MODULATION INSTABILITY AND PERIODIC SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

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A very simple exact analytic solution of the nonlinear Schrödinger equation is found in the class of periodic solutions. It describes the time evolution of a wave with constant amplitude on which a small periodic perturbation is superimposed. Expressions are obtained for the evolution of the spectrum of this solution, and these expressions are analyzed qualitatively. It is shown that there exists a certain class of periodic solutions for which the real and imaginary parts are linearly related, and an example of a one-parameter family of such solutions is given.

In recent years, much attention in the scientific literature has been devoted to periodic solutions of nonlinear partial differential equations [1] and, in particular, the nonlinear Schrödinger equation (NSE) (see [2-5] and the bibliography given there). This is due to the fact that periodic solutions are needed in a number of practically important problems, for example, the problem of the generation of picosecond pulses in an optical fiber [6,7], in the problem of self-focusing [8,9], in the theory of waves on deep water [10,11], and in many other cases. In the class of periodic solutions of the NSE a particular position is occupied by the solution describing modulation instability, i.e., growth of long-wave periodic perturbations on the background of a continuous wave of constant amplitude. The initial stage in the development of the instability in this problem has been investigated by the linearization method [8,9], and the further evolution of the field has been studied by numerical modeling of the solutions of the NSE on a computer [5,6,10,11]. These investigations have made it possible to clarify to a large degree the qualitative behavior of the solution. In particular, it has been established that the initial growth of the perturbation amplitude as a result of evolution of the field is replaced by its subsequent decrease and return to the original state of the plane wave, this being similar to the Fermi–Pasta–Ulam return in a system of coupled oscillators, and under certain conditions the solution becomes oscillatory in time too. However, an exact analytic solution of this problem was not given in the quoted studies, and this was the stimulus for our investigations. In the present work, we have obtained an exact solution to the problem of modulation instability, expressed in elementary functions. The solution is made by means of an ansatz that relates linearly the real and imaginary parts of the unknown function, in which the coefficients depend only on the time. We show that such a connection is valid for a large class of periodic solutions of the NSE, and we give an example of a one-parameter family of solutions periodic with respect to two variables found by means of it.

We write the spatially one-dimensional NSE in the form

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0. \tag{1}$$

The notation here is standard. For convenience of the following analysis, we make in (1) a change of variable by means of the relation $\psi(x, t) = u(x, t)e^{it}$, and then the equation for the complex function $u(x, t)$ will have the form

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}-u+|u|^2u=0. \tag{2}$$

Equation (2) has a very simple stationary solution in the form of a complex constant $u=\exp i\varphi$, which, as is well known [6-11], is unstable with respect to x-periodic perturbations of the form $\cos k(x - x_0)$. For such perturbations, the first term in the expansion of the solution near the point $u = 1$ (for $\varphi=0$) has the form

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$$u\approx 1+\left[a\left(1+i\frac{2\delta}{k^2}\right)e^{\delta t}+b\left(1-i\frac{2\delta}{k^2}\right)e^{-\delta t}\right]\cos k(x-x_0), \tag{3}$$

where a and b are two independent small parameters, x_0 is an arbitrary constant, and $\delta = k(1 - k^2/4)^{\frac{1}{2}}$ is the growth rate of the instability, real in the interval of wave vectors $0 < k < 2$ and taking its maximal value, equal to unity, at the point $k = \sqrt{2}$. Below, we shall restrict ourselves to considering the further evolution of perturbations of the type (3) only with this wave vector, since it has the greatest interest in practice.

To find the required solution of Eq. (2), we represent it in the form of an explicit complex function $u = v + iw$. Then Eq. (2) itself can be expressed as a system for the two real functions v and w:

$$v_t-w+{}^1/_2w_{xx}+(v^2+w^2)w=0,\quad -w_t-v+{}^1/_2v_{xx}+(v^2+w^2)v=0. \tag{4}$$

