Pseudorecurrence in Two-Dimensional Modulation Instability with a Saturable Self-Focusing Nonlinearity

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The modulation instability of a two-dimensional field with a saturable self-focusing nonlinearity is investigated numerically. We show that approximate recurrence to an initial homogeneous field occurs, but oscillation around the exact recurrence is observed. This pseudorecurrence arises only for a restricted range of spatial modulation frequencies.

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The nonlinear Schrödinger equation (NLS) is a universal model for nonlinear wave propagation that has been studied extensively. Of particular interest is the modulation instability (MI) that causes an initial homogeneous state to develop spatial structure. This was first investigated by Bespalov and Talanov[1] using perturbative methods in the context of nonlinear optical self-focusing in two dimensions, and independently by Benjamin and Feir[2] for the case of deep-water waves in one dimension. Numerical simulations of the one-dimensional MI were later carried out,[3-5] most recently in the area of pulse propagation in nonlinear optical fibers.[6,7]

Modulation instabilities belong to a class of periodic solutions of the NLS that have been studied using the inverse scattering method.[8-12] Analytic solutions in one dimension have been obtained in some special cases,[13,14] and have found applications in nonlinear fiber optics.[15,16] In particular, it has been shown that many initial conditions[1-3,7,15,16] can recur to the initial state to within a phase factor upon propagation. The interest in such periodic solutions is connected with the study of chaos in nonlinear propagation as they describe homoclinic orbits.[17] In the presence of small perturbations[18,19] the solution can shift from a homoclinic orbit to adjacent periodic ones whose periods are sensitive to the size of the displacement. The field propagation can then appear to be chaotic. Whether periodic solutions giving recurrence arise in two dimensions or for saturable nonlinearities has, surprisingly, received very little attention.

In this Letter we investigate numerically MI with a saturable self-focusing nonlinearity in two dimensions. Saturation is introduced to avoid catastrophic collapse of the field. We have discovered that this problem has solutions that are close to homoclinic orbits, but which oscillate around the exact recurrence. This pseudorecurrence arises only for a restricted range of spatial modulation frequencies.

We consider a propagation in a nonlinear self-focusing medium described by the intensity-dependent dielectric constant

$$\epsilon(I)=\epsilon_0+\Delta\epsilon I/(1+I)\,, \tag{1}$$

where $\Delta\epsilon>0$ is the saturated change in dielectric constant, and I is in units of the saturation intensity. Then, in the slowly varying envelope approximation, we obtain the following scaled equation for the electric-field envelope $A(x,y,z)$:[20]

$$2i\frac{\partial A}{\partial z}+\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}-q^2A+\frac{|A|^2A}{1+|A|^2}=0\,. \tag{2}$$

If we consider a homogeneous solution $A(x,y,z)=A_0\times\exp(i\phi)$, then

$$q^2=A_0^2/(1+A_0^2)\,. \tag{3}$$

The phase ϕ is clearly arbitrary and can be set equal to zero, but we retain it for later purposes. If we now represent the solution of Eq. (2) in the form

$$A(x,y,z)=[A_0+f(x,y,z)]e^{i\phi}\,, \tag{4}$$

and linearize around A_0, we obtain the following equation for the perturbation f:

$$2i\frac{\partial f}{\partial z}+\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{A_0^2}{(1+A_0^2)^2}(f+f^*)=0\,. \tag{5}$$

This equation admits transverse periodic solutions with $k_x=k_y=k$ of the form

$$f(x,y,z)=(\eta_1Fe^{\delta z}+\eta_2F^*e^{-\delta z})[\cos(k_xx)+\cos(k_yy)]\,. \tag{6}$$

Here the complex factor $F=(k+2i\delta/k)k_{\max}$ ($|F|=1$), $k_{\max}^2=2A_0^2(1+A_0^2)^2$, η_1 and η_2 are independently variable small parameters, and the instability growth rate is given by

$$\delta=\tfrac{1}{2}k(k_{\max}^2-k^2)\,. \tag{7}$$

A simple calculation shows that for $k<k_{\max}$, δ acquires its maximum value for $A_0=1$, and we adopt this value in the remainder of this Letter. The dependence of δ on k is shown in Fig. 1, and is valid in both one dimension and two dimensions. However, the form of perturbation in Eq. (6) is only valid for $k_x=k_y=k$, the case we consider here. In general, the perturbation should be in the form

