ON THE NONLINEAR SCHRODINGER LIMIT OF THE KORTEWEG–DE VRIES EQUATION

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Using the multiscale approach of Zakharov and Kuznetsov it is shown that the nonlinear Schrödinger periodic scattering data is related to the Korteweg–de Vries periodic scattering data via an average over the Korteweg–de Vries carrier oscillation. This allows a complete elucidation of the physical meaning of the nonlinear Schrödinger scattering data, conservation laws, theta function solutions and reality constraint.

1. Introduction

The Korteweg–de Vries (KdV) equation

\[ u_t + 6uu_x + u_{xxx} = 0 \quad (u_t = \partial u/\partial t, \text{etc.}) \quad (1) \]

is one of the central equations in the modern theory of nonlinear waves [1]. The behavior of solutions of eq. (1) when \( u(x,0) \) is small in amplitude and nearly monochromatic is of particular physical interest. Although the wave may initially (at \( t = 0 \)) be nearly monochromatic the nonlinearity can cause the exchange of energy between the carrier oscillation and its harmonics and sub-harmonics. Thus, in principle, the nonlinearity can pump energy from the carrier to much longer wavelengths. This process is called nonlinear self-modulation. In such a situation it is important to know whether the self-modulation results in a large distortion or breakup of the original wave train.

This basic nonlinear process can be studied by changing to an envelope description of the wave train. An evolution equation for the envelope can then be derived by averaging eq. (1) over the carrier oscillation. This technique is well known and results in the defocusing nonlinear Schrödinger equation (hereafter referred to as NLS):

\[ iq_t + \frac{1}{2}q_{xx} - |q|^2q = 0, \quad (2) \]

where the primes on the independent variables indicate that \( q \) is a function of slow time scales and long...
space scales. The relationship between these space-time variables and the physical space-time variables, as well as that between the $q$ of eq. (2) and the $u$ of eq. (1), will be made explicit in the next section.

Although in this paper we will focus on the relationship between eqs. (1) and (2) it should be noted that eq. (2) is a generic envelope equation and therefore many of the techniques described herein have broader application.

The KdV and NLS equations are not only physically important, they have the added attraction of being exactly solvable by the Inverse Scattering Transform (IST). The central feature of the IST approach is the relation of a given nonlinear evolution equation to an auxiliary pair of linear operators, called the Lax pair [1]. The Lax pairs for the KdV and NLS equations have been known for some time and superficially appear quite different. Through the work of Ablowitz, Kaup, Newell and Segur (AKNS), however, the KdV and NLS Lax pairs were shown to be special examples of a more general Lax pair [2]. More recently, Zakharov and Kuznetsov have shown how to derive the NLS Lax pair from the KdV Lax pair through multiscale techniques [3], a result that is, in retrospect, to be expected given the averaging derivation of the NLS.

An appealing aspect of the IST approach is that it can be thought of as a generalization of Fourier analysis to nonlinear systems. This insight can be used to give physical meaning to the mathematical objects which appear in the IST approach. For example: in the theory of the KdV equation with spatially periodic boundary conditions the IST solution is reminiscent of a simple Fourier decomposition into plane waves – in fact this is precisely what the IST solution approaches in the zero amplitude (linear) limit [4]. Thus, although the mathematical theory of the periodic KdV equation involves much that is obscure to the physicist, in the end the scattering data is something which can be given concrete physical meaning.

The IST can also be used to solve the NLS for spatially periodic boundary conditions. (However, the mathematical theory is more difficult than that of the periodic KdV.) The solutions are expressible in terms of a Fourier-like series of nonlinear oscillations, but the physical meaning of the variables is not immediately apparent. What is required is a method to explicitly trace the path which starts with the physical KdV wave and ends with the NLS scattering data. The purpose of this paper is to map out this path. We also show how to construct the NLS theta function solutions from those of KdV. This is an extremely singular limit, but it is possible to carry out all of the calculations explicitly.

Clarification of the physical meaning of the NLS scattering data and theta function solutions is desirable for one major reason. For a physicist the end goal of any theory is to know how the abstract mathematical objects of theory relate to the real world. This is essential for applications of the IST to experimental situations.

A word here is in order regarding the relationship between the present work and that of refs. [5] and [6] which also deal with multiscale averaging of KdV wavetrains. The present work deals strictly with the NLS limit and, therefore, is a small amplitude theory, while that of refs. [5] and [6] deal with finite amplitude KdV wavetrains.

The outline of the paper is as follows:

First, the multiscale approach of Zakharov and Kuznetsov is briefly discussed. The NLS is derived from the KdV. The NLS Lax pair is derived from the KdV Lax pair.

Second, the KdV scattering data is introduced as the zeros of the squared eigenfunction of the related scattering problem. The multiscale expansion approach of Zakharov and Kuznetsov is then applied. It is shown how to average the KdV scattering data and its relation to the NLS data. The form of the NLS scattering data used in the present paper is the same as that of ref. [7]. There is a straightforward relationship between this form of the scattering data and that of ref. [8]. For complete details the interested reader is referred to ref. [9].
The multiscale approach is then applied to the KdV conservation laws as a whole (via the ‘master’ conservation law for the squared eigenfunction) to derive the NLS conservation laws.

Finally, the NLS limit of KdV N-band solutions is computed directly. This leads to a very appealing interpretation of the ‘reality’ constraint which appears in the periodic NLS theory [7, 10, 11].

2. Multiscale averaging of KdV waves and Lax pair

Consider eq. (1) when \( u(x, t) \) is small in amplitude (\(|u(x, t)| \ll 1\)). In such a situation the nonlinear term is small compared to the linear terms. The linear dispersion relation is

\[
\omega = -k^3,
\]

and the linear form of eq. (1) has solutions

\[
(a e^{i\theta} + \text{c.c.})
\]

(here \( \theta = kx + k^3t \) and c.c. means complex conjugate). Starting with a small amplitude monochromatic solution of this form we can ask what the effect of the nonlinearity in eq. (1) would be. Since it is simple quadratic nonlinearity the answer is that, although the wave may initially be monochromatic, it will immediately develop harmonics:

\[
u(x, t) = U_n e^{i\theta}, \quad (3)
\]

where \( U_n^* = U_{-n} \) to insure reality. The coefficient \( U_1 \) is assumed to be \( \mathcal{O}(\epsilon) \) with \( \epsilon \ll 1 \). (The scaling with \( \epsilon \) for \( n \neq \pm 1 \) will be discussed momentarily.) Notice that only the carrier oscillation obeys the linear dispersion relation. The terms with \( n \neq \pm 1 \) are driven oscillations.

