Comparison of Lagrangian approach and method of moments for reducing dimensionality of soliton dynamical systems

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For equations that cannot be solved exactly, the trial function approach to modelling soliton solutions represents a useful approximate technique. It has to be supplemented with the Lagrangian technique or the method of moments to obtain a finite dimensional dynamical system which can be analyzed more easily than the original partial differential equation. We compare these two approaches. Using the cubic-quintic complex Ginzburg–Landau equation as an example, we show that, for a wide class of plausible trial functions, the same system of equations will be obtained. We also explain where the two methods differ. © 2008 American Institute of Physics.

[I. INTRODUCTION]

Many physical, chemical, and biological systems are described by partial differential equations which support localized soliton solutions. Usually, these cannot be obtained analytically and so we try to use a “trial function” which gives a good approximation to the pulse solution. This can provide direct insight into the nature of the system’s behavior for a wide range of input parameters. The Lagrangian approach or the “method of moments” can be used with the trial function for this purpose. The two techniques have been developed independently but never directly compared. A question arises: Do the two approaches result in the same dynamical system of coupled ordinary differential equations? This work provides at least a partial answer to this question. For a realistic and common form of the phase dependence, we find that the same resultant system is obtained in each case. For solitons which move, or have a more complicated phase dependence, the resulting systems are different.

The number of equations that admit soliton solutions is enormous. Originally, they represented the class of completely integrable partial differential equations (PDEs), such as the Korteweg–de Vries (KdV) equation and nonlinear Schrödinger equation (NLSE). Later, this class was extended to include nonintegrable equations. Localized traveling wave solutions—mostly the so-called “extended NLSEs”—were found and analyzed using some basic approaches from nonlinear dynamics. Even though they are not completely integrable, some of them admit soliton solutions in analytic form. However, many equations have soliton solutions that cannot be written analytically. One important example in physics is the complex cubic-quintic Ginzburg–Landau equation (CGLE). It has a variety of soliton solutions whose analytic forms are unknown. Because of the difficulties that arise due to the absence of exact solutions, approximate techniques are becoming increasingly popular and it is highly important to explicitly show the differences between them. This will help with the correct choice of trial functions in any specific analysis and this is our task in the present paper.

In order to find those solutions, one needs numerical simulations. It is also helpful to find certain approximations that will simplify the analysis of their regions of existence in the space of parameters of the original equation. The trial function approach allows us to make preliminary investigations in cases where it seems impossible to obtain an exact solution.

A trial function is a simple analytic function that has amplitude and phase profiles that are as close as possible to the real solution. Often, this is either a Gaussian or a sech-function with a finite number of parameters that control the shape and chirp of the soliton. These parameters are usually functions of time that evolve subject to the constraints of the system and finally converge to a fixed point or a limit cycle. The evolution of these parameters is governed by a finite-dimensional dynamical system (FDDS) that can be derived from the original PDE.

There are several techniques that can be used to derive the FDDS. The most popular one is the Lagrangian technique. It was first used in soliton theory by Anderson. Another technique, called “the method of moments,” was developed later by Maimistov. A third technique known to use trial functions is the “collective variable approach” (CVA). Clearly, when using different trial functions, the resulting FDDS must be different. However, a question arises as to whether using the same trial function will result in the same FDDS using three different techniques mentioned above.

In this paper, we compare the two main techniques, namely, the Lagrangian approach and the method of moments. These two use different physical principles to derive the FDDS. The Lagrangian approach is based on minimization of “action,” while the method of moments is based on studying the principle of deviations from the NLSE and applying the resulting evolution equations for the former “conserved quantities” of the NLSE and higher-order moments.
Thus, a priori, we would not expect them to result in the same FDDS. Nonetheless, actual calculations show that, for simple trial functions, the resulting FDDS turns out to be the same.10

The CVA technique for optical solitons, presented in Refs. 8 and 9, is based on writing the exact solution of a PDE as a trial function plus a remnant field and the subsequent minimization of the integral of the remnant. The authors then use a Gaussian approximation to study the interaction (between the tails) of two pulses in a fiber. This leads to coupled modified NLSEs which involve an interaction term which has been represented by an interaction potential in earlier perturbation studies.11 These equations of motion are then equivalent to an averaged Lagrangian approach. The authors note the valid point that a purely Gaussian trial function is equivalent to an averaged Lagrangian approach. The authors then use a Gaussian approximation to study the interaction between the tails perturbation studies.11 These equations of motion are then represented by an interaction potential in earlier studies.12

The original PDE with unknown soliton solutions considered here by us is the CGLE. For dissipative systems, there are no conserved quantities but we can write down the Lagrangian and derive soliton solutions.