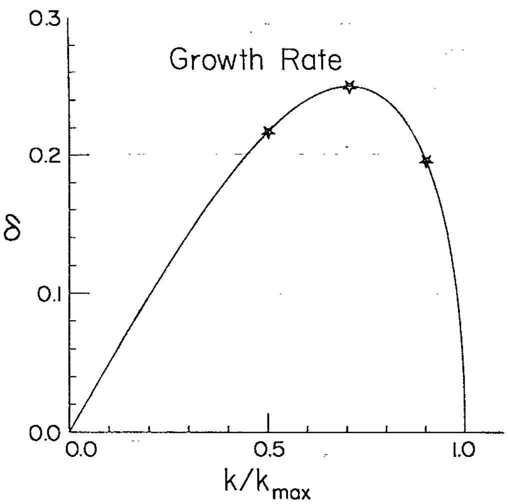

FIG. 1. Instability growth rate δ vs spatial modulation frequency k. The stars correspond to $k=0.5k_{\max}$, $k_{\max}/\sqrt{2}$, $0.9k_{\max}$, which are discussed in the text.

of a product $\cos(k_x x)\cos(k_y y)$.[1] We have chosen the form in Eq. (6) because it results in the smallest spatial dimensions of a unit cell when periodic boundary conditions are imposed. We note that significantly different results can be obtained for $k_x \neq k_y$.

We have obtained numerical solutions of Eq. (2) with periodic boundary conditions using the two-dimensional split-step method.[21] For a given $k<k_{\max}$ the spatial grid L is chosen such that $kL=2\pi$, and the number of mesh points varied from 8×8 to 128×128. The reproducibility of the results was established by varying the numerical parameters over a wide range in every example presented. In all cases we set $\eta_2=0$ and $\eta_1=10^{-4}$.

Numerical results were obtained for spatial modulation frequencies $0<k<k_{\max}$ in which the growth rate is positive (Fig. 1). To display the results, we chose to construct the phase-space diagrams $(\mathrm{Re}(A),\mathrm{Im}(A))$ at several points over the transverse plane with the longitudinal coordinate z parametrizing the evolution. In this representation the homogeneous solution $A=A_0\exp(i\phi)$ $(A_0=1)$ is a unit circle. The initial condition is then (1,0) along with the perturbation given in Eq. (6). Each point on the unit circle then represents a phase-shifted version of the initial condition.

Figure 2 shows the results obtained for $k=0.9k_{\max}$. We see that as the MI develops from the initial phase-space point (1,0) all phase-space trajectories are repelled from it as shown in Fig. 2(a). This is understood by noting that all points on the unit circle have the structure of a saddle point, and the initial condition is chosen to lie along the direction of maximum growth.[13] [The term proportional to η_2 in Eq. (2) corresponds to the direction in which the perturbation decays.] We have also used two-dimensional computer animation to follow the evolution of the full field profile. As the MI grows the field