Now suppose that this solution is modulated on a long spatial scale. We shall assume that the modulation is periodic: \( u(x + L, t) = u(x, t) \), with \( L \gg 1/k \). This can be dealt with by modifying eq. (3) to allow the expansion coefficients to depend on the longer spatial scale and a slow time scale:

\[
U_n = e^{\gamma_n x'} u_n(x', t'), \quad (4)
\]

where

\[
x' = \epsilon(x + 3k^2t) \quad \text{and} \quad t' = -6\epsilon^2 k t, \quad (5)
\]

and

\[
\gamma_0 = 2; \quad \gamma_n = |n| \quad \text{for} \quad n \neq 0. \quad (6)
\]

The coefficients \( u_n \) are \( \mathcal{O}(1) \). Notice that the definition of \( x' \) consists of a transformation to the frame moving with the group velocity of the wavetrain and a rescaling. The scaling with harmonic number, \( n \), in eqs. (4) and (6) is required by eq. (1). The next step is to insert the harmonic expansion for \( u(x, t) \) into eq. (1), making use of the fact that

\[
\frac{\partial}{\partial t} = -6\epsilon^2 k \frac{\partial}{\partial t} + 3k^2 \epsilon \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial x'}.
\]
After collecting terms with equal \( \exp(i n \theta) \) this gives

\[
\begin{align*}
-6e^2k \frac{\partial}{\partial t} + 3k^2e \frac{\partial}{\partial x} + i nk^3 \right] e^nu_n + \left[ e \frac{\partial}{\partial x} + ink \right]^3 e^nu_n \\
+ 3 \left[ e \frac{\partial}{\partial x} + ink \right] \sum_{m=-\infty}^{\infty} e^{i n + \gamma_{n-m}} u_m u_{n-m} = 0.
\end{align*}
\] (8)

As \( e \to 0 \) the lowest order terms in \( e \) must balance. Except for \( n = 0, \pm 1 \) this amounts to balancing \( (ink^3) \) from the first bracket, \( (ink)^3 \) from the second bracket, and a finite number of terms from the third expression:

\[
ink^3(1 - n^2) u_n + 3ink \sum_m u_m u_{n-m} = 0 \quad (n \neq 0, \pm 1),
\] (9)

where the prime on the summation indicates that only the values of \( m \) such that \( \gamma_m + \gamma_{n-m} = \gamma_n \) are included. Closer inspection reveals that the summation involves coefficients of lower order than \( n \), therefore eq. (9) is simply an algebraic recursion relation for the \( u_n \).

When \( n = 0 \) eq. (9) is trivially satisfied. In this case eq. (8) must be consulted again, leading to

\[
k^2 \frac{\partial u_0}{\partial t} + \frac{\partial (u_0 u_{-1})}{\partial x} = 0,
\]

which can be rewritten as

\[
u_0 = \frac{2}{k^2} |u_1|^2 + \kappa,
\] (10)

where \( \kappa \) is a constant of integration.

Now consider the cases \( n = \pm 1 \). Since \( u_{-1} = u_1^* \) it is sufficient to consider only the \( n = 1 \) term. The terms which are normally lowest order in \( e \) drop out when \( n = 1 \), therefore the balance must be found at the next highest order. A little algebra shows that \( u_1 \) satisfies

\[
\frac{1}{i} \frac{\partial u_1}{\partial t} + \frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} = -u_2 u_{-1} - u_1 u_0.
\] (11)

An expression is needed for \( u_2 \). This can be obtained from eq. (9) with \( n = 2 \):

\[
k^2 u_2 = u_1^2.
\]

Putting this all together gives

\[
\frac{1}{i} \frac{\partial u_1}{\partial t} + \frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} - \frac{1}{k^2} |u_1|^2 u_1 = -\kappa u_1.
\]

This can be converted to the standard form of the defocusing NLS by defining

\[
q(x', t') \equiv \left( \frac{u_1}{k} \right) e^{-i \kappa t'}.
\]

In what follows \( \kappa \) will not be important, so we will set it equal to zero for simplicity.
Zakharov and Kuznetsov applied the above multiscale asymptotic expansion to the Lax pair for the KdV and showed how to extract the Lax pair for the NLS. Here we will discuss the result for the spatial half of the Lax pair since the calculation for the time part is similar. The spatial scattering problem associated with the KdV equation is the one-dimensional time-independent Schrödinger equation with the KdV wave, \( u(x, t) \), playing the part of the potential \([1]\):

\[
\left[ \frac{\partial^2}{\partial x^2} + u + E^2 \right] \Phi = 0. 
\]  

When we average the conservation laws for the KdV equation we shall also need the time part of the Lax pair:

\[
\left[ \frac{\partial}{\partial t} + 4 \frac{\partial^3}{\partial x^3} + 6u \frac{\partial}{\partial x} + 3u_x \right] \Phi = 0. 
\]

Since the potential is periodic, we look for the Floquet solutions of eq. (12); in other words solutions with the property

\[
\Phi(x + L; E) = m(E) \Phi(x; E),
\]

where \( m(E) \) is the Floquet multiplier \([12,13,14]\).

Floquet's theorem states that there are two such solutions (call them \( \Phi \) and \( \Psi \)) for all \( E^2 \) except for a discrete set of values on the real axis. This set of points is called the nondegenerate main spectrum. These particular eigenvalues also have the property that the Floquet solutions associated with them are either periodic or anti-periodic over the spatial period of the potential \( u \). The spectral parameter \( E^2 \) is termed a stable, or unstable, eigenvalue if the Floquet solution corresponding to that eigenvalue is stable, or unstable, under spatial translation. The set of stable eigenvalues all lie on the real \( E^2 \)-axis, interspersed with segments of unstable eigenvalues. All values of \( E^2 \) not on the real axis are unstable. There is a minimal eigenvalue \( (E^2 = E^2_0) \); all of the values of \( E^2 \) to the left of \( E^2_0 \) on the real axis are unstable (see fig. 1).

Now consider the Floquet solutions \( \Phi \) and \( \Psi \) for fixed \( x \) and variable spectral parameter \( E^2 \). The zeros of these two functions coincide and lie within the unstable bands. There is one such zero in each band, except for the unstable band extending to \( E^2 = -\infty \). These zeros are called the auxiliary spectrum.