The Lagrangian approach is based on extending Euler–Lagrange least-action principles to dissipative systems. We formulate Lagrangian equations for the CGLE for a general form of it can be written as

\[ \text{amplitude, the width, the velocity, the chirp, etc.} \]

One fairly parameters in the problem at hand.12,13 These could be the nonlinear optical, biological or chemical system.

The dynamical equations can represent chaotic solitons. The dynamical equations can represent a system of coupled ordinary differential equations which describe the system. This means that any convenient trial function can be chosen, and this includes ones which do not give analytic values of the required integrals. Solving these shows up fixed points which correspond to stationary solutions of the PDE, limit cycles which correspond to pulsating solitons, and more complicated behavior, such as moving pulses, creeping solitons, and chaotic solitons. The dynamical equations can represent a nonlinear optical, biological or chemical system.

The trial function usually includes physically relevant parameters in the problem at hand.12,13 These could be the amplitude, the width, the velocity, the chirp, etc. One fairly general form of it can be written as

\[ \psi = g(x,z) \exp\left[\text{i}(c(z)x^2 - \theta(z))\right], \tag{1} \]

where \( g(x,z) \) is a localized real function of \( x \) while \( c(z)x^2 \) is the chirp across the soliton, and \( \theta(z) \) gives the phase evolution on propagation. Here, we have limited the generality of Eq. (1) by choosing a quadratic form of the phase chirp, as is widely used in the literature. It usually gives an accurate model of the exact phase. We define \( Q \) as the pulse energy

\[ Q(z) = \int_{-\infty}^{\infty} g^2(x,z) dx, \]

which has to be finite for localized solutions. We can introduce two other useful functions, \( f(z) \) and \( b(X) \), which normalize the transverse coordinate \( x \) and the function \( g \). Letting \( X = x/f(z) \), we can set \( g(X,z) = \sqrt{Q(z)/f(z)} b(x) \), so that \( \int_{-\infty}^{\infty} b^2(X) dX = 1 \). This normalization conveniently introduces the variable width of the soliton, \( f(z) \), which can evolve in propagation. We do not need to specify the form of the function \( g \) at the first stage of the calculations.

We use the trial function (1) to model the soliton solutions of the CGLE. In dimensionless form, it is given by

\[ i \psi_t + \frac{D}{2} \psi_{xx} + |\psi|^2 \psi + \nu |\psi|^4 \psi = + i \delta \psi + i e |\psi|^2 \psi + i \beta \psi_{xx}, \]

\[ + i \mu |\psi|^4 \psi. \tag{2} \]

We choose the dispersion, \( D = \pm 1 \), where \( \nu \) represents a nonlinearity which is of higher order than a Kerr law response for the system. The parameter \( \delta \), which is chosen to be negative, is responsible for linear losses; \( e \) (which is positive) is the cubic gain parameter; and \( \beta \), which is also positive, defines spectral filtering, while \( \mu \) (which is negative) describes the saturation of the nonlinear gain. The choice of signs for \( \epsilon, \beta, \mu, \) and \( \delta \) is explained in Ref. 14.

The use of the variational integral and Euler equations has been explained in Ref. 15, where the Lagrange density is formulated Lagrangian equations for the Schrödinger equation and other conservative equations is also given. Following Ref. 15, we employ the Lagrangian (action)

\[ L = \int_{-\infty}^{\infty} L_d dx, \]

where \( L_d \) is the Lagrangian density and we have

\[ L = -\frac{i}{2} \int_{-\infty}^{\infty} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) dx + \frac{D}{2} W - \frac{1}{2} S_4 - \frac{\nu}{3} S_6, \tag{3} \]

for field \( \psi(x,z) \). The quantities \( S_4 \) and \( W \) here are defined as follows:

\[ S_4 = \int_{-\infty}^{\infty} |\psi|^4 dx \text{ and } W = \int_{-\infty}^{\infty} \left\{ \frac{\partial \psi}{\partial x} \right\}^2 dx. \]