We assume that v and w are related linearly:

$$v=\eta w+\mu, \tag{5}$$

in which the coefficients η and μ depend only on the time variable. In this paper, we consider the case $\eta = \mu = -\tanh t$ and take

$$v=-\operatorname{th} t(w+1). \tag{5a}$$

For functions v and w satisfying the NSE, substitution of (5) in (4) must lead to two equivalent equations for the function w. Eliminating from these two equations the second derivative with respect to x, we obtain after simple transformations an equation of first order for w (Bernoulli equation):

$$w_t+w^2\operatorname{th} t+w\operatorname{th} t=0. \tag{6}$$

The general solution of this equation can be represented in the form

$$w=c/(\operatorname{ch} t-c), \tag{7}$$

where $c = c(x)$ is a constant of integration. To find its dependence on x, we substitute (7) in one of the equations obtained by substituting (5a) in (4). In both cases, we obtain the equation

$$C_{xx}+\frac{2C_x^2}{\operatorname{ch} t-C}-2\frac{1-C^2-C\operatorname{ch} t}{\operatorname{ch} t-C}=0, \tag{8}$$

and this indicates the validity of the above ansatz. It is easy to show that a function that satisfies (8) for all t must solve the equation

$$C_x^2=1-2C^2, \tag{9}$$

from which we find

$$C=\frac{1}{\sqrt{2}}\cos\sqrt{2}(x-x_0). \tag{10}$$

Substituting (10) in (7), and then (7) in (5a), we obtain the real and imaginary parts of the function u and, therefore, the complete solution of Eq. (2) in the form

$$u(x,t)=\frac{-\sqrt{2}\operatorname{sh} t+i\cos\sqrt{2}(x-x_0)}{\sqrt{2}\operatorname{ch} t-\cos\sqrt{2}(x-x_0)}. \tag{11}$$

This solution is the simplest in the class of periodic solutions of the NSE. It is easy to show that in the limit $t \to -\infty$ the solution (11) is identical to the expansion (3), in which the coefficient $b = 0$. In the limit $t \to +\infty$, the solution (11) is identical to formula (3), in which $a = 0$ and the factor $e^{i\pi}$ has been added, i.e., the arbitrary phase φ in the given case is π. Thus, as a result of the development of the instability the amplitude of the modulation increases from zero to the maximal value at $t = 0$, and then as $t \to \infty$ returns to a stationary solution with the original amplitude but opposite phase φ.

For periodic solutions, the evolution of the spectrum of the original wave is an important question. In the given case, we can also write down exact expressions for the spectrum. We represent the solution (11) in the form of the Fourier expansion

$$u(x,t)=f_0(t)+2\sum_{n=1}^{\infty}f_n(t)\cos[n(x-x_0)\sqrt{2}]. \tag{12}$$

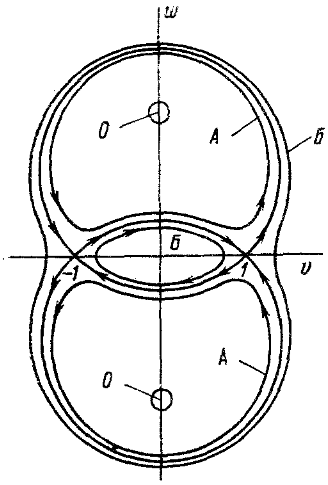

Fig. 1

The coefficients in (12) can be readily calculated:

$$f_0(t)=\frac{1}{\sqrt{2}\,\pi}\int_0^{\sqrt{2}\,\pi}u(x,t)\,dx=\frac{i\sqrt{2}\,\mathrm{sh}\,t+\sqrt{2}\,\mathrm{ch}\,t-\sqrt{2\,\mathrm{ch}^2 t-1}}{\sqrt{2\,\mathrm{ch}^2 t-1}} \tag{13}$$

$$f_n(t)=\frac{1}{\sqrt{2}\,\pi}\int_0^{\sqrt{2}\,\pi}u(x,t)\cos(nx\sqrt{2})\,dx=\sqrt{2}\,\frac{i\,\mathrm{sh}\,t+\mathrm{ch}\,t}{\sqrt{2\,\mathrm{ch}^2 t-1}}\left(\sqrt{2}\,\mathrm{ch}\,t-\sqrt{2\,\mathrm{ch}^2 t-1}\right)^n. \tag{14}$$

For the sum of the moduli of the coefficients $f_n(t)$ we have the relation

$$|f_0(t)|^2+2\sum_{n=1}^{\infty}|f_n(t)|^2=\frac{1}{\sqrt{2}\,\pi}\int_0^{\sqrt{2}\,\pi}|u|^2\,dx=1. \tag{15}$$

It is obvious that as $t \to \pm\infty$ all $f_n(t)$ vanish except for $|f_0(t)|^2 = 1$. For arbitrary t, all $f_n(t)$ are nonzero, but the energy of the higher harmonics decreases in accordance with the law of geometric progression. At $t = 0$, the squares of the moduli of the coefficients are equal to $|f_0|^2=(\sqrt{2}-1)^2$, $|f_n|^2=2(\sqrt{2}-1)^{2n}$. In this case, the maximal energy is concentrated in the first sideband.

