Partially Coherent Solitons of Variable Shape

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We present analytical and numerical results related to partially coherent solitons (PCS) of a novel class. In effect, there is freedom to choose the shape of a PCS as it is governed by $2N - 1$ parameters, where N is the number of linear modes contributing to the soliton. In particular, we show that a PCS may have an asymmetric shape. The PCS shape becomes arbitrary in the limit of complete incoherence. Another remarkable new feature of a PCS is that collisions in Kerr-like media cause the PCS to change its shape, although each beam remains a stationary soliton after the collision. [S0031-9007(98)07677-7]

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The notion of temporal incoherent solitons was introduced by Hasegawa some 20 years ago in a series of works [1–3], both for plasma waves and for nonlinear pulses in multimode fibers. However, as the creation of incoherent solitons in optical fibers requires unrealistically high pulse energies, photorefractive materials are the medium of choice for experimental studies, as they generally exhibit very strong nonlinear effects with extremely low optical powers [4,5]. The first experimental observation of partially incoherent solitons has been made by Mitchell *et al.* [6].

A theoretical description of spatial incoherent solitons, based on the so-called "coherent density approach," where the partially coherent beam is represented as a superposition of mutually incoherent components, has been developed by Christodoulides *et al.* [7,8]. For the special case of the logarithmic nonlinearity, the symmetric solutions can be written in analytic form [7]. The description of a partially coherent stationary soliton (PCS) as a multimode self-induced waveguide [9–12] has been especially fruitful. In that view, stationary soliton propagation is obtained by proper population of various mutually incoherent linear modes of the self-induced waveguide. Because of mutual incoherence, the total light intensity is a direct sum of the intensities of all excited modes. Thus mode beating, which is a signature of coherent excitation, is eliminated. On the other hand, we found that this approach is not sufficient to explain all the properties of the PCS. We claim that an additional viewpoint, seeing the PCS as multisoliton complexes, gives us more information about their shape and collisions.

Complex soliton structures, in the context of temporal vector solitons, have been investigated earlier in [13,14]. Higher-order vector solitons have been studied in [15]. The concept of a vector soliton as multimoded waveguide self-induced by its linear modes has been suggested by Snyder *et al.* [16,17] in the study of spatial soliton structures. As we mentioned above, the PCS is a similar object physically. It is a multimode self-induced waveguide in a slowly responding medium, so that its linear modes are mutually incoherent. Experimental demonstration of multimode spatial solitons formed by incoherent superposition of their linear modes in photorefractive crystals has been reported recently in [18].

The diffractionless ray optics limit for treating spatial incoherent solitons has been proposed by Snyder and Mitchell [19]. This approach is accurate when the size of the PCS is much larger than the optical wavelength. The latter have been nicknamed [19] "big incoherent solitons." In terms of a multimode waveguide, this limit is valid when the number of modes goes to infinity, so that the soliton becomes completely incoherent. The interaction of incoherent and partially coherent solitons is an interesting area of research, and it has only been addressed in the recent papers [10,11].

Most of the above-referenced works showed only the existence of symmetric solutions for PCSs. On the other hand, Hasegawa [1] (in the case of 1D solitons in Kerr-like media) and Snyder *et al.* [19] (in the case of 3D solitons in media with arbitrary nonlinearity) pointed out that incoherent solitons in general may have an arbitrary shape (at least in the regime of complete incoherence). In the present paper, we investigate this problem using the dual character of PCSs as self-induced linear waveguides, as well as multisoliton complexes. Thus, we have found that PCSs can have a profile which is variable to a certain degree and which is governed by a finite number of parameters. The number of parameters depends on the number of linear modes comprising the PCS. At one extreme, when the PCS forms a single-mode waveguide, the soliton is coherent, its shape is symmetric, and it is described by the sech function. This is the case of a single fundamental soliton. At the other extreme, when the number of modes goes to infinity, the number of parameters which control the shape is also infinite. In this limit the soliton effectively has an arbitrary profile.