![Fig. 1. The complex \( E^2 \) plane showing a schematic representation of a typical scattering spectrum for a narrow banded, small amplitude potential. The point \( E^2_0 \) is within \( o(E^2) \) of \( E^2 = 0 \), as discussed in the text. All of the other relevant spectrum clusters within \( o(E^2) \) of the resonance at \( k_0^2/4 \). In general there would be many other unstable bands present surrounding the higher harmonics \( (E^2 = n^2k_0^2/4, n = 1, 2, \ldots) \), but they are not relevant for the NLS limit. In both figures the stable bands are indicated by lines of light weight, while the unstable bands are indicated by heavy wavy lines. The main spectrum is indicated by X's.](image-url)
There are two important limits which occur simultaneously as $\varepsilon \to 0$:

1) since the amplitude of $u(x)$ is $O(\varepsilon)$ this implies that the width of the unstable bands is also $O(\varepsilon)$.
2) the length scale of the modulation (related to $x'$) is $O(1/\varepsilon)$ which implies that the unstable bands are only $O(\varepsilon)$ apart.

Readers can convince themselves of these assertions by considering the nature of the main spectrum for $u(x) = 0$. In this case all of the unstable bands are degenerate and it can be seen that if the scattering calculation is done in a box of length $O(1/\varepsilon)$ then the eigenvalues where the Floquet solutions are periodic or antiperiodic in the given box length are separated by $\Delta E = O(\varepsilon)$. These eigenvalues constitute the main spectrum. When the amplitude of the potential is no longer zero but $O(\varepsilon)$, the degeneracies of the main spectrum are broken to the same order, which is a standard result from perturbation theory.

When the spectral parameter $E$ is approximately an integral multiple of $k/2$ the Floquet eigenfunctions can resonate with the potential. The strongest resonance occurs when $E^2 = k^2/4$. There will be many unstable bands with widths of $O(\varepsilon)$ in an $O(\varepsilon)$ neighborhood around this point. Inside each of these unstable bands is a member of the auxiliary spectrum, therefore the Floquet functions have many zeros in a small region clustered around the first resonance. If there are $N$ unstable bands in the neighborhood surrounding the resonance then the Floquet solutions obey

$$\Phi \left( x; E^2 = \frac{k^2}{4} + \varepsilon\lambda \right) = O(\varepsilon^{N/2}),$$

with a similar relation for $\Psi$. The Floquet solutions are $O(\varepsilon^{N/2})$ in this region because the zeros are square root branch points [1].

The scattering problem, eq. (12), can be expanded about the resonance by writing

$$E^2 = \frac{k^2}{4} + \varepsilon\lambda.$$  \hfill (15)

Then, in the region around the resonance the Floquet solutions are expanded as

$$\Phi \left( x; E^2 = \frac{k^2}{4} + \varepsilon\lambda \right) = \varepsilon^{N/2} \sum_{n=\infty}^{\infty} e^{\alpha_n} \phi_n(x'; \lambda) e^{in\theta/2}, \quad \phi_0 \equiv 0,$$

where the $\phi_n$ are $O(1)$ quantities and $\alpha_n = (|m| - 1)/2$. Notice that the scaling assumed here differs from that of ref. [3] by the overall factor of $\varepsilon^{N/2}$. This is due to the fact that we focus on Floquet solutions. The ansatz above is inserted into the scattering problem, eq. (12) (the harmonic expansion for $u(x)$ is also used) and terms with like $\varepsilon$ behavior are collected:

$$e^{(|n|+3)/2} \frac{\partial^2 \phi_n}{\partial x'^2} + i n k e^{(|n|+1)/2} \frac{\partial \phi_n}{\partial x'} + (1 - n^2) \frac{k^2}{4} e^{(|n|-1)/2} \phi_n$$

$$+ e^{(|n|+1)/2} k \phi_n + \sum_{m=-\infty}^{\infty} e^{\alpha_m + (|n-2m|-1)/2} u_m \phi_{n-2m} = 0,$$  \hfill (17)
This expression is then evaluated as $\epsilon \to 0$ and the leading order terms required to balance. The results are, once again, algebraic recursion relations for $n \neq \pm 1$:

$$
(1 - n^2) \frac{k^2}{4} \phi_n + \sum_m \gamma_m \phi_{n-2m} = 0,
$$

(18)

and a pair of differential equations for $n = \pm 1$:

$$
i k \frac{\partial \phi_1}{\partial x'} + k \lambda \phi_1 = u_1 \phi_{-1},
$$

$$
- i k \frac{\partial \phi_{-1}}{\partial x'} + k \lambda \phi_{-1} = u_{-1} \phi_1.
$$

(19)

With the same change of variables used earlier in the NLS derivation, this becomes the spatial scattering problem for NLS. The temporal part of the Lax pair is more involved algebraically, but the approach remains essentially unchanged.

3. Averaging of the KdV scattering data

At this point the simplest way to proceed is to introduce the squared eigenfunction $R(x, E)$:

$$
R(x, t; E) = \Phi(x, t; E) \Psi(x, t; E).
$$

(20)

From now on we will suppress the $t$-dependence in writing $R$ for the sake of clarity. Using eq. (12) a little algebra shows that $R$ satisfies

$$
R_{xxx} + 4(E^2 + u)R_x + 2u_x R = 0.
$$

This can be integrated once, after first multiplying by $R$, which leads to

$$
RR_{xx} - \frac{1}{2}R_x^2 + 2(E^2 + u)R^2 = C(E),
$$

(21)

where $C(E)$ is a constant in $x$ (it is also constant in $t$). For special potentials there will only be a finite number of nondegenerate unstable bands (the so-called $N$-band potentials) [1]. In such a case it is possible to find solutions of eq. (21) which are finite polynomials in $E$. Assume that $u(x)$ is such a potential and expand $R$ as

$$
R(x; E) = \sum_{j=0}^{N} R_j(x) E^{2j}.
$$

The integration constant, $C(E)$, may also be expanded in a similar fashion. Inserting these expansions into eq. (21) leads to a recursion relation for the $R_j$'s, which for $j = N$ gives

$$
2R_{N}^2 = C_{4N+2},
$$
while \( j = N - 1 \) leads to
\[
4R_N R_{N-1} + 2uR_N^2 = C_{4N}.
\]

Using the normalization \( R_N = 1 \) results in
\[
C_{4N+2} = 2 \quad \text{and} \quad u(x) = -2R_{N-1}(x) + \frac{1}{2}C_{4N}.
\] (22)

At this point it is useful to rewrite \( R(x, E) \) and \( C(E) \) in product form:
\[
R(x, E) = \prod_{j=1}^{N} (E^2 - \mu_j), \quad C(E) = 2 \prod_{k=0}^{2N} (E^2 - E_k).
\] (23)