Clearly, \( S_4 = Q^2 B_4/\sqrt{f} \) and \( S_6 = Q^3 B_6/\sqrt{f^3} \), where

\[ B_4 = \int_{-\infty}^{\infty} b^4(X) dX \text{ and } B_6 = \int_{-\infty}^{\infty} b^6(X) dX. \]

Simple calculations show that

\[ W = B_{\delta} Q f^2 + 4 B_3 e^2 Q f^3, \tag{4} \]

where

\[ B_{\delta} = \int_{-\infty}^{\infty} [b'(X)]^2 dX \tag{5} \]

and

\[ B_3 = \int_{-\infty}^{\infty} X^2 b^2(X) dX. \tag{6} \]

Similarly, we can find the first integral term in Eq. (3) and write the general form of the normalized Lagrangian \( L/Q \).

\[ L = B_{\delta} f^2 + B_3 f^2 [2 D e^2 + c'(z)] - \frac{B_4 Q}{2 f} - \frac{B_6 e^2 Q^2}{3 f^3} - \theta'(z). \tag{7} \]

If we apply the Euler–Lagrange equations to the CGLE, we obtain

\[ \frac{L}{Q} = B_{\delta} f^2 + B_3 f^2 [2 D e^2 + c'(z)] - \frac{B_4 Q}{2 f} - \frac{B_6 e^2 Q^2}{3 f^3} - \theta'(z). \]
\[ \frac{\partial}{\partial z} \left( \frac{\partial L}{\partial \phi^*_x} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \phi^*_x} \right) - \frac{\partial L}{\partial \phi^*} = J, \]

where a subscript \( x \) or \( z \) means derivative with respect to that variable, the star * indicates complex conjugate and \( J \) indicates the dissipative terms of the CGLE, viz.,

\[ J = i \left[ \delta \psi + \epsilon |\psi|^2 \psi + \mu |\psi|^4 \psi + \beta \psi_{xx} \right]. \]

For a trial solution containing several parameters \( p_j, j = 1, 2, \ldots \), the standard variational approach can be modified to allow for dissipative terms. \(^{12}\) We have, from Eq. (3),

\[ \frac{d}{dz} \left( \frac{\partial L}{\partial p_c} \right) - \frac{\partial L}{\partial p_c} = 2 \operatorname{Re} \left( \int_{-\infty}^{\infty} J \frac{\partial \phi^*}{\partial \psi} dx \right) \]

(8)

for each parameter \( p \).

The set of Eqs. (3) comprise the reduced dynamical system for the soliton parameters. In other words, the pulse evolution is described by the variation of total energy \( Q(z) \), pulse width, \( f(z) \), chirp factor \( c(z) \), and the overall phase, \( \theta(z) \).

### II. COMPARISON OF THE TWO FORMALISMS

The Lagrangian formalism can be used to find stationary soliton solutions, i.e., solutions which have the form of Eq. (1), of the CGLE. Then, using Eq. (8), we have, for a general parameter \( p \),

\[ \frac{\partial L}{\partial p_c} - \frac{d}{dz} \left( \frac{\partial L}{\partial p_c} \right) = 2 \operatorname{Re} \left( \int_{-\infty}^{\infty} \frac{1}{i \psi^*} \frac{\partial \psi^*}{\partial \psi} dx \right) \]

\[ \times \left[ \delta g^2 + \epsilon g^4 + \mu g^6 + \beta \psi_{xx} \psi^* \right] dx. \]

(9)

The essence of the method of moments lies in the claim that if we know all the moments of a function and the manner in which they evolve in \( z \), then we can deduce the function itself, together with its evolution in \( z \). On the other hand, if we know a finite number of the moments, then we can recover the function to a certain accuracy. Higher-order field moments (called \( I_1, I_2, \) and \( I_3 \)) have been introduced in Ref. 6. Some are essentially conserved quantities of the NLS equation. The number of higher-order generalized moments is infinite, but, as explained above, we can limit ourselves to a finite number of them; the resulting ODEs give an accurate description of the dynamics resulting from the PDE.\(^{16-18}\) Using the original equation (2), we can derive the evolution equations for the generalized moments. We assume that the solutions are not moving, so the center of mass, \( x_0 \) is zero. Then the relevant generalized moments\(^{6,19}\) are the energy \( Q \) and

\[ I_2 = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = B_2 f^2 Q, \]

\[ I_5 = \int_{-\infty}^{\infty} x (\psi^* \psi_x - \psi \psi^*_x) dx \]

\[ = 4 i B_0 f^2 Q \]

\[ = 4 i c I_2. \]

(10)

### A. Evolution equation for the energy \( Q(z) \)

Let the parameter \( p \) in Eq. (9) be the phase, i.e., \( p = \theta(z) \).

