Invited paper

Dissipative soliton interactions inside a fiber laser cavity

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Abstract

We report our recent numerical and experimental observations of dissipative soliton interactions inside a fiber laser cavity. A bound state, formed from two pulses, may have a group velocity which differs from that of a single soliton. As a result, they can collide inside the cavity. This results in a variety of outcomes. Numerical simulations are based either on a continuous model or on a parameter-managed model of the cubic-quintic Ginzburg–Landau equation. Each of the models provides explanations for our experimental observations.

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1. Introduction

Soliton interaction is one of the most exciting areas of research in nonlinear dynamics. The unusual features of collisions in systems described by the Korteweg–de Vries (KdV) equation [1] were the starting point of these intensive studies. It was discovered that solitons in this system pass through each other without changing their amplitude and
velocity [1]. As a consequence, solitons can exist in such systems and their total number remains constant.

Dissipative systems can also support stable solitons [2]. However, their properties are completely different from those described by the KdV equation. Firstly, dissipative soliton solutions are fixed. Their shape, amplitude, velocity are all fixed and defined by the parameters of the system [3] rather than by the initial condition. Secondly, dissipative solitons exist only when there is a continuous energy supply to the system. Whenever the energy supply is stopped, solitons “die.” Dissipative solitons have an internal energy exchange mechanism which makes them similar to biological objects. Dissipative solitons may have pulsating dynamics [4]. All these properties make them attractive objects for research.

The formation of soliton pairs is another aspect of the problem of soliton interactions. Here the differences between integrable and dissipative systems become significant. In integrable systems, such as the one modeled by the nonlinear Schrödinger equation, two bright temporal solitons having different amplitudes and velocities can undergo elastic collisions, but they cannot form a stable bound state of two solitons. In contrast, in the case of dissipative solitons, it was found theoretically that stable asymmetric soliton pairs can be generated when we model a system by the complex cubic-quintic Ginzburg–Landau equation (CCQGLE) [5]. These stable soliton pairs were later observed experimentally in a fiber ring laser [6]. The key feature of these soliton pairs is that the phases of the two components are in quadrature. Consequently, the group velocity of the pair is different from that of a single soliton. When a soliton pair (doublet) and a soliton ‘singlet’ appear simultaneously in a cavity, they must collide at some stage.

In Ref. [7], we presented a preliminary study of these interactions in a fiber laser. Similar interactions have been observed in the experimental work [8]. In the present article, we give the results of massive numerical simulations for a laser model with distributed parameters, as well as for a model with parameter management. For each model, the simulations show a variety of outcomes after the collision. These give qualitative explanations for the results observed experimentally. In particular, we observed “elastic” collisions between a pair and a single soliton, as well as the formation of a stable triplet soliton state.

2. Soliton collisions in the continuous model

One of the simplest models for a passively mode-locked laser with a fast saturable absorber is the cubic-quintic complex Ginzburg–Landau equation. For this context, this equation has the following form [9,10]:

\[
i \psi_z + \frac{D}{2} \psi_{tt} + |\psi|^2 \psi + \nu |\psi|^4 \psi = i \delta \psi + i \varepsilon |\psi|^2 \psi + i \beta \psi_{tt} + i \mu |\psi|^4 \psi,
\]

(1)

where \( z \) is the propagation distance traversed in the cavity, \( t \) is the retarded time, \( \psi \) is the normalized envelope of the field, \( D \) is the group velocity dispersion coefficient, \( \delta \) is the linear gain-loss coefficient, \( i \beta \psi_{tt} \) accounts for spectral filtering or linear parabolic gain (\( \beta > 0 \)), \( \varepsilon |\psi|^2 \psi \) represents the nonlinear gain (which arises, e.g., from saturable absorption), the term with \( \mu \) represents, if negative, the saturation of the nonlinear gain, while the one with \( \nu \) corresponds, also if negative, to saturation of the nonlinear refractive index.
As the system we are considering is not integrable, we cannot use the powerful technique of the inverse scattering or its equivalent to solve the initial value problem. Instead, we will use the notions of nonlinear dynamics developed for dissipative systems. We have to remember that the system we are dealing with is infinite-dimensional [11].

We are interested in stable stationary solitons and their interactions. Stable solitons exist in finite regions of the five-dimensional parameter space, i.e., in the space of $\beta$, $\mu$, $\nu$, $\epsilon$, and $\delta$. The values that these parameters may take can be restricted in order to have solutions in the form of stable localized structures. Specifically, we take $\epsilon > 0$, $\delta < 0$, $\mu < 0$, $\beta > 0$, and $\nu \leq 0$. The condition $\delta < 0$ is required to keep the zero background stable. The condition $\epsilon > 0$ ensures that there is positive gain in the system. The condition $\mu < 0$ allows us to saturate the nonlinear gain in order to limit the soliton amplitude from above and $\nu < 0$ saturates the nonlinear Kerr effect. Finally, the condition $\beta > 0$ provides stability of the soliton in the frequency domain.