For the sake of greater clarity, we analyze the solution (11) on the complex (v, w) plane (see Fig. 1). To be definite, we fix the variable $x = x_0$, but this in no way restricts the generality. By analogy with the terminology adopted in the theory of nonlinear vibrations, the point $u = 1$ on this plane is a saddle, as can be readily seen by constructing near this point the trajectories described by formula (3). The parameter of these trajectories is t, and the parameter of the family of trajectories is the ratio a/b. Two trajectories emanate from the point $u = 1$ at angles 45° and −135° to the v axis, two enter it at the angles −45° and 135°, and the remaining trajectories have near this point the form of hyperbolas. The trajectories near the point $u = -1$ have the same behavior, as can be readily seen by multiplying (3) by $e^{i\pi}$. The trajectory described by the solution (11) is the upper part of the circle

$$v^2+(w-1)^2=2, \tag{16}$$

and it joins the points ±1 along one of the directions indicated above. The solutions connecting the points ±1 along the three other directions can be readily obtained from (11) by changing the sign in front of u and/or making the translation $x \to x_0 + \pi/\sqrt{2}$. All such trajectories are represented in Fig. 1.

In the case when the trajectory passes near the points ±1 without arriving at them, the solution is periodic in t, and the complete trajectories of such solutions, obtained by numerical methods (like those described in [12]), are represented in the figure by closed curves close to the separatrix curves (16). For solutions with period with respect to x independent of the parameter a/b, the phase trajectories do not intersect. It can be seen from Fig. 1 that the separatrix solutions (11) separate two qualitatively different types of t-periodic solutions. Pairs of trajectories of type A, symmetric with respect to the v axis, correspond to one and the same solution obtained by translation with respect to x by $\pi/\sqrt{2}$. The trajectories of the type B, symmetric with respect to the origin and situated beyond the separatrices, also have an analog obtained by translation and situated within the separatrices. For periodic trajectories of type A there is a "center point" O, which corresponds to a stationary solution in the form of a Jacobi elliptic function:

$$u_0(x)=i\alpha k_0\,\mathrm{cn}[\alpha(x-x_0),\,k_0], \tag{17}$$

where $\alpha=(k_0^2-{}^1/_2)^{-1/2}$, and the modulus of the elliptic function k_0 for the period with respect to x that we consider is determined from the equation $\pi/\sqrt{2k_0^2-1}=2\mathbf{K}(k_0)$, in which $\mathbf{K}(k_0)$ is a complete elliptic integral of the first kind.

By numerical modeling of the periodic solutions of the NSE we have established that for solutions periodic with respect to the two variables a linear connection of the general form (5), in which a definite solution corresponds to certain functions $\mu(t)$ and $\eta(t)$, is valid. If it is borne in mind that a linear connection of the real and imaginary parts of u is also valid for the known "simple" solutions of the NSE (families of single-soliton solutions with the soliton width as parameter, and also families of stationary solutions of the type (17)), it becomes clear that there exists a fairly large class of solutions of the NSE possessing the property (5). A regular procedure for finding the coefficients η and μ in (5), and with them the complete class of solutions, is the subject of a separate paper. In the present paper, we restrict ourselves to the example of the one-parameter family of solutions found by means of the ansatz (5), in which $\eta = -\mathrm{sn}(t,\,k)\mathrm{cn}(t,\,k)/\mathrm{dn}(t,\,k)$ and $\mu = -\mathrm{sn}(t,\,k)$ can be expressed in terms of Jacobi elliptic functions with modulus k as parameter. For these values of the coefficients, the solution of the NSE has the form