Noting that

\[ \frac{\partial L}{\partial \theta} = 0, \quad \frac{\partial L}{\partial \theta'} = -Q(z), \quad \text{and} \quad \frac{1}{i \psi^*} \frac{\partial \psi^*}{\partial \theta} = 1 \]

from Eq. (9), we obtain

\[ \frac{dQ}{dz} = 2 Q \left( \frac{\delta f^2 - \beta B_d - 4B_a \beta c^2 f^4 + B_4 \epsilon f Q + B_6 \mu Q^2}{f^2} \right). \]

(12)

This result represents the so-called “balance equation”\(^{14}\) for energy \( Q \). Plainly, the right-hand side is zero for a stationary solution.

On the other hand, using the method of moments, the equation for the first moment is

\[ \frac{dQ}{dz} = i \int_{-\infty}^{\infty} (\psi R^* - \psi^* R) dx, \]

(13)

where \( R = J - \epsilon |\psi|^4 \psi \). Simple calculations show that Eq. (13) gives the same result as Eq. (12). Thus, with the rather general form of the trial function (1), the two methods produce the same equation for the evolution of the soliton energy.

### B. Evolution equation for the width \( f(z) \)

To obtain the equation for the width, \( f(z) \) using the Lagrangian approach, we take as the parameter \( p \) the chirp, i.e., \( p = c(z) \). Now, noting that

\[ \frac{\partial L}{\partial c} = 4 B_0 D c f^2 Q, \quad \frac{\partial L}{\partial c'} = B_2 f^2 Q, \]

and

\[ \frac{1}{i \psi^*} \frac{\partial \psi^*}{\partial c} = -x^2, \]

(14)

we obtain

\[ 4 B_0 D c f^2 Q - B_2 \frac{d}{dz} (f^2 Q) \]

\[ = -2 \operatorname{Re} \int_{-\infty}^{\infty} x^2 \left[ \delta g^2 + \epsilon g^4 + \mu g^6 + \beta \psi_{xx} \psi^* \right] dx. \]

(15)

This result can be written explicitly for the derivative of \( f(z) \). We know \( Q'(z) \), so we can find \( f'(z) \),

\[ f(z) \frac{df}{dz} = \frac{1}{2} \frac{d(f^2)}{dz} \]

\[ = \beta \left( B_d + \frac{K_d}{B_s} \right) + 2 D c f^2 + 4 \beta c^2 f^4 \left( B_s - \frac{K_6}{B_s} B_s \right) \]

\[ + \epsilon f Q \left( \frac{K_4}{B_s} - B_s \right) + \mu Q^2 \left( \frac{K_6}{B_s} - B_s \right), \]

(16)

where
where we obtain the left-hand side of the equation, while the right-hand side integral turns out to be zero. Hence, we find that this is the same result as Eq. (16). Thus, for the trial function (1), with the specific lowest-order chirp dependence, the two methods give the same equation for the soliton width.

C. Evolution equation for the chirp c(z)

In order to obtain the equation for the chirp, c(z) from the Lagrangian approach, we use the parameter \( p = f(z) \) in Eq. (8). Now, noting that

\[
\frac{\partial L}{\partial f} = Q \left( 4B_d D f c^2 - \frac{B_d D}{f^3} + \frac{B_0 Q}{f} + \frac{B_0 \nu Q^2}{3f^3} + 2B f c' \right)
\]

and

\[
\frac{\partial L}{\partial f'} = 0,
\]

we obtain the left-hand side of the equation, while the right-hand side involves

\[
\frac{1}{\psi'} \frac{\partial \psi'}{\partial f} = \frac{1}{2f} \frac{\partial g}{\partial f} \frac{dx}{dx} = \frac{1}{2f} X b'(X).
\]

Calculating the integrals leads directly to the result for \( c'(z) \),

\[
\frac{dc}{dz} = \frac{B_d D}{2B f^4} + \frac{B c}{B f^2} (1 - 4K_2) - 2D c^2 - \frac{B_0 Q}{4B f^4} = \frac{B_0 \nu Q^2}{3B f^4},
\]

where

\[
K_2 = \int_{-\infty}^{\infty} X^2(b'(X))^2 \, dx.
\]

To find the equation for the chirp using the method of moments we need the third moment equation,

\[
\frac{dI_3}{dz} = 4i(c'I_2 + c' I_2).
\]

We know \( I_2 \) from Eq. (19), so we can substitute R into Eq. (22) and hence find \( c'(z) \). The result turns out to be identical with Eq. (20). Thus, the Lagrangian approach and the method of moments produce the same set of ODEs for the dynamical system governing the soliton parameters.