By using eq. (20), eq. (21) can be shown to be the Wronskian of \( \Psi \) and \( \Phi \). This would be zero only if \( \Psi \) and \( \Phi \) were dependent. Thus, the zeros of \( C(E) \) are precisely the nondegenerate main spectrum, while the zeros of \( R(x, E) \) are the auxiliary spectrum. The 'scattering data' for the periodic KdV problem consists of the nondegenerate main spectrum and the auxiliary spectrum. The auxiliary spectrum, denoted as \( \mu_j \), are \( x \)- and \( t \)-dependent and oscillate between the two neighboring main spectral eigenvalues. Using these product representations in eq. (22) we get
\[
u(x) = 2 \sum_{j=1}^{N} \mu_j(x) - \sum_{k=0}^{2N} E_k.
\] (24)

This is a Fourier-like representation of \( u(x) \) in terms of the nonlinear oscillations \( \mu_j(x) \). Marchenko [13] proved that any \( u(x) \) can be approximated as closely as desired by an \( N \)-band potential. Therefore the \( \mu_j \) can be thought of as a 'complete' set in that sense. During the discussion of the \( \Theta \)-function solutions of KdV we will need to make use of the evolution equations for the auxiliary variables. These can be obtained by evaluating eq. (21) at \( E^2 = \mu_j \) and using eqs. (23):
\[
-\frac{1}{2}R_x^2(\mu_j) = C(\mu_j),
\]
using
\[
R_x(\mu_j) = \left[ -\prod_{m \neq j} (\mu_j - \mu_m) \right] \mu_{jx} \quad \text{and} \quad C(\mu_j) = 2 \prod_{n=0}^{2N} (\mu_j - E_n),
\]
leads to
\[
\mu_{jx} = \frac{\pm 2i \left[ \prod_{n=0}^{2N} (\mu_j - E_n) \right]^{1/2}}{\prod_{m \neq j} (\mu_j - \mu_m)}, \quad j = 1, 2, \ldots, N.
\] (25)

These equations should be interpreted as describing the motion of the \( \mu_j \)'s on the two-sheeted Riemann surface associated with \( C(E) \) [14]. The choice of plus or minus sign in eq. (25) is equivalent to fixing which
The time derivatives can be obtained in the same manner by first computing the time derivative of \( R(x, t; E) \), and then evaluating it at \( E^2 = \mu_j \). The result is

\[
\mu_{jt} = -4 \left( \sum_{m \neq j} \mu_m - \frac{1}{2} \sum_{n=0}^{2N} E_n \right) \mu_{jx}.
\]

(26)

In the particular situation we consider in this paper all of the main spectrum and the auxiliary spectrum of interest cluster in the neighborhood of \( E^2 = k^2/4 \). The auxiliary variables oscillate between neighboring main spectral eigenvalues. If we average \( u(x, t) \) over the rapid oscillations, using eq. (24), we will be left simply with \( E_0 \). Therefore \( E_0 \) represents the uniform background of the wavetrain and must be \( O(\varepsilon^2) \) (recall that, in the harmonic expansion of \( u(x, t) \), \( U_0 = O(\varepsilon^2) \)). Thus,

\[
E_0 = \varepsilon^2 \lambda_0; \quad E_k = \frac{k^2}{4} + \varepsilon \lambda_k, \quad k = 1, 2, \ldots, 2N,
\]

\[
\mu_j(x) = \frac{k^2}{4} + \varepsilon \zeta_j(x), \quad j = 1, 2, \ldots, N,
\]

(27)

These relations define the \( \lambda_k \) and \( \zeta_j \). The \( \lambda_k \) \( (k = 1, 2, \ldots, 2N) \) are the NLS main spectrum, but the \( \zeta_j \) are not the NLS auxiliary variables since they still contain the carrier oscillation and its harmonics. Using these definitions, and eq. (23), we find that in an \( O(\varepsilon) \) neighborhood around the resonance the squared eigenfunction \( R \) has the following behavior:

\[
R(x; \lambda) \equiv R \left( x; E^2 = \frac{k^2}{4} + \varepsilon \lambda \right) = \varepsilon^N \prod_{j=1}^{N} \left( \lambda - \zeta_j(x) \right).
\]

(28)

We are particularly interested in the relationship between the KdV auxiliary variables and the NLS auxiliary variables. The auxiliary variables for the NLS are the zeros of the associated NLS squared eigenfunctions. In the NLS case, however, there are two spinor components to the eigenfunctions, which means that there are three independent squared eigenfunctions.

Precisely how is the single set of KdV auxiliary variables related to the three sets of NLS auxiliary variables?

This can be deduced by utilizing the expansions for the Floquet solutions to find a multiscale expansion for the KdV squared eigenfunction, \( R(x; E) \). Before this is done, however, it is useful to note that the algebraic recursion relations for expansion coefficients (eq. (18)) are such that all of the coefficients for \( n \neq \pm 1 \) can be gotten from \( \phi_1 \) and \( \phi_{-1} \) (or \( \psi_1 \) and \( \psi_{-1} \)). In fact, the recursion relations are such that coefficients with positive \( n \) relate only to \( \phi_1 \) and those with negative \( n \) only to \( \phi_{-1} \), and so forth. This means that we can write

\[
\phi_n(x'; \lambda) = F_n(x') \phi_1(x'; \lambda), \quad n \geq 1,
\]

\[
\phi_n(x'; \lambda) = G_n(x') \phi_{-1}(x'; \lambda), \quad n \leq -1,
\]

(29)

where the \( F_n(x') \) and \( G_n(x') \) are constructed from the \( u_k(x') \) (the expansion coefficients of \( u(x) \)). Since both Floquet solutions obey the same recursion relations the functions \( F_n \) and \( G_n \) will be identical for \( \Phi \).
and $\Psi$. Using this insight the Floquet solutions can be expanded as

$$
\Phi(x; \lambda) = e^{N/2} \left[ \sum_{n=1}^{\infty} e^{n} F_n(x') e^{i n \theta / 2} \right] \phi_1(x'; \lambda) + e^{N/2} \left[ \sum_{n=-\infty}^{-1} e^{n} G_n(x') e^{i n \theta / 2} \right] \phi_{-1}(x'; \lambda),
$$

with a similar expansion for $\Psi$. This equation can be further simplified as

$$
\Phi(x; \lambda) = e^{N/2} \left[ F(x, x'; \epsilon) \phi_1(x'; \lambda) + G(x, x'; \epsilon) \phi_{-1}(x'; \lambda) \right].
$$

It is important to note the function $F(x, x'; \epsilon)(G(x, x'; \epsilon))$ contains only positive (negative) harmonics of the carrier. The KdV squared eigenfunction, $R$, can now be written as

$$
R(x; \lambda) = e^{N} \left[ 2F(x, x'; \epsilon)G(x, x'; \epsilon)f(x'; \lambda) + F^2(x, x'; \epsilon)g(x'; \lambda) + G^2(x, x'; \epsilon)h(x'; \lambda) \right],
$$

where we have used the definitions

\begin{align*}
&f(x'; \lambda) = \frac{1}{2} (\phi_1 \psi_{-1} + \psi_1 \phi_{-1}), \\
g(x'; \lambda) = \phi_1 \psi_1, \\
h(x'; \lambda) = \phi_{-1} \psi_{-1}.
\end{align*}

The functions $f$, $g$, and $h$ are the NLS squared eigenfunctions [7, 11].