$$u(x,t)=k\frac{-\mathrm{sn}(t,k)+i(1/\sqrt{1+k})\,\mathrm{cd}(\sqrt{1+k}x,\sqrt{(1-k)/(1+k)})\,\mathrm{cn}(t,k)}{1-(1/\sqrt{1+k})\,\mathrm{cd}(\sqrt{1+k}\,x,\sqrt{(1-k)/(1+k)})\,\mathrm{dn}(t,k)} \tag{18}$$

where $\mathrm{cd}(z) = \mathrm{cn}(z)/\mathrm{dn}(z)$. For this family of solutions, the periods with respect to t and with respect to x depend on the parameter k. In the limit $k \to 1$, (18) is identical to the solution (11), and in the limit $k \to 0$ it degenerates into a soliton solution with fixed width $u = 2ie^{it}/\cosh 2x$. Since the limiting cases of the family (18) are solutions of other families with fixed parameters, the class of solutions possessing the property (5) is determined by not less than two parameters. The number of these parameters is, however, limited, since two- and, in general, N-soliton solutions, and also periodic solutions with two and more periods with respect to x do not satisfy the relation (5). Thus, the relation (5) separates from the complete manifold of NSE solutions those that can be called first-order solutions.

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EFFECTIVE ACTION FOR SUPERSYMMETRIC CHIRAL ANOMALY

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It is shown that consistency conditions of the type of the Wess–Zumino conditions are necessary and sufficient conditions for local integrability of the supersymmetric chiral anomaly. It follows from the requirement of global integrability that the coefficient of the anomalous action is discrete. Explicit expressions are obtained for consistent anomalies and the corresponding functionals, which depend on superfields of various types.

Introduction. In the construction of low-energy effective Lagrangians describing the behavior of composite objects, anomalous terms must be included if the low-energy phenomenology is to be correctly described. The form of the invariant part of the effective chiral Lagrangian is fixed in the case of quantum chromodynamics (QCD) more or less uniquely by considerations of global invariance under $U(3)_L \times U(3)_R$ and minimality with respect to the derivatives and fields that describe the mesons. The obtaining of the anomalous term in the effective action is a much more difficult problem, and requires for its solution not only allowance for the symmetries of the theory but also the use of global topological arguments. In the case of QCD, a number of authors have shown [1-4] that the anomalous action can be taken in the form

$$\Gamma_{WZW}=-(i/240\pi^2)c\varepsilon_{ijklm}\int_{\mathscr{D}} d^5x\,\mathrm{Tr}(U\partial_iU^+\,U\partial_jU^+U\partial_kU^+U\partial_lU^+U\partial_mU^+), \tag{1}$$

where the integration is over a five-dimensional disk $\mathscr{D}$.

The proof of integrality of the coefficient c [1] is based on the nontriviality of $\pi_5(SU)3))$, whereas $\pi_4(SU(3)) = 0$. Several ways of obtaining Γ_{WZW} are known. Besides the geometrical method based on the use of the cohomology group [1,2] and the method associated with using the index theorem [5], there exists a method of "direct" calculation of the effective action by integrating the anomaly, which satisfies consistency conditions [3,4] and can be obtained in some regularization scheme.

The problem of calculating the consistent supersymmetric chiral anomaly has been discussed in a number of studies [6,7,8]. In the first section, we formulate the problem of constructing the supersymmetric analog of the anomalous Wess–Zumino–Witten action, which reproduces the consistent chiral anomaly. The expression for the anomaly is here calculated in the presence of a gauge field and additional chiral multiplets. The contribution of these additional chiral fields to the expression for the anomaly is equal to the variation of a local functional. The expression for the topologically nontrivial part of the single-loop effective action is not changed. In the second section, we generalize to the supersymmetric case the method of integrating the consistent chiral anomaly developed in [4]. It is shown that consistency conditions of the Wess–Zumino type are necessary and sufficient conditions of local integrability of the anomaly. Integrality of the coefficient of the supersymmetric effective action arises from the requirement of global integrability of the anomaly. In the Appendix, a formula for the consistency supersymmetric anomaly is derived.

1. In connection with the appreciable progress in supersymmetric theories, a number of authors have recently discussed the problem of constructing supersymmetric effective Lagrangians [9]. In the construction of a supersymmetric analog of the anomalous action (1), the choice of the parametrization of the group manifold constitutes a certain problem. We assume that the meson multiplet is described by the pseudoscalar component $\pi_a(x)$

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