{1-\varkappa^2\operatorname{sn}^4(t,\varkappa)}}.$$

The solution of Eq. (15), which varies in the interval $Q_2 \leq Q \leq Q_1$, can be conveniently expressed in the given case in the form

$$Q=\frac{\varkappa[\sqrt{1+\varkappa}\operatorname{sn}^2(t,\varkappa)\operatorname{dn}(t,\varkappa)+\operatorname{cn}^2(t,\varkappa)A(x)]}{\sqrt{2}\sqrt{1-\varkappa^2\operatorname{sn}^4(t,\varkappa)}[\sqrt{1+\varkappa}-\operatorname{dn}(t,\varkappa)A(x)]}, \tag{44}$$

where

$$A(x)=\frac{\mathrm{cn}\left(\sqrt{\frac{1+\varkappa}{2}}x,\sqrt{\frac{1-\varkappa}{1+\varkappa}}\right)}{\mathrm{dn}\left(\sqrt{\frac{1+\varkappa}{2}}x,\sqrt{\frac{1-\varkappa}{1+\varkappa}}\right)}.$$

The solution of the NSE corresponding to (44) can also be calculated in accordance with formula (36) and takes the form

$$\psi'(x,t)=\frac{\varkappa}{\sqrt{2}}\frac{A(x)\,\mathrm{cn}(t,\varkappa)+i\sqrt{1+\varkappa}\,\mathrm{sn}(t,\varkappa)}{\sqrt{1+\varkappa}-A(x)\,\mathrm{dn}(t,\varkappa)}e^{it}. \tag{45}$$

The solution (45) is periodic in x, and with respect to the variable t the solution has a double frequency: a duty cycle frequency, determined by the exponential function, and a modulation frequency, determined by the elliptic functions. Depending on $\varkappa$, the latter may vary from zero to the duty cycle frequency. In the limit $\varkappa\to 1$, this solution reduces to the solution (39), while in the limit $\varkappa\to 0$ (45) has as limit the soliton solution

$$\psi'(x,t)=\frac{\sqrt{2}}{\mathrm{ch}\,\sqrt{2}x}e^{2it}. \tag{46}$$

With allowance for (31), this solution can be written in the form $\psi=\frac{\sqrt{2}q}{\mathrm{ch}\,\sqrt{2}qx}e^{2iq^2t}$. Other types of solutions in the case $a_1 + a_2 = a_3$ can be obtained from (45) by shifts by a quarter period with respect to each of the variables, and we shall not give them here.

3. Solitons with nonvanishing asymptotic behavior at infinity. Suppose $a_1 \leqq a_2 = a_3 = \frac{1}{2}$. Then z(t) can be expressed in terms of trigonometric functions:

$$z(t)=\frac{a_1\sin^2\mu t}{1-2a_1\cos^2\mu t}, \tag{47}$$

where $\mu=2\sqrt{1-2a_1}$, and the function φ has the form

$$\varphi=2a_1t+\mathrm{arctg}\left(\frac{2}{\mu}\mathrm{tg}\,\mu t\right). \tag{48}$$

The roots Q_i in (23) take the form

$$Q_{2,3}=-f_1\sqrt{a_1}\cos\mu t,\quad Q_{1,4}=f_1(\sqrt{a_1}\cos\mu t\pm\sqrt{2}), \tag{49}$$

where $f_1=\sqrt{1-2a_1}/\sqrt{1-2a_1\cos^2\mu t}$, and the root $Q_2 = Q_3$ is multiple. It is convenient to write the solution of Eq. (15) in the form

$$Q=f_1\frac{\sqrt{2}(1-a_1\cos^2\mu t)\mp\sqrt{a_1}\cos\mu t\,\mathrm{ch}\,2px}{-\sqrt{2a_1}\cos\mu t\pm\mathrm{ch}\,2px} \tag{50}$$

where $p=\sqrt{{}^1/_2-a_1}$. Besides (50), there also exists a solution stationary with respect to x: $Q = Q_2$. The solution of the NSE corresponding to this value of Q has the form

$$\psi'=-\sqrt{a_1}e^{2ia_1t}, \tag{51}$$

while the solution corresponding to (50) can be written in the form

$$\psi'(x,t)=\frac{2(1-a_1)\cos\mu t\mp\sqrt{2a_1}\,\mathrm{ch}\,2px+i\mu\sin\mu t}{-2\sqrt{a_1}\cos\mu t\pm\sqrt{2}\,\mathrm{ch}\,2px}e^{2a_1t}. \tag{52}$$

The solution (52) describes a solitary wave doubly asymptotic with respect to the time, having at infinity (as $x \to \pm\infty$) the behavior (51). The solutions with the two different signs in (52) go over into each other under a shift with respect to the variable t by the half-period of the modulation: $\mu t \to \mu t + \pi$. If the upper sign in front of cosh 2px is chosen, the limit of the function (52) as $a_1 \to a_2$ is the rational solution (40), while in the limit $a_1 \to 0$ formula (52) degenerates into the soliton solution (46).