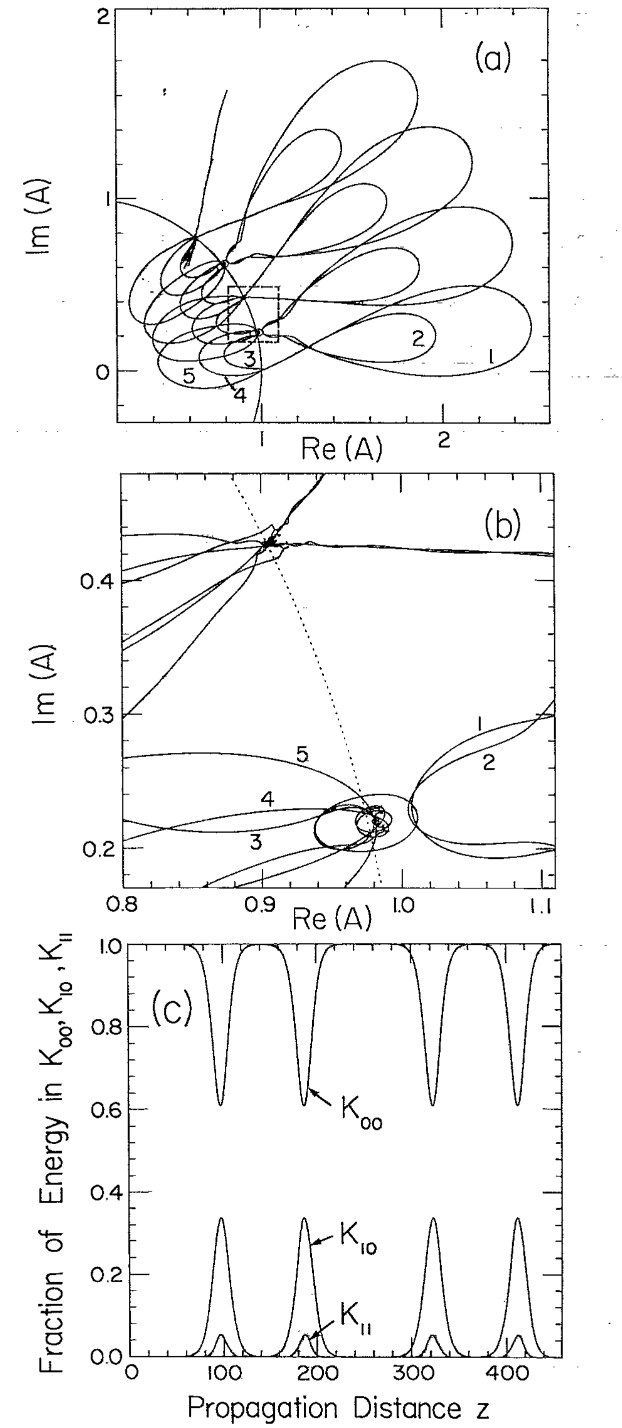

FIG. 2. Phase-space trajectories $(\mathrm{Re}(A),\mathrm{Im}(A))$ for several points in the (x,y) plane including the field maximum (curve 1) and minimum (curve 5) for a spatial modulation frequency $k=0.9k_{\max}$ (denoted by the right-most star in Fig. 1). (a) Starting near the point (1,0), the field evolves through four pseudorecurrences on the unit circle in the counterclockwise direction. (b) Enlarged picture of trajectories inside the dashed box of (a) showing the first two recurrences; the dashed line is the unit circle. (c) The fraction of energy in the components K_{00}, K_{01}, and K_{11} vs propagation distance.

develops a peak in one quadrant of the transverse plane that reaches a maximum and then starts to decrease. This is illustrated in Fig. 2(a) by the large-amplitude trajectory (curve 1), which is at the peak of the field. As the peak continues to decrease in amplitude, we see that the phase-space trajectories tend to regroup back to a new point on the unit circle. This result is clearly indicative of periodic behavior in which the initial condition is reconstructed but phase shifted. Figure 2(a) shows that this cycle can be repeated several times, and we are tempted, perhaps, to claim that we are observing recurrence in which the phase-space trajectories are close to homoclinic orbits. However, it is clear from Fig. 2(a) that the first and third recurrences are different in nature from the second and fourth. Figure 2(b) shows an enlarged section of the phase-space trajectories around the first and second recurrences, and demonstrates that there is no true recurrence. Some phase-space trajectories oscillate around the exact saddle point before diverging again; others are repelled, as would be the case for a true saddle point. We have termed this effect *pseudorecurrence*, since it only has the appearance of a true recurrence from afar [Fig. 2(a)]. We remark that these results can be reproduced reliably and are not numerical artifacts. Pseudorecurrence also appeared in the one-dimensional variation of this problem, and our computer code correctly reproduced the exact recurrences that arise without saturation in one dimension.

To investigate the nature of the pseudorecurrence further, we have also calculated the transverse spatial-frequency profile $\bar{A}$ of the field A using Fourier transformation. In the numerical scheme the spatial frequencies are integer multiples of k, so we label the elements of $\bar{A}$ by two integers m and n, and define $|\bar{A}|^2 = K_{mn}$. K_{00} corresponds to the dc component and must equal 1 on the unit circle, that is, at points of recurrence. To a high degree of accuracy, we have found that for all $K_{mn} \neq K_{00}$, K_{mn} is symmetric under interchange of m and n, and changing the signs of m and n. We therefore include a factor of 4 to account for this degeneracy in the following discussion. Figure 2(c) shows the evolution of K_{00}, K_{01}, and K_{11}; the other components are nonzero but not clearly visible on this scale. Here we clearly see the pseudorecurrence as K_{00} returns close to 1. The other components become populated as K_{00} is depleted. Note that, consistent with Fig. 2(a), K_{00} only appears to repeat after every second cycle. This suggests that a "period 2" oscillation may be occurring, but so far we have found no evidence of a period doubling or other type transition to chaos as k is varied. This possibility is currently being explored in more detail.

Having introduced the phenomenon of pseudorecurrence for a specific example, we now consider the effect of variation of the spatial modulation frequency k. Figure 3 shows the phase-space trajectories at the peak of the field obtained for (a) $k = k_{max}/\sqrt{2}$, which corresponds to the peak of the instability curve in Fig. 1, and