The above restrictions allow us to limit the boundaries in the search for single solitons. Specific regions where single stable solitons exist are much narrower than those defined above. To some extent, these regions have been found in Ref. [12]. In turn, soliton bound states exist in regions that are much smaller than those where single solitons are stable. Thus far, there is no classification of these areas. Hence, we choose a set of parameters based on our previous experience. This may not completely represent all the properties of the bound states. However, the specific cases studied here may serve as starting points for wider research in this area.

In Ref. [13], a technique has been developed for the investigation of soliton bound states in dissipative systems. It is based on the assumption that the profile of each single soliton in the dissipative system is hardly modified by the interaction. Hence, the problem can be reduced to a two-dimensional dynamical system with the variables $\rho$, as the soliton separation, and $\phi$ as the phase difference between the two solitons. These two variables can be represented on an interaction plane in which $(\rho, \phi)$ are the polar coordinates. The trajectories in this plane not only show the fixed points corresponding to bound states, but also the stability properties of these bound states. An example, taken from Ref. [14] is shown in Fig. 1. For a given set of parameters of the CGLE, we have two stable fixed points, $F_1$ and $F_2$. They correspond to two stable two-soliton solutions that have a $\pm \pi/2$ phase difference between the solitons and a separation of approximately $\rho \approx 1.8$. Any initial condition consisting of two solitons in close proximity will converge to one of these bound states, as the figure shows.

The bound state is an asymmetric solution of the CGLE because of the phase difference, $\pi/2$, between the solitons. As a consequence, it moves, relative to the single soliton, with non-zero group velocity. When a two-soliton bound state and a single soliton exist simultaneously, they may collide. The result of the collision depends mainly on the phase difference, $\phi_{2-s}$, between the pair and the single soliton, as this is the only parameter in the initial condition, provided that the soliton pair is exactly at one of the points $F_i$. If a bound state is not reached before the collision, the result will also depend on where the point is located on the trajectory that converges to the bound state (see Fig. 1).

Below, we present numerical simulations of collisions between a soliton pair at the point $F_1$ and a single soliton. Due to the symmetry of Eq. (1) relative to the transformation $t \rightarrow -t$, the same collision occurs when the soliton pair is at $F_2$. This simply requires a
Fig. 1. Trajectories showing the evolution of two-soliton solution on the interaction plane for the parameters shown inside the figure. The two singular points $F_1$ and $F_2$ are stable foci. The central part of the figure, where $\rho$ is less than a single soliton width, does not describe a valid bound state. Trajectories converging to the center describe the merging of two solitons. Three other singular points, $S_1$, $S_2$, and $S_3$, are saddles that correspond to unstable bound states.

change from $\psi(t)$ to $\psi(-t)$. The pair denoted by $F_1$ moves to the right. Hence, in order to collide, the single soliton must be located at the right-hand-side of the pair.

In all plots, we assign the numbers 1, 2, and 3 to the solitons, counting them from left to right (see Fig. 2). In this section, initially the solitons 1 and 2 form the stable pair. The soliton 3 in the initial condition is well separated from the pair, so that there is no interaction with it. Nevertheless, we can plot the separation between the solitons 2 and 3 and the phase difference between them on the interaction plane in the same way as for the solitons 1 and 2. This plot will represent the interaction plane of the initial conditions for the pair and a single soliton before the collision. As the pair initially is located at the point $F_1$ of Fig. 1, the interaction plane of the solitons 2 and 3 is a convenient way to classify the collisions in the continuous model.

A few scenarios for the collision are possible. We made a number of simulations where we started with the pair (solitons 1 and 2 in Fig. 2) and the single soliton (soliton 3) at a fixed distance from the pair (where the interaction was negligible) and changed the initial phase difference between the solitons 2 (i.e., the pair) and 3.

One possible scenario is the complete destruction of the bound state and the resulting soliton fusion. Of the three solitons in the initial condition, one or two may disappear. The result after the collision then would be just one or two solitons at a distance where they interact weakly. As a result, they have zero velocity. These two examples of collisions are shown in Fig. 2. The initial values for the phase difference $\phi_{2-3}$ and fixed separation between solitons 2 and 3 in the initial conditions which were used to obtain these results will be described later.

The second scenario that we observed is the merging of all solitons into a three-soliton bound state, which can be of two different types. They are shown in Fig. 3. In Fig. 3a,
Fig. 2. Collision of two-soliton bound state with a single soliton. We refer to this case as “soliton fusion” or “annihilation.” Depending on the initial phase difference between the bound state and the single soliton, the output is either (a) one or (b) two weakly-interacting solitons.