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We restrict ourselves to PCSs in Kerr-like media so that the wave evolution can be represented by a set of N coupled Manakov equations. Examples of the lowest-order symmetric solutions and their interactions have been presented in recent works [10,11]. However, those solutions contain only one free parameter. They are symmetric solutions of a given (sech) profile and have variable amplitude (and width) for any particular N. We show in this work that the actual PCS profiles are multiparameter families of solutions and that their shape and amplitude may vary. For finite N, the number of parameters governing the stationary PCS profile is at least $2N - 1$. We also investigate collisions of PCS. We find that, in the case of Kerr-like media, PCSs change their shape after collisions, but nevertheless remain stationary solutions.

The set of equations describing propagation of N self-trapped mutually incoherent wave packets in media with Kerr-like nonlinearity is

$$i\,\frac{\partial\psi_i}{\partial z} + \frac{1}{2}\,\frac{\partial^2\psi_i}{\partial\tau^2} + \alpha\,\delta n(\psi_i)\psi_i = 0\,, \tag{1}$$

where ψ_i denotes the ith component of the beam, α is the coefficient representing the strength of nonlinearity, τ is the transverse coordinate, z is the coordinate along the direction of propagation, and

$$\delta n(\psi_i) = \sum_{i=1}^{N} |\psi_i|^2 \tag{2}$$

is the change in refractive index profile created by all incoherent components of the light beam. The response time of the nonlinearity is assumed to be long compared to temporal variations of the mutual phases of all components, so the medium responds to the average light intensity, and this is just a simple sum of modal intensities expressed by the relation (2). As a result, the set of equations (1) is a generalized Manakov set which has been shown to be integrable [20]. This means that all solutions, in principle, can be written in analytical form.

Stationary solutions of (1) are given by

$$\psi_i(\tau, z) = \frac{1}{\sqrt{\alpha}}\,u_i(\tau)\,\exp\!\left(i\,\frac{\lambda_i^2}{2}\,z\right), \tag{3}$$

with real functions u_i, so that the set of Eqs. (1) reduces to the set of ODEs:

$$\frac{\partial^2 u_i}{\partial\tau^2} + 2\left[\sum_{i=1}^{N} u_i^2\right]u_i = \lambda_i^2 u_i\,, \tag{4}$$

which is also completely integrable for an arbitrary set of real λ_i. It can be shown, using Poisson brackets, that the set of ODEs (4) has N conserved quantities, namely the Hamiltonian H:

$$H = \sum_{i=1}^{N}(\dot{u}_i^2 - \lambda_i^2 u_i^2) + \left(\sum_{i=1}^{N} u_i^2\right)^2 = \text{const}\,, \tag{5}$$

and $N - 1$ additional integrals I_k

$$I_k = \sum_{i\neq j}(\dot{u}_i^2 u_j^2 - u_i^2\dot{u}_j^2)^2 - \sum_{i\neq k}^{N-1}\Delta\lambda_{ki} \times\left(\dot{u}_i^2 + u_k^2\sum_{m\neq i}^{N-1} u_m^2 + u_i^4 - \lambda_i^2 u_i^2\right) = \text{const}\,, \tag{6}$$

where $\Delta\lambda_{ki} = \lambda_k^2 - \lambda_i^2$ and each dot over u_i denotes a derivative with respect to τ. For zero background solutions, the integrals must be equal to zero ($H = 0$ and $I_k = 0$).

We note, from (4), that the constants λ_i have a dual physical meaning. First, they can be considered as the amplitudes of partial fundamental solitons in the multisoliton complex. Suppose that all fundamental solitons are separated in space and all u_i but one are close to zero at a certain $\tau = \tau_0$. Then the amplitude of that soliton of the single equation which is left is λ_i. Second, if we consider $[\sum |u_i|^2]$ as a given self-induced refractive index profile, then each λ_i is an eigenvalue related to a certain mode of the self-induced waveguide. In our analysis it is important that the number of linear eigenvalues N equals the number of fundamental solitons in the multisoliton complex.