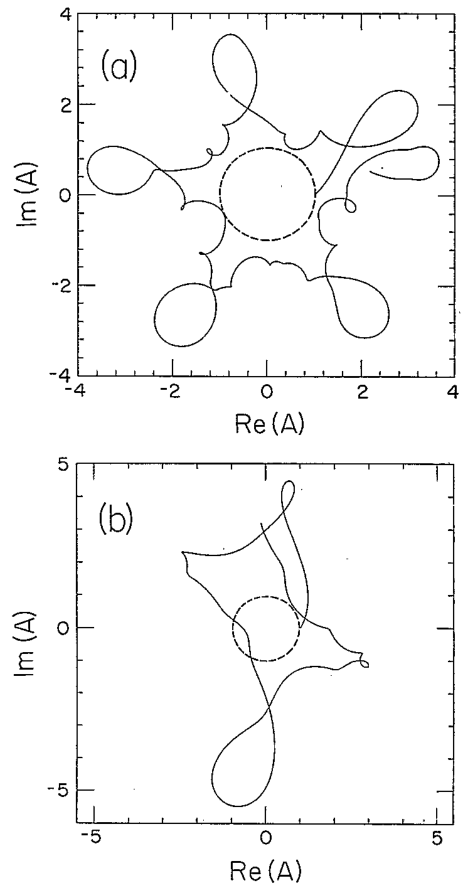

FIG. 3. Phase-space trajectories for (a) $k = k_{max}/\sqrt{2}$ (denoted by the middle star in Fig. 1), and (b) $k = k_{max}/2$ (denoted by the left-most star in Fig. 1). For clarity, only the field trajectory with the maximum amplitude is shown in each case.

(b) $k = 0.5k_{max}$, which is below the maximum. For case (a) there is still pseudorecurrence, though the oscillatory structure around the exact recurrence is clearly larger than in Fig. 2. In contrast, for case (b), the pseudorecurrence is absent and the evolution is irregular. Thus, pseudorecurrence does not appear for all spatial modulation frequencies. To quantify the degree of recurrence we have calculated K_{00} and $\Sigma = K_{00} + K_{10} + K_{11}$ at the first recurrence (first peak in K_{00} after it has been depleted) as a function of k, and the results are shown in Fig. 4. When Σ falls below 1, higher spatial frequencies than those in the sum are excited; for a perfect recurrence $K_{00} = \Sigma = 1$. Figure 4 shows that for $k > 0.6k_{max}$, there is a high degree of recurrence after the first cycle, both K_{00} and Σ being larger than 0.98. (The second recurrence is not necessarily so good.) For $k < 0.6k_{max}$, both K_{00} and Σ drop suddenly and the pseudorecurrence is destroyed. Indeed, in this case the phase-space trajectories become highly irregular, showing no signs of recurrence [see Fig. 3(b)]. Multiple pseudorecurrences of a high degree only occur for values of $k \gtrsim 0.8k_{max}$. For $k < 0.8k_{max}$ the pseudorecurrences

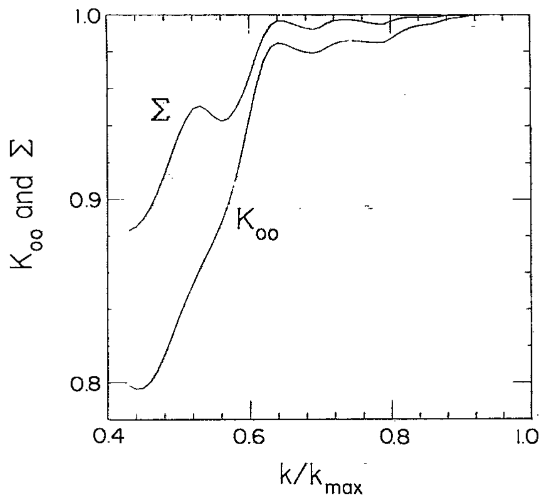

FIG. 4. Energy in the lowest spectral component K_{00} and the sum $\Sigma = K_{00}+K_{10}+K_{11}$, measured at the first recurrence and plotted as a function of the spatial modulation frequency k.

do not persist for as many cycles before irregular behavior ensues. In addition, the size of the oscillatory motion around the exact recurrence increases as k is decreased below $0.9k_{max}$ [see Fig. 3(a)].

Figure 4 provides a clue to the mechanism of the destruction of the pseudorecurrences with decreasing k. We see that for $k < 0.6k_{max}$ more energy is present in the higher spatial frequencies at the first pseudorecurrence since Σ is smaller. We have found that more and more energy transfers to higher spatial frequencies with increasing number of cycles (pseudorecurrences). For $k > 0.8k_{max}$ the transfer of energy is slow, thus allowing several complete cycles. As k is reduced, the rate of energy transfer increases, and eventually no cycles can be completed and irregular behavior follows. The pseudorecurrences are therefore destroyed by the gradual transfer of energy to high spatial frequencies. This behavior was also observed in the one-dimensional version of the problem. In contrast, in the one-dimensional problem with no saturation, which is integrable, this energy transfer is reversible and exact recurrence is obtained. The appearance of pseudorecurrences is therefore a consequence of the saturable nonlinearity which renders the system nonintegrable.

In conclusion, we have discovered that modulation instability of a two-dimensional field with a saturable self-focusing nonlinearity leads to pseudorecurrences rather than exact recurrences. These pseudorecurrences occur only for a restricted range of spatial-frequency modulations, and are a consequence of the saturable nonlinearity which makes the problem nonintegrable. These results should be of great significance in the study of chaos in nonlinear wave propagation.

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