Fig. 3. Formation of a three-soliton bound state after the collision of a two-soliton bound state with a single soliton. We refer to this case as the “formation of a triplet.” The phase differences between consecutive solitons are $-\pi/2$ and $\pi/2$, respectively. The resulting three-soliton bound state is an oscillatory solution, so each soliton changes its position and relative phase periodically around fixed values. After the collision, on the interaction plane, the corresponding resulting trajectory for solitons 1 and 2, as well as the trajectory for solitons 2 and 3, converges to a periodic orbit (limit cycle) around the points $(\rho, \phi) = (1.8, \pm \pi/2)$, respectively. In Fig. 3b, the two phase differences between solitons are both positive. In this case, the corresponding trajectories on the interaction plane, after the collision for solitons 1 and 2, and 2 and 3, converge to fixed points, which are very close, but do not coincide exactly with the point $F_1$. The bound state of three solitons moves with a velocity which is almost equal to the velocity of the pair.

The last possible scenario that we obtained in the continuous model is the destruction of the initial pair and the formation of a new soliton pair after the collision, with an exchange of the soliton in the middle. This case is shown in Fig. 4. We refer to this case as an “elastic collision.” This scenario of collision may seem to correlate with the laws of clas-
Fig. 4. (a) Exchange of a soliton in the bound state after the collision of a soliton pair with a single soliton. (b) Trajectories on the interaction plane for the solitons 1–2 (dashed line) and 2–3 (solid line). The black solid dot represents the initial point for the separation and phase difference between the pulses 2–3. The circle at the point $F_1$ is the initial point for the pulses 1–2 and the final point for the pulses 2–3. The first trajectory is pushed out of the point of equilibrium after the collision. The second trajectory is attracted to the point of equilibrium after the collision.

Classical mechanics, where two particles of the same mass, moving in one direction, collide with a third one, resulting in motion of the third particle, along with one of the incident ones. However, there are no conserved quantities such as momentum or energy in a dissipative system. Moreover, the binding energy of solitons is non-zero. On the other hand, we know that stationary solutions must be fixed. Therefore, the difference in velocities between the pair and the singlet is fixed and must be the same before and after collision. This is the main reason for the results with “elastic collisions” of dissipative solitons shown in Fig. 4.

Figure 4b shows the separations and phase differences between the solitons on the interaction plane, before, during and after a collision. The solid line is for solitons 2 and 3, while the dashed line is for solitons 1 and 2. The solid curve ends up at the point $F_1$, while the dashed line starts at this point, thus indicating that the pair formed by solitons 2 and 3 is exactly the same as the initial pair formed by the solitons 1 and 2.

Due to the fact that dissipative solitons are fixed, the outcome is always an integer number of solitons with fixed amplitude and velocity. Therefore, qualitative considerations lead us to the conclusion that no other scenario of collision between a pair and a singlet is possible. This would be a conclusion for any other set of parameters of the CGLE where stable soliton pairs exist.

A summary of our results is presented in Fig. 5. Initially, the point on the interaction plane for the solitons 2 and 3 is located far away from the origin at a fixed distance ($\approx 6$) in all our simulations. For the relative phase difference, $\phi$, we choose 36 values, with 10 degrees angular difference between neighbors. All 36 initial points are shown as small dots in Fig. 5. In this way, we cover a circle of initial conditions on the interaction plane in a relatively dense way. Each of these initial conditions leads to one of the outcomes that we described above. The simulations allowed us to establish a correspondence between the angular position of the initial point and the result of the collision.
Fig. 5. Schematic of the evolution of the solitons 2 and 3 on the interaction plane in the three-soliton initial conditions. The result of the collision of the soliton pair 1–2 with the single soliton 3 depends on the relative phase between the solitons 2 and 3 before the collision. This plot shows the outcomes of the collision for 36 initial conditions, which are represented by small dots on a circle of radius \( \approx 6 \). Parameters of the simulations are the same as in Fig. 1.

Specifically, the points in the lower left quadrant result in the annihilation (or fusion) of one or two solitons. This is indicated in the figure close to each set of initial conditions. This evolution is similar to the examples presented in Fig. 2. Fusion to a single soliton occurs when the initial point is in the upper part of this quadrant. Fusion of three solitons into two occurs when the initial point is in the lower part of this quadrant.

The rest of the circle of initial conditions can be divided into three arcs depending on the collision outcome. The final and initial points in this plane are joined schematically with the straight lines. These are not the actual trajectories. The trajectory may rotate around the origin before the collision. We obtain oscillating soliton triplets (as in Fig. 3a) when the initial point is in the lower right quadrant of the interaction plane. As indicated in that figure and mentioned above, the final state in these cases is a limit cycle rather than a point. Moving soliton triplets (as in Fig. 3b) are obtained when the initial point is in the upper right quadrant. Finally, an “elastic collision” (similar to that in Fig. 4a) occurs when the initial point is in the upper part of the circle. This last case is well illustrated in Fig. 4b.

