Waves that appear from nowhere and disappear without a trace

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A R T I C L E   I N F O

A B S T R A C T

The title (WANDT) can be applied to two objects: rogue waves in the ocean and rational solutions of the nonlinear Schrödinger equation (NLSE). There is a hierarchy of rational solutions of 'focussing' NLSE with increasing order and with progressively increasing amplitude. As the equation can be applied to waves in the deep ocean, the solutions can describe "rogue waves" with virtually infinite amplitude. They can appear from smooth initial conditions that are only slightly perturbed in a special way, and are given by our exact solutions. Thus, a slight perturbation on the ocean surface can dramatically increase the amplitude of the singular wave event that appears as a result.

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"Rogue waves" [1], "freak waves" [2], "killer waves" and similar names have been the topic of several recent publications related to giant single waves appearing in the ocean "from nowhere". Hitherto, we do not have a complete understanding of this phenomenon due to the difficult and risky observational conditions. Those who experience these phenomena while being on a ship would be busy saving their lives rather than making measurements. It is difficult to explain the high amplitudes that can occur in the open ocean using linear theories based on the superposition principles.

Nonlinear theories of ocean waves [3–6] are more likely to explain why the waves can "appear from nowhere" than linear theories. The reason for the phenomenon can lie in the instability of a certain class of initial conditions that tend to grow exponentially and hence have the possibility of increasing up to very high amplitudes. Zakharov [7] was the first to show that the focussing nonlinear Schrödinger equation (NLSE) is applicable in deep water, and that envelope solitons can appear there. In contrast, the defocussing equation would apply in shallow water, and this only has 'hole' envelope solutions and no 'bright' solitons.

The best-known examples of exponential growth of waves are due to the Benjamin–Feir instability [8], which is also known as Bespalov–Talanov instability [9]. This instability was originally considered by Turing [10] with an application to biological objects. The growth of instability is initially exponential, but it was noticed [11] that it saturates during later stages and decays back to the starting condition. Thus, in nonlinear theory of conservative systems,

every wave that appears from nowhere must disappear without a trace.

The first ideas in this regard can be attributed to Fermi et al. [12]. In the Fermi–Pasta–Ulam model, which is a set of nonlinearly-coupled oscillators, the energy which has been launched into a certain mode will eventually return to the same mode instead of being equi-partitioned – i.e. dispersed equally between the modes. Exact solutions of the nonlinear Schrödinger equation [11] showed that recurrence is indeed a specific property of nonlinear systems [13]. The return to the initial condition occurs both in one-dimensional and (less accurately) in two-dimensional systems [14]. The type of nonlinearity can also be quite general [14]. The latter case is more applicable to the ocean waves.

Recently [15,16], it has also been found that rogue waves can be generated in optical systems, thus creating another possibility for generating highly energetic optical pulses. Although, the technique has a probabilistic nature, i.e. only one pulse among thousands can have high energy [15], it can be useful as another alternative to the existing techniques such as passively mode-locked lasers [17].

The integrability of the nonlinear Schrödinger equation was discovered by Zakharov and Shabat [18]. As numerous applications were found, the equation became quite popular. In normalized form, it can be written as:

$$i\frac{\partial \psi}{\partial x} + \frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + |\psi|^2 \psi = 0 \tag{1}$$

where x is the propagation distance and t is the transverse variable. These notations are standard in nonlinear fiber optics [19] and in the theory of ocean waves [5]. Note that ψ is an envelope of the physical solution and in optics its squared modulus represents a measurable quantity – intensity. For the ocean waves,

Fig. 1. Akhmediev breather.