As $\lambda \to \infty$ it can be shown, from the product form of $R(x; \gamma)$, that

$$
\lim_{\lambda \to \infty} R(x; \lambda) = \epsilon^N \lambda^N.
$$

The NLS squared eigenfunctions have the following asymptotic behavior:

\begin{align*}
\lim_{\lambda \to \infty} \begin{pmatrix} f(x'; \lambda) \\ g(x'; \lambda) \\ h(x'; \lambda) \end{pmatrix} &= \begin{pmatrix} \lambda^N \\ q(x') \lambda^{N-1} \\ q^*(x') \lambda^{N-1} \end{pmatrix},
\end{align*}

which immediately implies that $2F(x, x'; \epsilon)G(x, x'; \epsilon) = 1$. Putting this together we have

$$
\epsilon^{-N} R(x; \lambda) = f(x'; \lambda) + F^2(x, x'; \epsilon)g(x'; \lambda) + G^2(x, x'; \epsilon)h(x'; \lambda),
$$

which is our central result.

Notice that, since the function $F(x, x'; \epsilon)(G(x, x'; \epsilon))$ involves only positive (negative) harmonic numbers, the functions $F^2$ and $G^2$ have no zero-wavenumber components. Thus, if we average the KdV squared eigenfunction over a carrier oscillation:

$$
\langle \langle R(x; \lambda) \rangle \rangle = \frac{k}{2\pi} \int_{x_0}^{x_0 + 2\pi/k} R(x; \lambda) \, dx,
$$

we are left with

$$
\epsilon^{-N} \langle \langle R \rangle \rangle = f(x'; \lambda).
$$
The functions $F^2$ and $G^2$ can be evaluated. Expand both sides of eq. (30) as finite order power series in $\lambda$. Consider the $\lambda^{N-1}$ terms in particular:

$$e^{-N}R_{N-1} = f_{N-1} + F^2g_{N-1} + G^2h_{N-1}.$$ 

It turns out that $f_{N-1}$ can be related to the NLS main spectrum by a useful identity [7, 11]:

$$f_{N-1} = -\frac{1}{2} \sum_{k=1}^{2N} \lambda_k.$$ 

Now use eqs. (27) and (28) to get

$$e^{-N}R_{N-1} = -\frac{1}{2} \sum_{k=1}^{2N} \lambda_k + F^2q(x') + G^2q^*(x').$$ 

This can be rewritten, using eqs. (24) and (27), as

$$-\frac{1}{2\epsilon} u(x, t) - \epsilon^2\lambda_0 = F^2q(x') + G^2q^*(x').$$

Since $u(x, t)$ is a real function, this implies that $F = G^*$. The harmonic expansions of $F^2$ and $G^2$ can be determined by multiplying the above result with each Fourier harmonic, averaging over the short spatial scale, and using the known harmonic expansion for $u(x, t)$.

4. Averaging of the KdV conservation laws

It is well known that the KdV equation has an infinite number of conservation laws. These can all be summarized as a single conservation law for the squared eigenfunction $R(x, t; E)$:

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial x} \left[ 6(u + 2E^2)R + 2 \frac{\partial^2 R}{\partial x^2} \right].$$

This can be verified by using $R = \Phi\Psi$ and eqs. (12) and (13). By expanding this as a power series in $E$ it is possible to recover the entire set of KdV conservation laws. We wish to average this relation and extract the NLS conservation laws. First, $R$ is written as the product of the two Floquet eigenfunctions and expanded as before. The parameter $E^2$ is expanded about $k^2/4$ as usual. Averaging over the rapid oscillations then yields

$$\left[ \epsilon^2 \frac{\partial}{\partial t} + 2\epsilon^2 \lambda + \epsilon^3 \frac{1}{3} \frac{\partial^3}{\partial x^3} \right] \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{\alpha_n + \alpha_m} \psi_n \phi_m + \frac{\partial}{\partial x} \left[ \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} e^{\gamma_n + \alpha_p + \alpha_r} \frac{1}{k} \int u \phi_p \psi_r \right] = 0.$$

Recall that $\gamma_n = |n|$ for $n \neq 0$ and that $\gamma_0 = 2$. Also $\alpha_m = (|m| - 1)/2$ and $\phi_0 = \psi_0 = 0$. The lowest order
terms in the expansion above are $\mathcal{O}(\epsilon^2)$. Collecting these we find
\[
\left[ \frac{\partial}{\partial t} + 2\lambda \frac{\partial}{\partial x} \right] (\psi_{-1}\phi_1 + \phi_{-1}\psi_1) + \frac{\partial}{\partial x} \left[ \frac{u_1}{k} \phi_{-1}\psi_{-1} + \frac{u_{-1}}{k} \phi_1\psi_1 \right] = 0.
\]

Using the definitions of the NLS squared eigenfunctions, and $kq(x') = u_1$, we find
\[
\frac{\partial}{\partial t} [f] + \frac{\partial}{\partial x} \left[ 2\lambda f + \frac{1}{2}(q^*g - qh) \right] = 0,
\]
which is the fundamental conservation law for the NLS equation.

5. NLS solutions as singular limits of KdV solutions

The reader who is unfamiliar with the theta function representation for $N$-band solutions of the KdV and NLS equations is referred to refs. [7], [11], [14] and [15] for introduction. Here we are concerned only with extracting the NLS solutions from the KdV solutions, not with their initial construction.

An $N$-band solution of the KdV equation can be written in the form
\[
u(x,t) = \left[ 2 - \frac{\partial^2}{\partial x^2} \ln \Theta(x,t) \right] + \Lambda, \tag{31}
\]
where $\Theta(x,t)$ is the Riemann Theta function:
\[
\Theta(s|\tau) = \sum_{m \in \mathbb{Z}^N} \exp \left( 2\pi im \cdot s + 2\pi m \cdot \tau \cdot m \right), \quad s \in \mathbb{C}^N,
\]
$\tau$ is an $N \times N$ complex matrix and the constant, $\Lambda$, will be defined in a moment.