If we select an initial distance between the solitons 2 and 3 which is different from the value 6, the whole circle of initial conditions may rotate around the origin thus moving our classification scheme around. However, the relative location of the various outcomes of the collision will remain unchanged.

3. Laser model with parameter management

The model with continuous parameters is relatively simple. It is clear that it is not quite adequate to describe a real laser system. This is because the internal periodicity of the system which is related to the round-trip time is averaged out in the continuous model. In this
sense, an essential improvement would be a model whose parameters change periodically with $z$. Hence, we model the fiber laser using the cubic-quintic complex Ginzburg–Landau equation (Eq. (1)) as before, but now allow the values of the parameters to vary with $z$.

The real laser cavity typically consists of several pieces of fiber which connect the elements and a mode-locking device (see Section 5). The optical properties of the media where the pulse propagates are different, so that the material parameters experienced by the pulse vary periodically with propagation distance. As a result, the coefficients in Eq. (1) must be periodic functions of the distance $z$. We set the coefficients in Eq. (1) to be periodic step-wise functions of $z$, since this is quite a simple model, but it still takes into account the periodicity imposed by the cavity.

This technique for modeling the fiber laser can be called “parameter management.” The term comes from the theory of “dispersion-managed solitons” [15,16] which uses the nonlinear Schrödinger equation (NLSE) with a step-wise coefficient for the second-order derivative term. The periodic change of the group velocity dispersion induces a change in the soliton profile. This is usually chaotic, but may become periodic, provided the initial condition is chosen in a special way.

We use a similar approach for our laser system, but, instead of the nonlinear Schrödinger equation, we use the cubic-quintic Ginzburg–Landau equation with coefficients that are all periodic step-wise functions of $z$. Each period in this model naturally describes one round-trip of the optical pulse. In contrast to the “dispersion-managing” of the NLSE, after a certain number of round trips, pulse evolution in our model generally does not depend on the initial conditions. In most cases, we use a sech-like profile to start the simulations with a single pulse. The pulse evolves into the solution, provided its amplitude is above a certain threshold given by the parameters of the system. If the amplitude of the initial condition is below this threshold, it decays and quickly disappears. When we found a localized stable solution, we looked for bound states of two pulses. As a final stage in this process, we used the single soliton solution and the bound state together as initial conditions to simulate collisions in the same way as in Section 2.

The model is illustrated in Fig. 6 [7]. In each section, the full CGL equation is used, though the parameter values differ. In comparison with the continuous model, the model with parameter management is more complicated in the sense that the number of parameters involved increases considerably. Instead of the 6 parameters in the continuous model, we potentially have 6 parameters in each region of the map. The lengths of each part of the map give two more parameters. Hence, the model in general has 14 free parameters. Consequently, the search process for the parameters of the system is more complicated than in the continuous model. In this task we aim to meet the following objectives:

1. The 14 parameters chosen in our model must lead to the generation of stable localized solutions. In general, these solutions should not depend on the initial condition after the pulse has propagated for several round trips. In other words, the pulse should converge to the same “stationary” soliton, independent of the input pulse. This means that the correctly-tuned laser has entered the regime of stable pulse train generation.

2. In addition to single solitons, the model must provide the generation of stable soliton pairs. This requirement greatly narrows the region of the system parameters, making the search much more difficult.
(3) The model must provide collisions of pairs with single solitons that are similar to collisions in the continuous model.

(4) In addition, the model must have features in common with the experimental setup. In particular, we took values of dispersion $D$ having opposite signs in the two sections of the laser, with an average value slightly on the normal dispersion side.

(5) The model should show the effects that we observe in the fiber laser experiment.

After a lengthy search procedure, we chose the set of parameters shown in Fig. 6. This set is not a unique case. No doubt, other sets that satisfy the criteria listed above can be found. However, there is no analytical procedure that would simplify this optimization procedure. The only possibility is to check every new set of parameters in direct numerical simulations. Hence, this optimization requires a lot of computer time. Analytical methods reducing the number of degrees of freedom in the system [17] may help in attempts to find a more regular technique to narrow the range of the parameter set. However, even with the reductions, direct numerical simulations cannot be eliminated.