4. Now suppose $a_1 = 0 \leqq a_2 \leqq a_3 = \frac{1}{2}$. In this case, a solution of the NSE exists only for z = 0. The roots (23) do not depend on the time:

$$Q_{1,2}=\frac{1\pm k}{\sqrt{2}},\qquad Q_{3,4}=-\frac{1\mp k}{\sqrt{2}},\tag{53}$$

where $k = \sqrt{2a_1}$. The solution of the NSE can be expressed directly in the form

$$\psi'(x,t)=\pm\frac{1+k}{\sqrt{2}}\operatorname{dn}\left(\frac{1+k}{\sqrt{2}}x,\frac{2\sqrt{k}}{1+k}\right)e^{i(1+k^2)t}.\tag{54}$$

This solution describes a stationary envelope wave and is singly periodic with respect to the time. In the limit $a_2 \to 0$, the solution (54) with the lower sign degenerates into the stationary solution (37), while as $a_2 \to a_3$ the function (54) has as limit the soliton solution (46).

The four cases we have considered exhaust all possibilities for simplifying the general formulas (22), (24), and (25), which determine together with (3) the solution of the NSE for real parameters a_i. We now turn to the case of complex parameters a_i.

5. Suppose $a_1=a_2^*={}^1/_4+i\eta$, $a_3={}^1/_2$. Then the function z(t) can be represented in the form

$$z(t)=\frac{\operatorname{sn}^2(\mu t,k)\operatorname{dn}^2(\mu t,k)}{2[1-k^2\operatorname{sn}^4(\mu t,k)]},\tag{55}$$

where $\mu=\sqrt{1+16\eta^2}$, $k=1/\mu$. At the same time φ is determined by the expression (33), where

$$\gamma=\frac{k\operatorname{sn}(\mu t,k)\operatorname{cn}(\mu t,k)}{\operatorname{dn}(\mu t,k)}.\tag{56}$$

The roots of the polynomial (18) can be expressed in the form (28), where

$$b=\frac{\operatorname{cn}(\mu t,k)}{\sqrt{2}[1-k^2\operatorname{sn}^4(\mu t,k)]},\qquad d,c=\sqrt{\frac{1\pm k}{2k}\,\frac{1\mp k\operatorname{sn}^2(\mu t,k)}{1\pm k\operatorname{sn}^2(\mu t,k)}}.\tag{57}$$

It is convenient in this case to represent the solution of Eq. (15) in the form

$$Q=\frac{k^2\operatorname{sn}^2\left(\frac{t}{k},k\right)\operatorname{cn}\left(\frac{t}{k},k\right)+\sqrt{\frac{k}{1+k}}\operatorname{cn}\left(\frac{x}{\sqrt{k}},\sqrt{\frac{1-k}{2}}\right)\operatorname{dn}^2\left(\frac{t}{k},k\right)}{k\sqrt{2}\sqrt{1-k^2\operatorname{sn}^4\left(\frac{t}{k},k\right)}\left[1-\sqrt{\frac{k}{1+k}}\operatorname{cn}\left(\frac{x}{\sqrt{k}},\sqrt{\frac{1-k}{2}}\right)\operatorname{cn}\left(\frac{t}{k},k\right)\right]},\tag{58}$$

and then the complete solution of the NSE takes the form

$$\psi'(x,t)=\frac{\sqrt{\frac{k}{1+k}}\operatorname{cn}\left(\frac{x}{\sqrt{k}},\sqrt{\frac{1-k}{2}}\right)\operatorname{dn}\left(\frac{t}{k},k\right)+ik\operatorname{sn}\left(\frac{t}{k},k\right)}{k\sqrt{2}\left[1-\sqrt{\frac{k}{1+k}}\operatorname{cn}\left(\frac{x}{\sqrt{k}},\sqrt{\frac{1-k}{2}}\right)\operatorname{cn}\left(\frac{t}{k},k\right)\right]}e^{it}.\tag{59}$$

This solution is an analytic continuation of the solution (45) to $\varkappa>1$ ($\varkappa=1/k$). As $k \to 1$, the solution (59) has as its limit formula (39).