For the case $N = 2$ we have:

$$\begin{aligned} u_{\tau\tau} + 2(u^2 + v^2)u &= \lambda_1^2 u\,,\\ v_{\tau\tau} + 2(u^2 + v^2)v &= \lambda_2^2 v\,, \end{aligned} \tag{7}$$

where we consider u and v to be real. This set of ODEs has two conserved quantities, which follow from (5),(6). Namely, the Hamiltonian is

$$H = u_\tau^2 + v_\tau^2 - \lambda_1^2 u^2 - \lambda_2^2 v^2 + (u^2 + v^2)^2 = \text{const}\,, \tag{8}$$

and the second integral [21] (for the momentum $M = uv_\tau - u_\tau v$) is

$$I = M^2 + \Delta\lambda_{12}(u_\tau^2 + u^4 + u^2 v^2 - \lambda_1^2 u^2) = \text{const}\,. \tag{9}$$

A solution of (7) which describes PCSs is [22,23]:

$$u = \lambda_1\sqrt{\Delta\lambda_{12}}\,\cosh(\lambda_2\tau_2)/D\,, \tag{10a}$$

$$v = \lambda_2\sqrt{\Delta\lambda_{12}}\,\sinh(\lambda_1\tau_1)/D\,, \tag{10b}$$

$$D = \lambda_1\cosh(\lambda_1\tau_1)\cosh(\lambda_2\tau_2) - \lambda_2\sinh(\lambda_1\tau_1)\sinh(\lambda_2\tau_2)\,,$$

where $\tau_1 = \tau - \tau_{01}$, $\tau_2 = \tau - \tau_{02}$, and λ_1, λ_2, τ_{01}, τ_{02} are arbitrary real constants. The change in refractive index profile self-induced by the beam is given by

$$\delta n = \Delta\lambda_{12}[\lambda_1^2\cosh^2(\lambda_2\tau_2) + \lambda_2^2\sinh^2(\lambda_1\tau_1)]/D^2. \tag{11}$$

The solution (10) has three nontrivial free parameters λ_1, λ_2, and $\Delta\tau_{12} = \tau_{02} - \tau_{01}$. Two examples of this soliton, e.g., its intensity profile [which also represents the refractive index profile (11) of the corresponding

waveguide], as well as the functions u and v, are shown in the inset of Fig. 1. It is easy to see that one of the components (u) is a zeroth order linear mode of the self-induced waveguide and that the other one (v) is the first order linear mode. For the two examples shown in Fig. 1, the first soliton is chosen to be a symmetric function of the spatial coordinate ($\Delta\tau_{12} = 0$), while the second one is an asymmetric function ($\Delta\tau_{12} = 2$). In the latter case, the functions u and v are also asymmetric. This is a new striking feature of PCSs, and is in contrast to the solutions found in [10,11]. It should be emphasized that even for $\Delta\tau_{12} = 0$, the shape, the amplitude and the width of the PCS are still arbitrary and are defined by the two other parameters, λ_i. The symmetric solution of [10,11] for $n = 2$ is only a very particular case of (10),(11) when $\lambda_1 = 2\sqrt{2\beta_0}$ and $\lambda_2 = \sqrt{2\beta_0}$, where β_0 is a parameter defined in [10,11].

An alternative view of the solution (10) is that it is actually a two-soliton solution of the Manakov set of