Naturally, the laser model that we use is one of the simplest models using parameter management, and does not have a one-to-one correspondence with the real laser cavity. However, the model is reasonably close to the experiment, since it includes the minimum number of physical effects needed to reproduce some of the behaviors observed experimentally. Dispersion has opposite signs in the two sections of the map, as it does in the laser cavity. As in our real laser, the average dispersion is taken as normal and close to zero. The model takes into account the most important feature, namely the round-trip periodicity of the effects that the pulse undergoes during its propagation inside the laser cavity. As an alternative, we could use a lumped model, where certain devices, such as the mode-locker are introduced at a point rather than in an interval. The results we would obtain would be qualitatively similar, as long as we explicitly take the round-trip periodicity into account in the model.

In a real laser system, the output pulse is taken at a certain position of the cavity, and it has always the same shape provided the laser is in a stable regime of operation. Similarly, we plot the pulse shape at a certain position in our model, thus eliminating pulsations with
the round-trip time. In mathematical terms, this technique is known as a Poincaré map [18]. All the plots in Section 4 are made using this technique. Thus, the solution is monitored every round trip at the end of Section 2.

4. Soliton collisions in the laser model with parameter management

Parameters in the model are chosen so that we ensure the existence of stable bound states. Indeed, two solitons can form stable asymmetric soliton pairs similar to those in the continuous model of Eq. (1) [5]. Figure 7 shows half of the interaction plane for two solitons. Due to the inversion symmetry $t \rightarrow -t$, the lower half is a mirror image of the upper one. Periodic width variations result in solitons interacting at larger separations than in the case of the continuous model. A distinctive feature of the model is the existence of three stable foci on the upper half plane, replacing the one in the continuous case. These correspond to three stable two-soliton bound states. They are indicated in Fig. 7 by the arrows and denoted $A$-, $B$-, and $C$-type bound states.

Trajectories close to any of these points converge to them. The rate of convergence to the point $A$ is the fastest, whereas that for point $C$ is the slowest. In fact, one needs thousands of times more round trips to converge to the point $C$ than to the point $B$ or $A$. We can say that solitons in the bound state $C$ are weakly-bound. The phase difference between the two solitons for each of these bound states is close to $\pi/2$, but is not exactly equal to it. The distances between the solitons are also different. As a result, the three types of bound states have different group velocities. Here we present numerical simulations for collisions involving only two types of coupled states: $A$ and $B$. There is also an unstable fixed point on the upper half-plane, denoted by the letter $D$. Any trajectory that starts near this point diverges from it. Other unstable foci may exist at $\phi = 0$ and $\phi = \pi$.

In numerical simulations, we have followed the same procedure as for the continuous model, namely we took a pair (either $A$ or $B$) located at a fixed distance from the ‘singlet,’

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Fig. 7. Interaction plane for two solitons in the model with parameter management. Only the upper half of the interaction plane is shown. The variable $\rho$ represents the separation between the two solitons, while $\phi$ is their relative phase. In contrast to the continuous model of Section 3, there are three stable soliton pairs in the upper half of the interaction plane. They are represented by stable foci $A$, $B$, and $C$. The unstable focus $D$ corresponds to an unstable soliton pair.
and with a certain phase difference with respect to it; we vary this to cover phase differences from 0 to $2\pi$. The possible results of the collision are illustrated in the following figures. We were able to obtain all types of scenarios of the previous section except soliton annihilation (fusion) for the type B bound states.

As in the continuous model, the asymmetric soliton pairs have group velocities that differ from the group velocity of the ‘singlet’ soliton. The plus or minus sign with the phase difference defines the direction of motion of the pair relative to the single pulse. In the cases shown in the following figures the phase difference is $\pi/2$; this makes the A-type pair move to the right and the B-type one move to the left. In each plot, they are initially located at the side of the single soliton which allows them to collide with it.

Figure 8 shows a collision involving a soliton pair of either A- or B-type and a single soliton. The distance $\rho$ between the soliton centers in the pair is around 4 dimensionless units for the A-type bound state and around 6 for the B-type. The phase difference is

![Fig. 8](image1.png)

Fig. 8. “Elastic” collision of a pair of coupled dissipative solitons with a soliton ‘singlet.’ The parameters for this simulation are shown in Fig. 6. The case (a) is for the soliton pair A in Fig. 7. The case (b) is for the soliton pair B. Each plot can be inverted via transformation $t \rightarrow -t$.

![Fig. 9](image2.png)

Fig. 9. (a) An example of soliton collision (a) for the soliton pair A in Fig. 7 and (b) for the soliton pair B. The outcome in each case is a soliton triplet. The parameters for this simulation are shown in Fig. 6. Each plot can be inverted via transformation $t \rightarrow -t$. 
Fig. 10. Two more examples of soliton collision. (a) Type A two-soliton bound state reverses the sign of its velocity after the collision and attracts the third soliton. (b) Two solitons forming a pair of type B are “repelled” by the single soliton with zero velocity. The parameters for this simulation are shown in Fig. 6. Each plot can be inverted via transformation $t \rightarrow -t$.