6. Now suppose $\alpha_1 = \alpha_2^* = \rho + i\eta$, $\alpha_3 = 0$. In this case it is not possible to normalize the roots by means of $2\alpha_3$, and the solution is represented in terms of the original parameters α_i. Equation (17) has only the trivial solution $z = 0$, and by means of (29) we can express the solution of the NSE in the form

$$\psi(x,t)=\frac{mq_0}{\sqrt{2m^2-1}}\operatorname{cn}\left(\frac{q_0x}{\sqrt{2m^2-1}},m\right)e^{iq_0^2t},\tag{60}$$

where $q_0=4\rho$, $m^2={}^1/_2[1+\rho/\sqrt{\rho^2+\eta^2}]$. The modulus of the elliptic function varies in the range $\frac{1}{2} \leqq m^2 \leqq 1$ for $\infty > \eta > 0$. Like (54), the solution (60) describes a stationary envelope wave and is singly periodic with respect to the time.

5. Solutions of the NSE of Higher Order

We define solutions of the NSE of n-th order as solutions for which the real and imaginary parts satisfy a relation

$$P_n(u, v)=0, \tag{61}$$

where P_n is a polynomial of n-th degree in u and v with coefficients that depend only on the time. Specifying the order of the polynomial n, we can in principle construct by the method we have described a dynamical system corresponding to the given order and find its solution, this being equivalent to the finding of a solution of the NSE. Such calculations in general form go beyond the scope of the present paper. Here, we limit ourselves to the simplest example of finding a particular solution of second order.

We consider the simplest case of a curve of second order and assume that the points (u, v) lie on a circle with center at the origin and with a radius that depends on the time:

$$u^2(x, t)+v^2(x, t)-R^2(t)=0. \tag{62}$$

The solutions of the NSE that satisfy the condition (62) can be represented in the form

$$\psi(x, t)=R(t)e^{i\Phi(x,t)}. \tag{63}$$

Substituting (63) in (1) and separating the real and imaginary parts, we obtain the system of equations

$$R_t+R\Phi_{xx}=0, \quad 2R^3-R\Phi_x^2-R\Phi_t=0. \tag{64}$$

After integration of the first equation and division of the second by R, the system takes the form

$$\Phi_x^2=\frac{2S}{R}-\frac{2R_t}{R}\Phi, \quad \Phi_t=2R^2-\Phi_x^2, \tag{65}$$

where S = S(t) is a constant of integration. Using the condition of compatibility $\Phi_{xt} = \Phi_{tx}$ and equating the coefficients of different powers of Φ, we obtain the dynamical system

$$S_tR-3SR_t-2R^3R_t=0, \quad 3R_t^2-RR_{tt}=0. \tag{66}$$

The solution of the system (66),

$$R=Ct^{-1/2}, \quad S=-C^3t^{-3/2}\ln t, \tag{67}$$

where C is a constant of integration, together with the solution of the system (65) for the function Φ,

$$\Phi=x^2/4t+2C^2\ln t, \tag{68}$$

determines the NSE solution

$$\psi(x,t)=\frac{C}{\sqrt{t}}\exp i\left[\frac{x^2}{4t}+2C^2\ln t\right]. \tag{69}$$

This solution was given earlier in [5] by Sall'.

Besides the solution (69), there exist solutions of higher orders. For example, the solutions doubly asymptotic as $t \to \pm\infty$ found in [7] are solutions of third order, as we have confirmed by numerical calculations. In the general case, the possibility of constructing solutions of the NSE of n-th order together with the corresponding dynamical system can be seen from the following considerations. Let P(u, v; t) = 0 be the equation of an algebraic curve whose parameters depend explicitly on the time in the space of pairs of functions u and v satisfying Eq. (1). In this case, a simple consequence of Eq. (1) is the relations

$$u_x^2+v_x^2+(u_x^2+v_x^2)^2+\int\frac{\dot P}{P_uP_v}(P_u\,du-P_v\,dv)=h(t), \tag{70}$$

$$u_tP_v-v_tP_u-F(u_x, v_x)+2(u^2+v^2)(uP_u+vP_v)=0, \tag{71}$$

where $F=P_{uu}u_x^2+2P_{uv}u_xv_x+P_{vv}v_x^2$, $\dot P=\partial P/\partial t$ is defined for u = const, v = const, the curvilinear integral in (70) is taken along the curve P = 0, and the derivatives u_x, v_x, u_t, and v_t are subject to the restrictions $u_xP_u+v_xP_v=0$, $\dot P+u_tP_u+v_tP_v=0$. The conditions of compatibility of Eqs. (70) and (71) determine the time evolution of the parameters of the algebraic curve P = 0 (i.e., the corresponding finite-dimensional dynamical system) and the function h(t), which redetermines the dependence of the solutions u and v on the spatial variable.