D. Evolution equation for the phase \( \theta(z) \)

To derive the equation for phase evolution, \( \theta(z) \), using the Lagrangian approach, we use as a parameter \( p = Q(z) \). Now, noting that

\[
\frac{\partial L}{\partial Q} = \frac{B_d D}{f^3} + \frac{5B_0 Q}{4f} - \frac{4B_0 \nu Q^2}{3f^3}
\]

and

\[
\frac{\partial L}{\partial Q'} = 0,
\]

we obtain the left-hand side of Eq. (8), while the right-hand side of it involves

\[
\frac{1}{\psi} \frac{\partial \psi}{\partial Q} = -\frac{i}{2Q}.
\]

The right-hand side integral turns out to be zero. Hence, we find that the phase evolution is governed by

\[
\theta'(z) = \beta c(1 - 4K_2) + \frac{B_d D}{f^3} - \frac{5B_0 Q}{4f} - \frac{4B_0 \nu Q^2}{3f^3}.
\]

The equation for phase evolution obtained from the method of moments has been discussed in Ref. 20,

\[
i \int_{-\infty}^{\infty} (\psi \psi'_{\alpha} - \psi^2 \psi_{\alpha}) \, dx = \int_{-\infty}^{\infty} (D |\psi_{\alpha}|^2 - 2|\psi^2|) \, dx
\]

\[
+ i \int_{-\infty}^{\infty} (\psi R^2 + \psi^2 R) \, dx.
\]

Direct calculations lead to an expression involving \( c' \). Substituting the expression for \( c' \) from Eq. (20), we can find \( \theta'(z) \). It turns out to be the same as Eq. (23). The conclusion is that FDDS obtained using the two techniques with the trial function (1) do coincide for any amplitude profile, \( g(x) \), of the trial function. This conclusion can be verified using the sech-function, Gaussian, or higher-order Gaussian amplitude profiles.

III. SPECIFIC EXAMPLES OF THE AMPLITUDE PROFILE

Let us apply the above results to specific cases of the amplitude profile (see Fig. 1). One of the profiles that allows us to advance in writing the FDDS in analytical form is a quartic-Gaussian-type function,
The Gaussian trial function is obtained from Eq. (25) by taking $m=\infty$, so $A=(2/\pi)^{1/4}$. This leads to the Lagrangian $L$ and system equations which are given in Ref. 10. Specifically, we have $B_1 = 1/(\sqrt{\pi})$, $B_2 = 2/(\sqrt{\pi}\sqrt{3})$, $B_3 = 1$, $B_4 = K_1 = 1/4$, $K_2 = 1/(8\sqrt{\pi})$, $K_3 = 1/(6\sqrt{\pi})$, and $K_4 = 3/16$, while $K_5 = 3/4$.

For the sech-function trial function $b(X) = (1/\sqrt{2})\text{sech}(X)$. Then, for the coefficients, we have $B_1 = B_4 = 1/3$, $B_6 = 2/15$, $B_7 = \pi^2/12$, $K_1 = (\pi^2/6)/36$, $K_2 = \pi^2/90$ $- 1/12$, $K_3 = 7\pi^4/240$, and $K_4 = (24 - \pi^2)/36$, while $K_5 = (12 + \pi^2)/36$.

For all the above cases, the method of moments and the Lagrangian approach produce the same three-parameter dynamical system.

IV. EXTENSION TO FIVE PARAMETERS IN THE TRIAL FUNCTION

The comparison can also be extended to the case where five or more parameters are used in the trial function. Increasing the number of parameters can be used to either increase the accuracy of the approximation or to describe more complicated soliton behavior, such as pulsations, transverse movements, etc. Adding more parameters may result in complications with analytic forms of the coefficients. However, the integrals in the above equations can be evaluated numerically, as we do not need analytic forms for them to write down the FDDS in explicit form.