Fig. 11. (a) Interaction plane showing the initial distance ($\approx 20$) and relative phase between solitons 2 and 3 (the solitons 1 and 2 form an A-type pair) used in our 36 simulations. They are represented by different symbols, distributed along the circumference, indicating the various possible outcomes of the collision. These are: (1) The 3 solitons are fused and one soliton is left after the collision (open circles). (2) The solitons are fused and two solitons are left after the collision (filled circles). (3) A triplet is formed moving in the opposite direction to the initial A-type pair as in Fig. 10a (solid triangle). (4) A triplet is formed and it moves in the same direction as the initial A-type pair (open triangles). (5) Elastic collisions (filled squares), as in Fig. 8a. (b) Initial conditions for the solitons 1 and 2 (the 2 and 3 initially form a B-type pair) on the interaction plane. The result of the collision between the soliton pair 2–3 with the single soliton 1 depends on the relative phase between solitons 1 and 2 before the collision. This plot shows the outcomes of the collision for 36 initial conditions, which are represented by different symbols on a circle of radius $\approx 60$. These are: (1) The pair is reflected back (open circles), as in Fig. 10b. (2) A triplet is formed (solid triangles) as in Fig. 9b. (3) Elastic collisions (filled squares) as in Fig. 8b.

approximately $\pi/2$. The two cases illustrated in Figs. 8a and 8b show that a properly-chosen initial phase difference allows us to obtain “elastic” collisions for each type of soliton pair. As in the experiment, collisions happen on the scale of hundreds of round trips. Note also that a pair with a smaller separation moves faster (see the $z$-scale). When
periodic boundary conditions in $t$-variable are used, the same type of collision can be repeated indefinitely.

Soliton triplets exist on the same basis as pairs. They move with the same velocity, provided the phase difference between the outermost solitons is twice $\pi / 2$, i.e., $\pi$. When the phase difference between the outermost solitons is zero, i.e., $\pi / 2 - \pi / 2$, the soliton triplet has zero velocity. Variations of some parameters in the dispersion map can result in either of these states being excited after the collision. Figure 9 illustrates the formation of triplets as a result of the collision between a soliton pair, of either type $A$ (a) or type $B$ (b), and a singlet.

Two different examples of collisions are shown in Fig. 10. The behavior is quite unusual in these cases. In Fig. 10a, the two-soliton bound state of type $A$ inverts its direction of motion after the collision, at the same time attracting the third soliton and forming a three-soliton bound state. By way of contrast, for Fig. 10b, the two-soliton bound state of type $B$ is reflected by a single soliton. These two examples confirm that the "momentum" is not conserved in dissipative systems.

The classification of the collision outcomes for the model with parameter management differs from the case of the continuous model. Moreover, now we are dealing with two different kinds of pairs forming the initial conditions: $A$- and $B$-types. The results that we obtained in our simulations for these two types are summarized in Figs. 11a and 11b, respectively.

5. Dissipative solitons in interaction: experimental observation

The mode-locked laser (see Fig. 12), which emits at around 1.53 µm, is a dispersion-managed fiber ring laser [19,20]. The gain is provided by a 1.9-m long, 1400-ppm erbium-doped fiber (EDF) that features normal chromatic dispersion [$D = -40$ (ps/nm)/km]. The pumping source consists of four wavelength-multiplexed laser diodes around 980 nm, providing a coupled power of up to 350 mW. The path-averaged cavity dispersion is adjusted with the use of an appropriate length of an SMF – 28 fiber that has anomalous dispersion [$D = 16.5$ (ps/nm)/km]. A 50-cm long open air section is used to insert polarization components. Due to the nonlinear polarization evolution that takes place during propagation in the fibers, transmission through the polarizer $P_1$ is intensity-dependent, and an appro-

![Fig. 12. Fiber laser experimental setup.](image)
Fig. 13. Stable pulse train generation by the mode-locked fiber laser with a single soliton in the cavity.

appropriate adjustment of the preceding wave plates triggers the mode-locked laser operation. A polarization-insensitive optical isolator (WDM-IS) ensures unidirectional lasing.

Two other optional outputs are implemented in the cavity, and may be used, depending on which type of experiment is performed. First, a second polarizer (P2), preceded by a half-wave plate, provides a convenient variable output coupler. We can use the half-wave plate to continuously tune the amount of cavity loss in a given range, so that collisions can manifest accordingly. Second, a 10% fiber output coupler is inserted inside the cavity in order to splice a small length of dispersion-compensation fiber (DCF). This gives a convenient way of compressing the chirped pulses that propagate in the cavity. A recording of the optical autocorrelation from this fiber output is used to distinguish the doublet and triplet multisoliton complexes when the pulses are very close to each other. In the present work, the path-averaged dispersion is slightly normal \[D \approx -2 \text{ (ps/nm/km)}\] and the output coupling from polarizer P2 is set around 40%.