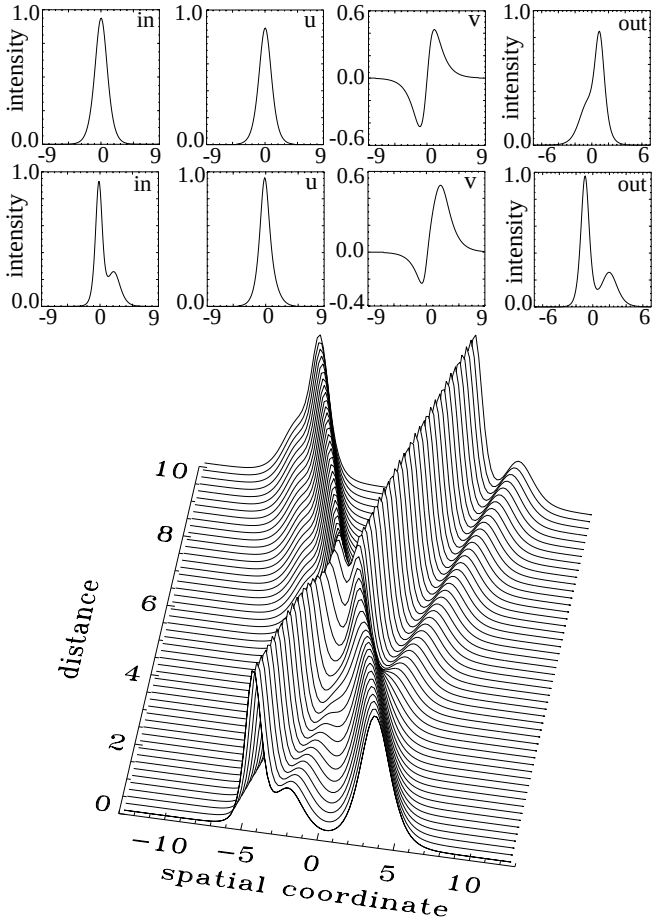

FIG. 1. Collision of a symmetric and an asymmetric PCS consisting of two linear modes. The inset shows the initial profiles of the solitons, amplitude profiles of both linear modes u and v of the corresponding waveguide as well as the final soliton profiles after the collision. Parameters chosen in calculations are $\lambda_1 = 1.0$, $\lambda_2 = 0.5$, $\Delta\tau_{12} = 0$ (for symmetric solution) and $\Delta\tau_{12} = 0.2$ (for asymmetric solution).

equations. When the parameter $\Delta\tau_{12}$ is big in comparison with the soliton width, the solution effectively separates into two independent solitons, with eigenvalues λ_1 and λ_2 arbitrarily located on the τ axis. As we explained above, the number of linear modes in the waveguide coincides with the number of fundamental solitons N in the PCS. Hence, the parameters which control the PCS shape are the N amplitudes and $N - 1$ distances $\Delta\tau_{ij}$.

Higher order (i.e., for $N > 2$) exact solutions of equations (1) can be obtained using dressing or direct methods [23,24]. Doing this, we keep in mind that the solution for $N = 3$ has 5 ($= 2N - 1$) free parameters governing its profile. These are the three eigenvalues, λ_i, and two relative distances $\Delta\tau_{ij}$ between the fundamental solitons of this nonlinear superposition.

An example of a PCS comprising three linear modes of its own waveguide is shown in the inset of Fig. 2. The eigenvalues λ_i in this case are $\lambda_1 = 1.22$, $\lambda_1 = 3.01$, and $\lambda_3 = 4.38$. The values $\Delta\tau_{ij}$ are small but nonzero. The PCS profile in this particular case is asymmetric and almost rectangular with three small peaks at the top. In general, the shape is variable and can be symmetric (when $\Delta\tau_{ij} = 0$) or even single-peaked, as in [10,11]. The three functions u_i (namely u, v, and w) are always the three linear modes of the self-induced waveguide.

The understanding of PCS as multisoliton complex suggests that collisions must reshape them. In fact, the N eigenvalues λ_i must be conserved during the collision, but the $N - 1$ relative distances $\Delta\tau_{ij}$ must change. As a result, the shapes of PCSs do not have to be preserved. This change can be calculated using the Manakov result [25] for pairwise collisions. Because of integrability, other components do not influence the results for pairwise interactions during which any two soliton components are always "orthogonal" to each other. All lateral shifts are then additive quantities. Adding up the shifts for each particular collision gives general collision induced shift for ith soliton in PCS:

$$\delta\tau_i = \frac{\sqrt{2}}{8\lambda_i} \sum_{k=1}^{N} \ln \sqrt{\frac{(\tan\theta_1 - \tan\theta_2)^2 + (\lambda_i + \lambda_k)^2}{(\tan\theta_1 - \tan\theta_2)^2 + (\lambda_i - \lambda_k)^2}}, \tag{12}$$

where θ_1 and θ_1 are angles of incidence of two PCS correspondingly. Clearly, these shifts are different for each soliton component. The net result is PCS reshaping. We should also mention that because of integrability of the model, collisions are elastic and radiation waves are not created. The output consists only of the reshaped PCS. Physically the reshaping phenomenon can also be understood as mutual refraction of PCS on the self-induced waveguides. Since all the consituent modes of PCS have different phase velocities, they experience different rates of refraction in the impact area of collision. Self-consistent reassembling of modes after the collision results in stationary output beam with another shape.