In particular, we can allow a soliton to move by adding a linear phase term in the exponential in Eq. (1), i.e., $a(z)y$, where $y = x - x_0(z)$. The phase part will then take the form $\exp[i(a(z)y + c(z)y^2 - \theta(z))]$. At the same time, the transverse variable $x$ in the amplitude profile function is replaced by $y$. The variable $x_0(z)$ here is the soliton center-of-mass, and it, too, is a function of $z$. If the amplitude profile is symmetric, then we still have

$$\frac{\partial L}{\partial c} = DL_j/i$$

and

$$\frac{\partial L}{\partial \tilde{c}} = I_2.$$  \hspace{1cm} (26)

Hence, $dL_2/d\tilde{z} + iDL_3$ can be written as

$$\frac{\partial L}{\partial c} = \frac{d}{d\tilde{z}} \left( \frac{\partial L}{\partial \tilde{c}} \right),$$

while the right-hand side of Eq. (9) contains $y^2$. Hence the Lagrangian equation for $p = c(z)$ is again obtained. Further, $\partial L/\partial x_0 = -aQ$ and so $a'$ can be found from the $p = x_0(z)$ equation. Then, $\partial L/\partial a = Q(Da - x_0)$ and so $x_0'$ can be found from the $p = a(z)$ equation.

In the formulation of the method of moments, $I_1 = Qx_0$ and momentum $P = -iaQ$. It turns out that identical equations with the Lagrangian approach are still obtained. So the reduced dynamical system for solitons that was obtained using the method of moments can also be obtained using the Lagrangian approach if the commonly used phase ansatz is employed. Thus, the two techniques are often equivalent, although the principles used to obtain them are different, and the resultant dynamical equations also differ when another phase ansatz is used.

In general, we would use an asymmetric amplitude function when the additional two parameters are being used, since the linear phase term $a$ causes the soliton to have non-zero velocity, $x_0'(z)$. In this case the situation is more complicated, as $I_1$ then contains a term $Qf$ and $P$ contains an additional term in $Qcf$. Due to this, we find that $\partial L/\partial c$ is no longer equal to $DL_2/i$ and the first-order equations from the Lagrangian approach differ from those derived using the method of moments. Only the $Q'(z)$ equation is the same for both cases.

V. OCCURRENCES OF NONEQUIVALENCE

A. Example where the difference occurs

The two methods always give the same form for equation with $Q'(z)$, regardless of the chirp form. However, the methods only agree fully for the commonly used chirp term, $x^2$. This occurs because of the right-hand side term $-x^2$ obtained when deriving Eq. (14) matches the $-x^2$ term which always exists in the second moment equation (19). Then $I_2 = \partial L/\partial c$, and $\partial L/\partial \tilde{c} = -iDL_3$, and as a result, the second moment equation is the same as the Lagrangian width equation.

For example, with quadratic chirp and $b(X) = \exp(-X^4)/j$ with $j = 2^{3/8}\sqrt{\Gamma(5/4)}$, we find that

$$I_2 = \frac{\Gamma(3/4)}{2\sqrt{\Gamma(1/4)}} j^2 Q,$$

while $I_3 = 4icI_2$. As expected, we then get the same equation for $f'(z)$ in each method.
\[
\frac{df}{dz} = \frac{0.3879\beta}{f(z)} + 2Dcf(z) - 1.13612\beta c^2 f^3 - 0.161569eQ - 0.138202\nu Q^2, \quad (27)
\]

However, this matching occurs only for the quadratic chirp. To give an example where the methods differ, we simply take the Gaussian amplitude with quartic chirp,
\[
\psi = (2/\pi)^{1/4}\exp(-X^2)\exp[i(c(z)x^4 - \theta(z))]. \quad (28)
\]
Then the Lagrangian approach gives
\[
\frac{df}{dz} = \frac{2\beta}{f(z)} + 5Dcf^3(z) - \frac{75}{2}\beta c^2 f^2(z) - \frac{3\epsilon Q}{8\nu} - \frac{8\mu Q^2}{9\pi^3 f(z)}, \quad (29)
\]
while the method of moments results in
\[
\frac{df}{dz} = \frac{2\beta}{f(z)} + 3Dcf^3(z) - \frac{45}{2}\beta c^2 f^2(z) - \frac{\epsilon Q}{2\nu} - \frac{4\mu Q^2}{3\pi^3 f(z)}. \quad (30)
\]
Clearly, the general forms of Eqs. (29) and (30) are similar. All terms have the same algebraic structure but the coefficients are different. This difference indicates that we have to be careful with the trial function technique implemented in different approximations.