Fig. 2. Maximal amplitude of solution given by Eq. (3) versus the parameter *a*.
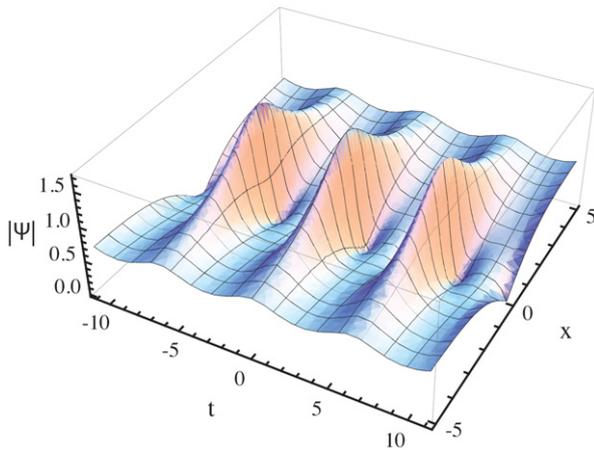
we assume [20] that there is a carrier wave with the wavelength $\lambda(\approx \omega^{-1})$ comparable to the central region of the envelope. So the actual water height, relative to the equilibrium sea-level, would be $|\psi|\cos(\omega t)$. The potential energy of a segment in t, of width Δt, is then proportional to $|\psi|^2 \Delta t$. Then, this value for ocean waves acts like intensity in optics.

Generally, Eq. (1) can be solved for arbitrary smooth initial conditions which are either periodic or have zeroes at infinity. One of the basic class of localized (in t) solutions of the NLSE consists of soliton solutions. Examples of another class of solutions of the NLSE are localized in x, i.e. they represent unique events in x. These solutions increase their amplitudes, either exponentially or according to a power law in x, then reach their maximum value and finally decay symmetrically to disappear forever. One of these solutions describes modulation instability [11]:

$$\psi = \frac{\cos(\sqrt{2}t) + i\sqrt{2}\sinh(x)}{\cos(\sqrt{2}t) - \sqrt{2}\cosh x}e^{ix}. \tag{2}$$

In recent publications [3,20,22,23], this was named the "Akhmediev breather" although the breathing actually happens only once. A plot of this solution is shown in Fig. 1. As we can see, the initially small periodic modulation along t increases to a maximum value at $x = 0$ and then decays back to a non-modulated state. The maximum of this solution at $t = 0$ and $x = 0$ is $1 + \sqrt{2} \approx 2.41$. The solution is a particular case of a family of solutions with an arbitrary period of modulation along t axis [27]:

$$\psi = \frac{(1 - 4a)\cosh(\beta x) + \sqrt{2a}\cos(pt) + i\beta\sinh(\beta x)}{[\sqrt{2a}\cos(pt) - \cosh(\beta x)]}e^{ix} \tag{3}$$

where $\beta = \sqrt{8a(1 - 2a)}$ and $p = 2\sqrt{(1 - 2a)}$. The maximal amplitude of this solution versus the parameter a is shown in Fig. 2. When $a = 1/4$, this solution is transformed into (2). This is the special case with the maximal value of the growth rate of modulation instability. When the period of modulation $(2\pi/p)$ increases to infinity ($a \to 0.5$), the solution becomes a rational one.

There is widespread belief that rational solutions can exist only in the case of self-defocussing NLSE [24]. Indeed, these solutions have been studied in great detail by Hone [24], and more recently by Clarkson [25]. One of the remarkable features of their rational solutions of the self-defocussing NLSE is that they are singular. These solutions contain infinities and thus do not represent physical situations. However, other works have quoted finite 'rational solutions' for the focussing case (see e.g. [26,27]). The confusion is due to the factor $\exp(ix)$ which appears in all solutions of (1). As this factor does not appear in the measurable quantities like intensity, in most cases of physical interest it is irrelevant. Here we look
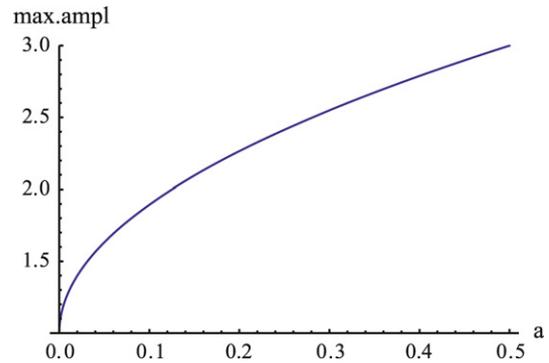
for solutions of the form $\frac{f}{g}\exp(ix)$, where f and g are binomials to get the finite solutions. Omitting the exponential factor and using only $\frac{f}{g}$ leads to singular solutions [24,25].