The output coupling is set to around 40%. Pulsed laser operation is analyzed by a homemade optical autocorrelator and by an optical spectrum analyzer (MS 9710B). Output intensity is also monitored on a 500-MHz digital phosphor oscilloscope. At a pumping power of around 150 mW, fundamental mode-locking is routinely achieved and can be stable for hours without the use of any external feedback. Due to the frequency chirping, the pulse duration is typically 1 ps at the open-air section outputs, and 150 fs at the compressed output port (see Ref. [7]). Intracavity energy of a single pulse is around 400 pJ. In a stable regime of laser operation, monitoring the output intensity displays the amplitude peaks that repeat at the cavity fundamental frequency of 36.6 MHz. An oscillogram for this regime is shown in Fig. 13.

The most stable regimes of operation of the mode-locked fiber laser consist of either a single soliton, or a bound state of several solitons (what we call a dissipative multisoliton state). They circulate around the cavity at the fundamental repetition rate. The type of
the soliton state that circulates depends mainly on the pumping power. Stable operation with a single soliton having a neat bell-shaped intensity profile requires limited pumping power. For a multisoliton state, the output energy monitored on an oscilloscope looks similar to the case of a single soliton (see Fig. 13). The separation between the solitons that form the multisoliton state is in the picosecond range, and therefore cannot be resolved at the nanosecond scale. The fine structure of multisoliton states can be found from optical autocorrelation and spectral measurements.

Let us mention a few key features that can be attributed to these regimes of operation. Stable regimes exist within relatively large intervals of cavity parameters. These regimes can be maintained for several hours without any external control. Transition from one regime to another is abrupt and shows hysteresis with respect to the altered parameter. In what follows, we show experimentally that interesting new regimes, such as coexistence of different types of solitons with a relative motion, can be observed at the edges of the parameter intervals that correspond to stable regimes.

Once mode-locking is achieved at a given pumping power and a given setting of waveplates, we have some latitude to vary one or several cavity parameters to observe changes in the dynamics of the output pulses. We know from previous works [7, 19] that multiple pulsing and the formation of multisoliton complexes can be favored in the cavity when the intracavity energy is increased. Multiple pulsing can be seen as a possible way of restoring the energy balance in the cavity and stabilizing the laser operation. For example, at \( P = 200 \text{ mW} \), a doublet soliton is formed. The newly-formed state can be analyzed using the recordings of its optical spectrum and autocorrelation trace. The stable soliton pair, or doublet, is characterized by highly contrasted fringes in its spectrum. The interfringe distances in the spectrum are inversely proportional to the temporal separation of the two pulses. This separation can be measured by the autocorrelation trace. At higher power levels we observe the creation of an additional third pulse. In particular, a stable multisoliton triplet is formed at around \( P \approx 300 \text{ mW} \). The spectrum in this case has a different appearance from the spectrum of the soliton pair. It has additional small peaks between the main fringes (see Fig. 14b).

![Fig. 14. (a) Recorded optical spectrum (solid line) and its baseline (dashed line), for the doublet and ‘singlet’ solitons in continuous relative motion. (b) Optical spectrum when the soliton triplet is formed.](image-url)
In order to observe the soliton collisions described by numerical simulations, we must operate with three dissipative solitons in the cavity. To reach this regime, we adjust (±5°) the orientation of the half-wave plate preceding P2, which is equivalent to changing the amount of linear loss in the cavity (±0.8 dB). For a small range of orientations, the oscilloscope traces feature two peaks per cavity round trip. The larger one is double the other one in amplitude and appears stably on the screen, since its level is used for trace synchronization. The second one moves forever with a constant relative velocity (at this scale of observation). This is illustrated by Fig. 15. The real motion can be seen in the first movie of Ref. [7]. The largest peak is identified as a doublet or soliton pair, whereas the second peak is identified as a single soliton.

As the ‘singlet’ moves, it takes 0.64 s to travel one round-trip, so the difference from the doublet represents 5.3 m of fiber length. This means the group velocity difference between the doublet and ‘singlet’ is 8.3 m/s, which is less than $10^{-7}$ of each soliton group velocity! Once it has been obtained, this relative motion can be stable for hours. This arrangement corresponds to the numerical results shown in Fig. 8.