Because of the singular nature of the limit being taken, care must be exercised in the choice of canonical loop cycles and differentials. All but one of the canonical $a$-cycles are chosen to loop around the unstable bands which cluster around the point $E^2 = k_0^2/4$ (see fig. 2). Here we have denoted the carrier wavenumber by $k_0$ to distinguish it from the wavenumbers $(k_j, j = 1, 2 \ldots N)$ in the $\Theta$-function. The $a$-cycles all orbit, as usual, in a counter-clockwise direction on the top sheet of the two-sheeted Riemann

Fig. 2. The canonical $a$- and $b$-cycles. The cycle $a_1$ is chosen to orbit the branch cut which extends from the origin to negative infinity. The other $a$-cycles orbit entirely within the interior region in the neighborhood of the resonance. This forces $b_1$ to cross between the interior and exterior regions. The solid lines indicate that the curve lies on the top sheet, dotted lines on the bottom sheet.
There are five fundamental sets of quantities which need to be evaluated to carry out the limit:

1) The matrix of \(a\)-periods (\(A\)), defined as

\[
[A]_{jm} = \int_{a_j} dU_m, \quad j, m = 1, 2, \ldots, N, \quad (33)
\]

where \(a_j\) is the \(j\)th \(a\)-cycle and \(dU_m\) the \(m\)th holomorphic differential (to be defined in a moment). The inverse of this matrix is called \(C\). Using the entries of the \(C\)-matrix the \('normalized'\) differentials can be constructed from the \(dU_m\) as follows:

\[
d\omega_n = \sum_{m=1}^{N} C_{mn} dU_m.
\]

2) The matrix of \(b\)-periods (\(B\)), defined as

\[
[B]_{jm} = \int_{b_j} dU_m, \quad j, m = 1, 2, \ldots, N, \quad (34)
\]

where \(b_j\) is the \(j\)th \(b\)-cycle. Multiplication of \(B\) by \(C\) gives the \(\tau\)-matrix which appears in the \(\Theta\)-function.

3) The Abel Transformation, defined as

\[
W_j(\mu) = \sum_{m=1}^{N} \sum_{n=1}^{N} C_{mn} \int_{p_0}^{\mu_n} dU_m = \sum_{n=1}^{N} \int_{p_0}^{\mu_n} d\omega_n, \quad j = 1, 2, \ldots, N, \quad (35)
\]

where the \(\mu_n\)'s are the zeros of the KdV squared eigenfunction, and \(p_0\) is some fixed \('base point'\) on the Riemann surface. For concreteness we will assume that \(p_0\) is in the neighborhood of the resonance at \(E^2 = k_0^2/4\).

4) The \(N\) Riemann constants, defined as

\[
K_j = \frac{k_j}{2\pi} (x-x_0) + \frac{\Omega_j}{2\pi} (t-t_0) + \frac{1}{2} \tau_{j0} - \sum_{m=1}^{N} \int_{a_m}^{\omega_m} d\omega_m + \sum_{m=1}^{N} \omega_j(\mu_{m0}), \quad j = 1, 2, \ldots, N, \quad (36)
\]

where \(k_j (\Omega_j)\) is the \(j\)th wavenumber (frequency) of the \(N\)-band wave and \(\mu_{m0}\) is the initial position of the \(m\)th auxiliary variable. The function \(\omega_j(p)\) is defined as

\[
\omega_j(p) = \int_{p_0}^{p} d\omega_j.
\]

5) The \"exterior\" constant \(\Lambda\):

\[
\Lambda = 2 \sum_{n=1}^{N} \int_{a_n} E^2 d\omega_n - \sum_{m=0}^{2N} E_m. \quad (37)
\]

As mentioned above, the \(a\)-cycles orbit the unstable bands on the top sheet. The \(b\)-cycle crosses between the sheets, intersecting only the \(a\)-cycle in the canonical manner. This requires the cycle \(b_1\) to extend from the region of the resonance to the origin where it can then cross back onto the lower sheet to
complete its loop (see, fig 2). As \( \varepsilon \to 0 \), two distinct regions develop:

1) The region within \( \mathcal{O}(\varepsilon) \) of the resonance at \( E^2 = k_0^2/4 \). Notice that all of the \( a \)- and \( b \)-cycles (except for \( a_1 \) and \( b_1 \)) remain entirely within this "interior" region.

2) The "exterior" region, which includes the origin. The \( a \)-cycle lies entirely within this exterior region.

Only the \( b_1 \)-cycle crosses between the two regions.

All that remains is to choose an appropriate basis for the holomorphic differentials. This needs to be done in such a way as to keep the matrix of \( a \)-periods finite. This method of attack throws any singularities into the \( b \)-matrix (eventually the \( \tau \)-matrix) where they can be dealt with. An appropriate choice of differentials is

\[
dU_j = \varepsilon^{N-j} \left( \frac{k_0^2}{4} - E_0 \right)^{1/2} \left( E^2 - \frac{k_0^2}{4} \right)^{i-1} \frac{d(E^2)}{\prod_{n=0}^{2N} \left( E^2 - E_n \right)^{1/2}}, \quad j = 1, 2, \ldots, N. \tag{38}
\]

In what follows, all of the results are quoted to leading order, the corrections of \( \mathcal{O}(\varepsilon) \) and higher are not stated explicitly for neatness' sake. However, they are needed in order to compute the higher order corrections to the wavenumbers and frequencies.

Now that the differentials, \( dU_j \), have been specified it is possible to compute the space and time derivatives of the Abel Transformation (eq. (35)):

\[
\frac{dW_j}{dx} = 2i \sum_{n=1}^{N} \sum_{m=1}^{N} C_{m,j} \varepsilon^{N-m} \left( \frac{k_0^2}{4} - E_0 \right)^{1/2} \left( \frac{k_0^2}{4} - m \right)^{m-1} \prod_{n=m}^{N} \left( \frac{k_0^2}{4} - \mu_n \right)
\]

The summations can be performed with the aid of the following identity:

\[
\sum_{n=1}^{N} \frac{\mu_m^{n-1}}{\prod_{n=m+1}^{N} (\mu_n - \mu_m)} = \frac{1}{2\pi} \int_{S} \frac{z^{m-1} \, dz}{\prod_{n=1}^{N} (z - \mu_m)},
\]

where the contour, \( S \), on the complex \( z \)-plane encloses all of the poles in a counter-clockwise manner. The integrand can also be expanded near infinity to find the residue there, with the result that the summation is zero unless \( m = N \). This finally gives

\[
\frac{dW_j}{dx} = 2i C_{N,j} \left( \frac{k_0^2}{4} - E_0 \right)^{1/2} \equiv \frac{k_j}{2\pi}. \tag{39}
\]