**B. Extension to higher-order chirp**

Let us consider an extension to a higher-order chirp term, \(ic(z)x^{2k}\), where \(k\) is a positive integer. The even exponent ensures that the chirp is symmetric and that the soliton has zero velocity. We introduce the following definitions which generalize the lower-order moments (10) and (11):
\[
I_{2k} = \int_{-\infty}^{\infty} x^{2k}\psi^2 dx \quad (31)
\]
and
\[
I_{2k+1} = k\int_{-\infty}^{\infty} x^{2k-1}(\psi^*\psi_x - \psi_x^*\psi) dx. \quad (32)
\]
Equations (10) and (11) are particular cases of Eqs. (31) and (32) when \(k=1\) and \(x_0(z) = 0\), so that \(y=x\). We need to generalize the moment equations. We restrict ourselves to symmetric trial functions, so that \(P=I_1=0\). The equations of the FDDS are now the following:
\[
\frac{dQ}{dz} = i\int_{-\infty}^{\infty} (\psi R^* - \psi^* R) dx, \quad (33)
\]
\[
\frac{dI_{2k}}{dz} = -iD_{2k+1} + i\int_{-\infty}^{\infty} x^{2k}(\psi R^* - \psi^* R) dx, \quad (34)
\]
\[
\frac{dI_{2k+1}}{dz} = i(k-2)\int_{-\infty}^{\infty} x^{2k-2}(2D|\psi_x|^2 - |\psi|^4) dx + 2ik\int_{-\infty}^{\infty} x^{2k-1}(\psi_x R^* + \psi_x^* R) dx + ik(2k-1)
\times \int_{-\infty}^{\infty} x^{2k-2}(\psi R^* + \psi^* R) dx. \quad (35)
\]
The first Eq. (33) retains the same form as Eq. (13). The \(k=1\) case of Eq. (34) gives the quadratic chirp equation for \(I_2\) [i.e., Eq. (19)], while the \(k=1\) case of Eq. (35) gives that for \(I_3\), viz., Eq. (22).

As mentioned, we allow for the chirp to follow an even power-law, i.e., \(x^{2k}\) for positive integer \(k\). For convenience, we consider a Gaussian amplitude function, \(2/\pi)^{1/4}\exp(-x^2/f^2)\), so that the field is
\[
\psi = (2/\pi)^{1/4}\sqrt{f/\pi} \exp[-x^2/f^2 + \nu \epsilon c(z) - \theta(z)].
\]
Then, the moments can be written in the form
\[
I_{2k} = \Gamma\left(k + \frac{1}{2}\right) f^{2k} \sqrt{Q} \quad (36)
\]
and
\[
I_{2k+1} = k^2 \frac{i}{\sqrt{\pi}} 2^{3-2k} \Gamma\left(2k + \frac{1}{2}\right) f^{4k-2} c^k Q, \quad (37)
\]
where \(\Gamma\) indicates the gamma function.

On the other hand, the Lagrangian, \(L\), for the above trial function takes the form
\[
\frac{L}{Q} = \frac{D}{2f^2} + \frac{D}{\sqrt{\pi}} 2^{2-2k} c(z)^2 f^{4k-2} \Gamma\left(2k + \frac{1}{2}\right) - \frac{Q}{2\sqrt{\pi}f}
\times \frac{2\nu Q^2}{3\sqrt{\pi} f^2} + \frac{2k}{\sqrt{\pi}} f^{2k} c(z) \Gamma\left(k + \frac{1}{2}\right) c' - \theta'. \quad (38)
\]
For this Lagrangian, the following conditions hold:
\[
\frac{\partial L}{\partial c} = DI_{2k+1}/i \quad \text{and} \quad \frac{\partial L}{\partial c^*} = I_{2k},
\]
thus generalizing Eq. (26).