If we vary the output coupling slightly, a stable triplet soliton state can be formed as a result of the interaction between the doublet and the ‘singlet’ states. This corresponds to the numerical simulations in Fig. 9. The triplet can be seen in the oscilloscope trace in Fig. 16 as a pulse with amplitude three times higher than that of the single soliton pulse. The triplet state is highly stable once obtained, but if we then vary the output coupling, the result is the decomposition of the triplet into doublet and ‘singlet’ solitons. The same cycle can be repeated over many times. The transition $2 + 1 \rightarrow 3$, although with some hysteresis, is reversible. We observe switching between the spectra in Figs. 14a and 14b when repeatedly forming and decomposing the triplet state. The switching can be also observed directly in the oscillograms. The second movie in Ref. [7] is an example of such switching cycles.

The appearance of singlet, doublet and triplet soliton states in the cavity finds more justification when we consider optical autocorrelation traces. Figure 17a represents the autocorrelation traces recorded at the variable output coupler. The solid curve is taken...
Fig. 16. Oscilloscope traces of stable triplet soliton. The small sub-peaks of amplitude around or less than one division are electronic artifacts due to imperfect impedance matching.

Fig. 17. Autocorrelation traces taken from (a) the variable output coupler and (b) the 10% coupler. Solid line is for doublet and singlet in continuous relative motion. Dashed line relates to time after the triplet soliton state has formed.

when the soliton pair and the soliton ‘singlet’ are moving with different group velocities, as in Fig. 15. We can see that the amplitude of the central peak is three times larger than the amplitude of the side peaks. This is compatible with the fact that the soliton pair or doublet is made of two equal pulses that have the same amplitude as the third moving pulse. The dashed curve is used when the triplet is formed, as in Fig. 16. This is clearly the autocorrelation function of three identical, equally-separated pulses.

The same analysis is confirmed by recording the autocorrelation function at a different location, namely after the erbium-doped fiber, at the 10%-output of the fiber coupler. It is
Solitons in integrable systems have the property of recovering their shapes after collisions. The only change as a result of a collision is a phase shift. This occurs despite the fact that the solution cannot be represented as a linear combination of two solitons during the collision. There is no radiation of small amplitude waves due to the collision or any other energy loss from the solitons.

In contrast, collisions in nonintegrable Hamiltonian systems have properties that are completely different from those in integrable systems such as systems described by the Korteweg–de Vries (KdV) or the nonlinear Schrödinger equations (NLSE). In nonintegrable Hamiltonian systems, collisions may cause the loss of soliton energy by emitting radiative waves. Solitons change the shape and other parameters after any collision. Even the number of solitons can change. Consequently, there is no closed theory of soliton collisions in nonintegrable models.

The subject of dissipative solitons is another emerging arena of scientific research. Even single dissipative solitons are objects with distinctive features. In particular, dissipative solitons have internal energy flows that make them similar to biological entities. Dissipative solitons cannot exist without external energy to pump the system. They will “die” if the energy supply is reduced. On the other hand, they can multiply when the energy supply is increased. It is no wonder that dissipative soliton collisions have nothing in common with the two previous Hamiltonian cases.

The variety of dissipative systems is enormous. Consequently, there is scope for assorted types of collisions. It is unlikely that a complete theory of collisions can be developed in this case. Nevertheless, the classification made in each particular case helps us to understand the general trends. In this sense, models based on the CGLE serve as a basic play-ground before considering more complicated systems.

In this paper, we considered two models of dissipative systems supporting solitons: the continuous CGLE and also the CGLE with parameter management. In the latter case, we intentionally designed the parameter map in order to obtain results similar to those of the continuous model. Moreover, in each case, we have chosen the parameters in such a way as to have special types of collisions between the three solitons, so that they would mimic the experimental results in a fiber laser. This way, we were able to predict and observe certain types of collisions involving soliton pairs. In particular, we observed, both numerically and experimentally, “elastic” collisions, as well as the formation of soliton triplets. Clearly, our systems with the chosen parameters represent very particular cases. For them, we were able to make a classification of the results of the collisions based on the initial conditions. Although the variety of cases can be large, our results give us some ideas about expectations for soliton collisions in other systems. In particular, a comparison between the two models based on equations with continuous parameters and with parameter management
shows that, in fact, the classifications can be very similar. We hope that our work will help in considering more complicated interactions of dissipative solitons when the number of solitons in the cavity exceeds 3 [21].

7. Conclusions

In conclusion, we have studied, both numerically and experimentally collisions of groups of dissipative solitons in a fiber laser cavity. We have considered two models of a laser system: continuous and with parameter management. We have compared the results obtained in each model and with experimental data. Two solitons can be coupled into a stable pair that has a group velocity different from the velocity of a single soliton. The collision of the pair with a single soliton produces various scenarios. In particular, we have shown that the collision may destroy the bound state, but we can observe formation of another pair that moves away with the same velocity, leaving one of the solitons of the previously-moving pair at rest. We have also shown that soliton triplets can be formed as a result of a collision.

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