A similar calculation for the time derivative leads to

\[
\frac{dW_j}{dt} = 4i \left( \frac{k_0^2}{4} - E_0 \right)^{1/2} \left( 2 \varepsilon C_{N-1,j} + \sum_{n=0}^{2N} E_n - (N - 1)\frac{k_0^2}{2} \right) C_{N,j} \equiv \frac{\Omega_j}{2\pi}. \tag{40}
\]
In the “interior” region \( (E^2 = k_0^2/4 + \varepsilon \lambda) \) surrounding the resonance:
\[
dU_j = \frac{\lambda^{j-1} d \lambda}{\left( \prod_{n=1}^{2N} (\lambda - \lambda_n) \right)^{1/2}}, \quad j = 1, 2, \ldots, N,
\]
as \( \varepsilon \to 0 \). Notice that \( dU_N \) is a singular differential while \( dU_j \) for \( j \leq N - 1 \) are the NLS holomorphic differentials [7, 11, 15]. In the “exterior” region:
\[
dU_j = \frac{k_0}{2} \frac{d(E^2)}{\sqrt{E^2 (E^2 - \frac{k_0^2}{4})}}, \quad j = N,
\]
in the same limit. Notice the pole at the resonance when \( j = N \).

Using the above differentials, the matrix of \( a \)-periods has the following asymptotic form:
\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & -2\pi i \\
& & & & \\
& & & \sigma_1 & \\
& & \vdots & & \\
\sigma_1 & \vdots & & \sigma_{N-1} & \\
A_{\text{NLS}} & & & \sigma_1 & \\
\end{bmatrix}
\]
where \( A_{\text{NLS}} \) is the \( (N - 1) \times (N - 1) \) matrix of NLS \( a \)-periods, and \( \sigma_m \) is the \( m \)th singular period. The inverse of the \( A \)-matrix, \( C \), has the following basic structure:
\[
C = \begin{bmatrix}
\Gamma_1 & C_{\text{NLS}}^{(N-1) \times (N-1)} & & \\
\vdots & \vdots & \ddots & \\
\Gamma_{N-1} & & & \\
\frac{-1}{2\pi i} & 0 & \cdots & 0 \\
\end{bmatrix}
\]

The matrix \( B \) has the following asymptotic form:
\[
B = \begin{bmatrix}
S_1^+ - S_1^- & \cdots & S_{N-1}^+ - S_{N-1}^- & -2 \ln(\varepsilon) + O(1) \\
& & & \\
B_{\text{NLS}}^{(N-1) \times (N-1)} & \beta_1 & \vdots & \beta_{N-1} \\
\end{bmatrix}
\]
where \( S_j^\pm = \int_{P_0}^{\pm} dU_j \).
These integrals are carried out in the interior region. The \( b_1 \)-cycle requires a loop around the origin \((E^2 = 0)\), which is in the exterior region (and therefore at infinity). One half of \( b_1 \) is on the top sheet, this becomes the integral to \( \infty + \), the other on the lower sheet, which becomes the integral to \( \infty - \). The generic form of \( C \) and \( B \) quoted above implies that \( \tau (= BC) \) looks like

\[
\begin{bmatrix}
-\frac{i}{\pi} \ln(e) + \mathcal{O}(1) & r_1^+ - r_1^- & \cdots & r_{N-1}^+ - r_{N-1}^- \\
r_1^+ - r_1^- & \tau_{NLS}^{(N-1)\times(N-1)} \\
\vdots & \vdots \\
r_{N-1}^+ - r_{N-1}^- & \\
\end{bmatrix}
\]

where \( r_j^\pm = \int_{p_0}^{\infty \pm} \omega_j \, d\omega_{j+1} \). Notice the special behavior of the first row and column.

The Abel Transformation can be evaluated as well. Since \( p_0 \) is assumed to be within the interior region all of the integrals in the Abel Transformation are finite and it simplifies to

\[
W_j(x, t) = \left\{ \begin{array}{ll}
-\frac{1}{2\pi} (k_0 x + k_0^3 t) + \mathcal{O}(\varepsilon), & j = 1, \\
\mathcal{O}(\varepsilon), & j = 2, 3, \ldots, N.
\end{array} \right.
\]

Thus, in the limit, all of the wavenumbers and frequencies become degenerate on the short space–time scale. The nonlinear wavenumber and frequency shifts may be extracted by carrying out the analysis to higher order. In particular, it needs be shown that the introduction of the variable \( x' \) (defined by eq. (5)) eliminates any \( t \)-dependence at \( \mathcal{O}(\varepsilon) \). This calculation is somewhat involved, and is therefore relegated to appendix 1.

Forging ahead, the Riemann constants, \( K_j \), need to be evaluated in several steps. Consider the following integrals:

\[
\sum_{k=1}^{N} \int_{a_k} \omega_j \, d\omega_k.
\]

With \( k \neq 1 \) (for any \( j \)) the integrals are entirely within the interior region, well defined, real and \( \mathcal{O}(1) \).

When \( j = 1 \) and \( k = 1 \) the integral can be evaluated explicitly:

\[
\int_{a_1} \omega_1 \, d\omega_1 = \int_{a_1} \left[ \frac{1}{2} \, d\omega_1^2 \right] = \left[ \frac{1}{2} \omega_1^2(p) \right]_{a_1} = \frac{1}{2} \left( \omega_1 + 1 \right)^2 - \omega_1^2 = \frac{1}{2},
\]

for all finite \( \varepsilon \) since \( \omega_1 \) is a normalized differential.

Now consider the case \( j \neq 1, k = 1 \). Recall that the \( a_1 \)-cycle lies entirely in the exterior region. The behavior of the functions \( \omega_j(p) \) when \( p \) is in the external region can be deduced by first integrating from \( p_0 \) to the origin on the bottom sheet, which gives \( -\tau_{1j}/2 \) plus a correction of \( \mathcal{O}(1) \), and then completing the integration from this origin to the point \( p \). (We choose the lower sheet for later simplicity.) Thus,

\[
\omega_j(p) - \frac{1}{2} \tau_{1j} + \xi_j(p),
\]
where $\xi_j(p) = O(1)$ for $p$ in the exterior region. Using this in the integrals above gives

$$
\int_{a_1} \omega_j \, d\omega_1 = \int_{a_1} \left[ -\tau_{1j} + \xi_j(p) \right] \, d\omega_1 = \frac{1}{2} \tau_{1j} + O(1), \quad j \neq 1.
$$

The remaining expression in eq. (36) involves integrals of the normalized differentials from $p_0$ to the initial positions of the auxiliary variables, $\mu_{m0}$. Since these integrals remain entirely within the internal region the results are finite.