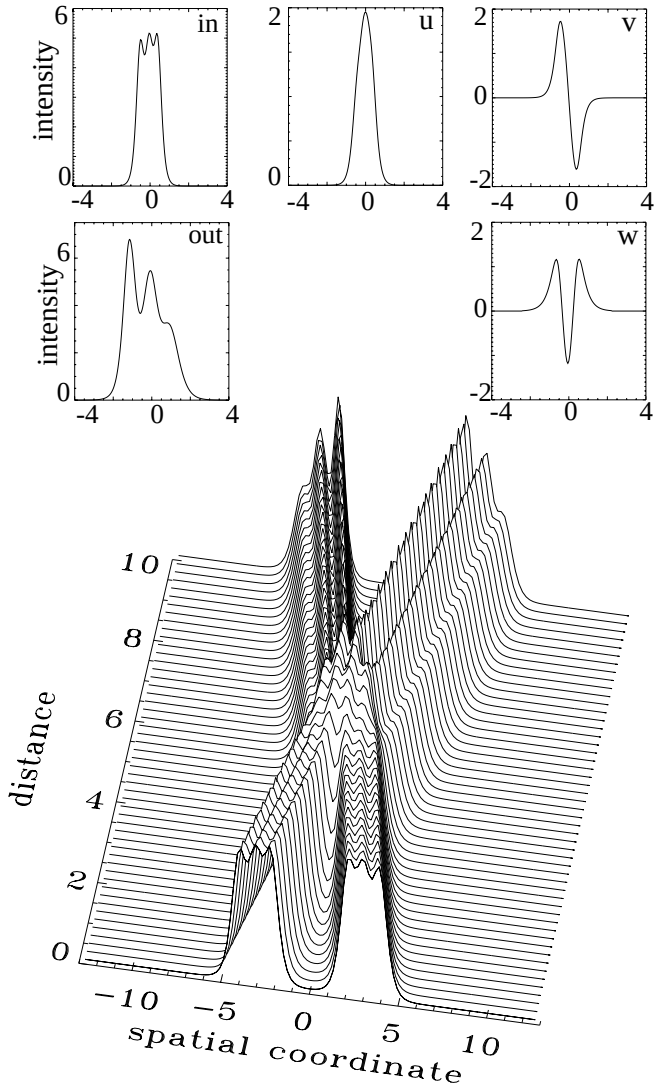

FIG. 2. Collision of PCSs formed by three linear modes. The inset shows the initial intensity profile of the soliton, amplitude profile of all three linear modes, u, v, and w, as well as the final intensity profile of the soliton after the collision.

We investigated collisions of PCSs using numerical simulations. A collision for the case $N = 2$ is shown in Fig. 1 and one for the case $N = 3$ is shown in Fig. 2. In the first case (Fig. 1), symmetric and asymmetric solitons collide, while in the second (Fig. 2), an asymmetric soliton collides with its mirror image. These simulations confirm unusual collisional properties of PCSs predicted above. The collision induces a dramatic change of shape of the solitons. After the collision, each beam remains a soliton but has an intensity profile different from the initial one. It is evident from Fig. 1 that an initially symmetric soliton becomes clearly asymmetric. The reshaping effect is even stronger for the collision shown in Fig. 2, where initial solitons develop strong multipeak structure after the collision. The values of shifts for each component can be estimated from numerical results and they are in agreement with Eq. (12). It follows from (12) that when the order of the PCS is higher, then the reshaping is stronger. This is a remarkable feature of a PCS collision, and it differs drastically from a standard collision between two fundamental bright solitons.

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