Now, let us compare the two techniques. We find that, for general integer \(k\), the equations giving \(Q'\) do coincide. In each case, we have
\[
Q'\Gamma/\Gamma = 2\delta - \frac{\beta}{\sqrt{\pi}} 2^{2-2k} c(z)^2 \Gamma\left(2k + \frac{1}{2}\right) f^{4k-2}(z)
\times \frac{2\epsilon Q(z)}{\sqrt{\pi} f(z)} + \frac{4\mu Q^2(z)}{\sqrt{\pi} f^2(z)} - \frac{2\beta}{f^2(z)}.
\]
Furthermore, the relation
\[
\frac{1}{i\psi^*} \frac{\partial \psi^*}{\partial c} = -x^{2k}
\]
generalizes Eq. (14) and shows that the Lagrangian width equation \([\mu = c(z)\epsilon]\) in Eq. (9)] is the same as the moment equation involving \(dI_{2k}/dz\). Hence the width equations also coincide. In each case, we have
we can expect to find them. It can also let us predict the approaches. They become different when higher-order terms obtain identical forms of the FDDS from each of the two

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However, the expressions for $c'(z)$ are the same only when $k = 1$. When $k > 1$, the equations are different. For example, when $k=2$, the Lagrangian approach gives

$$c'(z) = - 15 D c^2(z) f^3(z) - \frac{8 \beta c(z) Q(z)}{f^2(z)} - \frac{2 Q(z)}{3 \sqrt{\pi} f^6(z)}$$

$$- \frac{16 \mu Q^2(z)}{9 \sqrt{3} \pi f^8(z)} + \frac{4 D}{3 f^6(z)},$$

whereas the method of moments results in a different equation,

$$c'(z) = - 114 \beta c^2(z) f^3(z) - 9 D c^2(z) f^3(z) - \frac{56 \beta c}{5 f^2(z)}$$

$$+ \frac{12 D}{5 f^3(z)} + \frac{\epsilon c(z) Q(z)}{2 \sqrt{\pi} f^6(z)} + \frac{40 \mu c(z) Q^2(z)}{27 \sqrt{3} \pi f^7(z)}$$

$$- \frac{Q(z)}{5 \sqrt{\pi} f^7(z)} - \frac{16 \mu Q^2(z)}{45 \sqrt{3} \pi f^9(z)}.$$ (39)

As we can see, the difference in these two equations is not merely in the numerical coefficients. The right-hand sides of these equations have algebraically different terms.

The generalization here provides two equations of FDDS that match, but the remaining one, viz., the chirp derivative, does not match. This difference would result in different numerical outcomes. Clearly, this difference should be taken into account when modeling highly chirped CGLE soliton solutions.

VI. DISCUSSION AND CONCLUSION

Although the Lagrangian approach and the method of moments have been used for a long time to approximate soliton solutions for various PDEs, they have not been compared directly in terms of the equivalence of the reduced dynamical systems that they provide. Indeed, each technique uses a separate physical principle to derive the final reduced dynamical system. Thus, generally speaking, the final equations do not have to be the same. Our detailed calculations have shown that they may give different results. However, for general amplitude trial functions with quadratic chirp, we obtain identical forms of the FDDS from each of the two approaches. They become different when higher-order terms are included in the phase chirp.

The use of a trial function has great value when the soliton solutions of the original PDE are unknown. Furthermore, the technique allows us to estimate the regions where we can expect to find them. It can also let us predict the existence of new types of solitons. The accuracy of these estimates may not be high, but we overcome difficulties encountered when using massive direct numerical simulations of the original PDE. The final results have to be confirmed by the numerical simulations anyway (e.g., Ref. 18), at least for several particular cases.

In all our previous calculations, the reduced dynamical system was the same, independent of the technique chosen. However, when the detailed chirp of the pulse becomes complicated, the choice of the technique could also be important. This may happen in problems related to pulse generation by passively mode-locked lasers operating in the normal dispersion regime. It is known that the pulses in this case can be highly chirped, and so appropriate pulse modeling in that case could be the way to a more accurate analysis.

In conclusion, we have shown that, for a symmetric function with a chirp that only has quadratic terms, the results obtained using the Lagrangian approach are the same as those obtained using the “method of moments.” This includes commonly used trial functions. This coincidence occurs because of the specific mathematical quadratic ($x^2$) chirp form. Hence the methods differ when other chirps are used. It is not necessary for analytic values of the integrals to be available. When the trial function is asymmetric or the chirp is more complicated, then the ordinary differential equations obtained are not the same.

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