These assertions become more transparent if we go over from the implicit form of specification of the curve, $P(u, v; t) = 0$, to its parametric representation. Namely, let $Q(x, t)$ be a variable that uniformizes the curve $P = 0$. Then $u(x, t) = u(Q, t)$, $v(x, t) = v(Q, t)$, and Eqs. (70) and (71) take the form

$$DQ_x^2+(u^2+v^2)+\int^Q dq\,\frac{\dot{P}}{P_uP_v}\left(P_u\frac{\partial u}{\partial q}-P_v\frac{\partial v}{\partial q}\right)=h(t), \tag{72}$$

$$\left(\frac{\partial u}{\partial Q}P_v-\frac{\partial v}{\partial Q}P_u\right)Q_t+(u_tP_v-v_tP_u)-F\left(\frac{\partial u}{\partial Q},\frac{\partial v}{\partial Q}\right)Q_x^2+2(u^2+v^2)(uP_u+vP_v)=0, \tag{73}$$

where

$$D=\left(\frac{\partial u}{\partial Q}\right)^2+\left(\frac{\partial v}{\partial Q}\right)^2,\quad \frac{\partial u}{\partial Q}P_u+\frac{\partial v}{\partial Q}P_v=0.$$

Thus, uniformization of the curve $P(u, v, t) = 0$ leads to the system of equations (72) and (73), which determines Q_x and Q_t as functions of the uniformizing variable Q and the finite number of parameters of the algebraic curve $P = 0$ and their first derivatives with respect to the time. In this representation, $Q_{xt} = Q_{tx}$ is the condition of compatibility of the system (72)-(73). The reduction of this condition to a finite number of conditions determines the dynamical system in the phase space of the parameters of the algebraic curve $P = 0$, the functions $h(t)$, and their first derivatives.

6. Conclusions

The class of NSE solutions that we have obtained encompasses all the important cases, including the single-soliton solutions with zero and nonzero asymptotic behavior at infinity, the solutions that describe the evolution of the modulation instability, and also solutions periodic with respect to x and doubly periodic with respect to t. These solutions are of great importance from the point of view of applications — in the theory of optical communication (see [7,14] and the references quoted there), the theory of deep water waves, etc. For example, the solutions describing the evolution of a modulationally unstable wave with constant amplitude describe the generation of a periodic train of pulses in an optical fiber [7]. The solutions periodic with respect to two variables describe the propagation of periodic signals in a fiber. The obtained solutions can also be used in the theory of two-dimensional small-scale self-focusing for the description of the decay of a plane wave into individual filaments [13]. Since the solutions can be expressed in terms of elliptic Jacobi functions and elliptic integrals, for which there are standard calculation programs, any of the obtained solutions can readily be represented graphically, and this is of no small importance for concrete applications.

LITERATURE CITED

1. V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz., 118, 118 (1971).
2. M. V. Babich, A. I. Bobenko, and V. B. Matveev, Izv. Akad. Nauk SSSR, Ser. Mat., 49, 511 (1985).
3. A. R. Its and V. P. Kotlyarov, Dokl. Akad. Nauk SSSR, Ser. A, No. 11, 965 (1976).
4. Y. C. Ma and M. J. Ablowitz, Stud. Appl. Math., 65, 113 (1981).
5. M. A. Sall', Teor. Mat. Fiz., 53, 227 (1982).
6. N. N. Akhmediev and V. I. Korneev, Teor. Mat. Fiz., 69, 189 (1986).
7. N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Zh. Eksp. Teor. Fiz., 89, 1542 (1985).
8. V. M. Eleonskii, N. E. Kulagin, and N. S. Novozhilova, Zh. Eksp. Teor. Fiz., 89, 2174 (1985).
9. N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, in: Methods of the Qualitative Theory of Differential Equations [in Russian], Gor'ki University, Gor'ki (1985), p. 18.
10. A. Érdelyi et al. (eds), Higher Transcendental Functions (California Institute of Technology H. Bateman M. S. Project), Vol. 3, McGraw Hill, New York (1955).
11. G. A. Korn and T. M. Korn, Mathematical Handbook for Scientists and Engineers, McGraw Hill, New York (1961).
12. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover (1964).
13. V. I. Vespalov and V. I. Talanov, Pis'ma Zh. Eksp. Teor. Fiz., 3, 471 (1966).
14. D. Anderson and M. Lisak, Optics Letters, 9, 468 (1984).