Indeed, there is a hierarchy of rational solutions of the self-focussing NLSE (1) with increasing order that starts with the simplest one. We can describe them all with the following basic structure:

$$\psi = \left[1 - \frac{G + iH}{D}\right]e^{ix}. \tag{4}$$

Here D should have no zeros to ensure that the solution is finite everywhere. For the "first order" one, obtained by taking the limit $a \to 0.5$ in the modulation instability solution (3), we find $G = 4$, $H = 8x$ and $D = 1 + 4t^2 + 4x^2$. So the full solution is:

$$\psi = \left[1 - 4\frac{1 + 2ix}{1 + 4x^2 + 4t^2}\right]e^{ix}. \tag{5}$$

The maximum amplitude, $|\psi|$, of this solution occurs at $t = 0$ and $x = 0$ and is equal to 3 (see Fig. 2).

Note that the "intensity" minus background is

$$|\psi|^2 - 1 = 8\frac{1 + 4x^2 - 4t^2}{(1 + 4x^2 + 4t^2)^2}$$

and the integral of this (w.r.t. t) is zero for any x. On the line $x = 0$, we obtain $|\psi| = |3 - 4t^2|/(1 + 4t^2)$ so $|\psi| = 0$ for $t = \pm\sqrt{3}/2$. On this central line, $|\psi|^2 = 1$ for $t = \pm\frac{1}{2}$. The integral of $|\psi|^2 - 1$ from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$ is 4, while its integral over the rest of the line is -4. This shows clearly how the freak wave 'concentrates' energy in towards its central part.

There is also a lesser known higher-order rational solution. It has the form of Eq. (4) with G, H and D given by:

$$G = -\frac{3}{16} + \frac{3}{2}t^2 + t^4 + \frac{9}{2}x^2 + 6t^2x^2 + 5x^4$$

$$= \left(t^2 + x^2 + \frac{3}{4}\right)\left(t^2 + 5x^2 + \frac{3}{4}\right) - \frac{3}{4},$$

$$H = -\frac{15}{8}x - 3t^2x + 2t^4x + x^3 + 4t^2x^3 + 2x^5$$

$$= x\left(x^2 - 3t^2 + 2(t^2 + x^2)^2 - \frac{15}{8}\right),$$

$$D = \frac{3}{64} + \frac{9t^2}{16} + \frac{t^4}{4} + \frac{t^6}{3} + \frac{33}{16}x^2$$

$$- \frac{3}{2}t^2x^2 + t^4x^2 + \frac{9}{4}x^4 + t^2x^4 + \frac{x^6}{3}$$

$$= \frac{1}{3}(t^2 + x^2)^3 + \frac{1}{4}(t^2 - 3x^2)^2 + \frac{3}{64}(12t^2 + 44x^2 + 1).$$

This solution can be obtained using Darboux transformation [27,28] or equivalent techniques. The factorization shows that
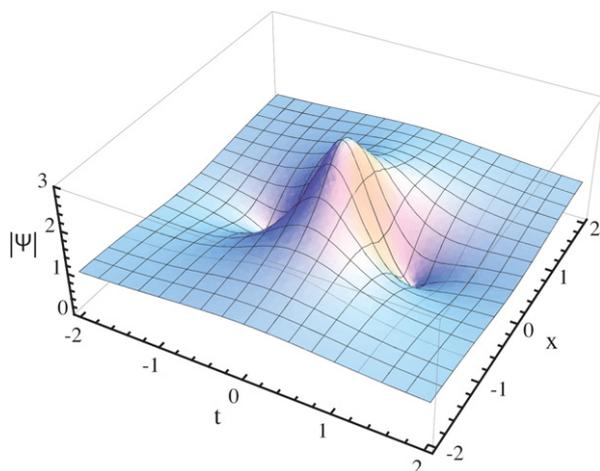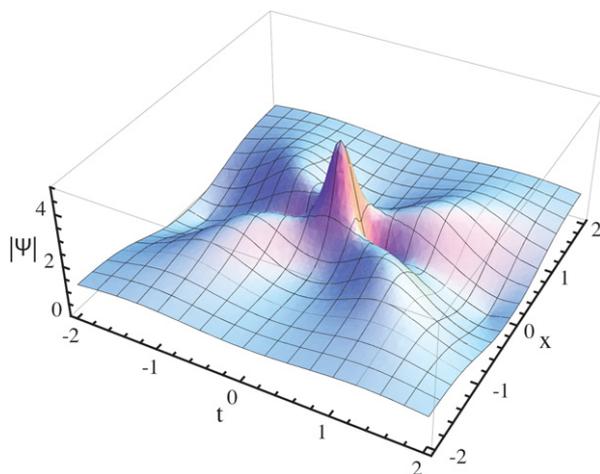
Fig. 3. First-order rational solution.

Fig. 4. Higher-order rational solution.

$D > 0$ for all x, t, so the solution is not singular and can represent a real wave. The maximum amplitude of this solution is 5, i.e. much higher than that of the first order rational solution, Eq. (5). On the line $x = 0$, we now have $|\psi| = 0$ at 4 points, viz. $t = \pm 0.465$ and $t = \pm 1.757$. Comparing the solution in Fig. 4 with the one in Fig. 3, we can say that the latter is more likely to break up a ship than the former one.

Are there WANDT solutions with even higher amplitude? Surely there are. The solution of any higher-order can be obtained using the same technique as for deriving the solution above, i.e. the Darboux transformations [28]. However, their explicit forms are far too complicated to be written on a single journal page. They are nothing other than nonlinear superpositions of the first-order rational solutions. The results obtained for the modulation instability solutions [28] show that the amplitude of these solutions progressively increases and can be virtually unlimited.

If we know exact solutions, we can create initial conditions at any particular negative x that would allow us to generate them. This would not be very helpful for ocean waves, but could be useful for the generation of "rogue waves" in optics. Presently, they are generated randomly [15] and their detection requires probabilistic equipment. Creating special initial conditions at the input of a fiber would provide much more deterministic way of dealing with them.

There is another misconception that the amplitude of solutions presented above is the absolute maximum that can be reached. We note that the NLSE has the so-called "scaling transformation" [21].

This transformation shows that any of its solutions can be transformed to provide even higher amplitudes. Then the event will happen on shorter x and t scales. Namely, if $\psi(x, t)$ is a solution of Eq. (1) then

$$\psi'(x, t, q) = q\psi\left(qt, q^2 x\right) \tag{6}$$

is also a solution of the same equation for an arbitrary real q. To be specific, if we use $q = 100$ in Eq. (5) we obtain:

$$\psi = 100\left[1 - 4\frac{1 + 2 \times 10^4 ix}{1 + (200)^2 t^2 + 4(10)^8 x^2}\right]e^{i10^4 x}. \tag{7}$$

The amplitude of this solution is 300. Needless to say, this event happens on much shorter scales in x and t. Such a wave would be highly unexpected for any captain in the sea. The scaling transform can be applied to every solution of the NLSE given above. Of course, in reality, we have to take into account viscosity, the finite depth of the sea, bottom friction [23], dissipation and many other effects that exist in real oceans. These all restrict the maximum height of the rogue waves.

Generally, these waves require longer time to form, as their growth rate has a power law rather than an exponential one. However, the ocean is big and there is enough space for them to be born. They also need special initial conditions to be created. Indeed, these initial conditions can be part of the chaotic small amplitude waves that the ocean is covered with. On the other hand, some parts of the ocean may have a specific bottom configuration that would make the proper initial conditions more likely than in other parts of the world. Maybe the Bermuda triangle has these specific properties?

In conclusion, we have presented examples of solitary wave solutions of the nonlinear Schrödinger equation that "appear from nowhere and disappear without a trace". We show that the amplitude of each of these solutions can take arbitrarily high values that depend on initial conditions. These properties of solutions make them a very likely explanation for the so-called "rogue waves" in the ocean, since these also may have unusually high amplitudes and appear "from nowhere". Another application for these results is the generation of exceptionally high amplitude optical pulses – "optical rogue waves".

Acknowledgements

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