Thus, in summary,

$$
K_j = \begin{cases} 
-\frac{1}{2\pi} \left( k_0 x + k_0^3 t \right) + \frac{1}{2} \tau_{1j} + \delta_1 + O(\epsilon), & j = 1, \\
\frac{1}{2} \tau_{jj} + \frac{1}{2} \tau_{1j} + \delta_j + O(\epsilon), & j = 2, 3, \ldots, N,
\end{cases}
$$

where all of the $O(1)$ terms discussed above have been lumped into the constants $\delta_j$.

The argument of the KdV $\Theta$-function [14] is

$$
\omega_j(\infty) - K_j = \frac{1}{2} \tau_{1j} - K_j + O(1), \quad j = 1, 2, \ldots, N.
$$

Using the results quoted above for $K_j$, the argument of the KdV $\Theta$-function becomes

$$
\omega_j(\infty) - K_j = \begin{cases} 
\frac{k_0}{2\pi} x + \frac{k_0^3}{2\pi} t + \delta_1 + O(\epsilon), & j = 1, \\
-\frac{1}{2} \tau_{jj} + \delta_j + O(\epsilon), & j = 2, 3, \ldots, N.
\end{cases}
$$

By carrying out the evaluation of this relation to $O(\epsilon^2)$ and using the definitions given in eqs. (5) for the long space and time variables, $x'$ and $t'$, the nonlinear wavenumber and frequency shifts can be computed (see appendix 1, where the calculation is carried out to first order, as an example). The result can be written as

$$
\omega_j(\infty) - K_j = \begin{cases} 
\frac{\theta}{2\pi} + \frac{1}{2\pi} \left[ k_0 x' + \nu_0 t' + \eta_0 \right], & j = 1, \\
\frac{1}{2\pi} \left[ k_{j-1} x' + \nu_{j-1} t' + \eta_{j-1} \right], & j = 2, 3, \ldots, N.
\end{cases}
$$

Here $\theta$ is the carrier oscillation, the $\kappa$'s and $\nu$'s will be the NLS wavenumbers and frequencies, and the $\eta$'s are constant phases. These quantities are defined by the relations above.

Since $\tau$ is singular many of the terms in the $\Theta$-series (eq. (32)) become small as $\epsilon \to 0$. The ordering of the terms is determined by the $\tau$ matrix as follows (using $\tau_{jk} = r_{kj}$ [14]):

$$
m \cdot \tau \cdot m = \sum_{j,k=1}^{N} m_j m_k r_{jk} = m_1 \tau_{11} + 2 m_1 \sum_{k=2}^{N} m_k \tau_{1k} + \sum_{j,k=2}^{N} m_j m_k \tau_{jk}
\begin{align*}
&= m_1 \left( \frac{-i}{\pi} \ln \epsilon + O(1) \right) + 2 m_1 \sum_{k=2}^{N} m_k (r_{k-1}^+ - r_{k-1}^-) + m' \cdot \tau_{\text{NLS}} \cdot m',
\end{align*}
$$
from which, using eq. (32), the $\Theta$-series can be ordered as

$$\Theta_{\text{KdV}}(x, t|\tau) = \Theta^0(x', t'|\tau_{\text{NLS}}) + \varepsilon \left[ e^{i\theta} + i\kappa x' + iv_0 t' \Theta^1(x', t'|\tau_{\text{NLS}}) + \text{c.c.} \right] + \mathcal{O}(\varepsilon^2).$$

The scaling is fixed by the value of $m_1$, namely the first entry in the summation vector $m$. The first term comes from the $m_1 = 0$ contributions (the phase $\eta_0$ has been suppressed for compactness), the second comes from $m_1 = \pm 1$, the last expression gives the generic form of the remaining terms. The function $\Theta^1$ is an $N - 1$ dimensional $\Theta$-function. It differs from $\Theta^0$ (which is also an $N - 1$ dimensional $\Theta$-function) by a shift in its argument by $r_1^+ - r_1^-$. Notice that both $\Theta$-functions depend only on the long space–time scales.

We have written the above expansion assuming that the $m_1 = -1$ contribution to the $\Theta$-sum is the complex conjugate of the $m_1 = +1$ contribution. This must be true if the KdV solution, $u(x, t)$, is to be a real function (see, for example, eq. (3) and the discussion immediately thereafter). However, if we, for the moment, imagine that we were given the two $\Theta$-functions generated by the above expansion and were asked to prove that they were complex conjugates of one another, this would be a difficult task. It can be shown that they would not be conjugate unless their $N - 1$ phase constants obeyed a set of constraints [7, 10, 11]. Thus, the above calculation shows that the 'reality' constraint which appears somewhat mysteriously in the periodic NLS theory has an underlying physical interpretation: it is needed to insure that the NLS solution is the envelope of a real (as opposed to complex) wavetrain.

The next step is to expand the logarithm in eq. (31). Using the Taylor expansion of $\ln(1 + a)$ for $a \ll 1$, we find that the logarithm of the KdV $\Theta$-function can be expanded as

$$\ln \left[ \Theta^0 + \varepsilon \left( e^{i\theta} + i\kappa x' + iv_0 t' \Theta^1 + \text{c.c.} \right) + \cdots \right] = \ln \Theta^0 + \ln \left[ 1 + \varepsilon \left( e^{i\theta} + i\kappa x' \Theta^1 \Theta^0 + \text{c.c.} \right) + \cdots \right]$$

$$= \ln \Theta^0 + \varepsilon \left( e^{i\theta} + i\kappa x' \Theta^1 \Theta^0 + \text{c.c.} \right) + \cdots.$$

The nonlinear wavenumber and frequency shift, $(\kappa_0 x' + v_0 t')$, have been written more compactly as $\varphi$. Only the lowest order terms are shown, obviously all of the terms in the series can be ordered in this way. The next step is to differentiate this twice with respect to $x$ (using eqs. (7) and the fact that the NLS $\Theta$-functions depend only on $x'$ and $t'$):

$$\frac{\partial^2}{\partial x^2} \ln \Theta_{\text{KdV}} = \varepsilon^2 \frac{\partial^2}{\partial x' \partial t'} \ln \Theta^0 - \varepsilon k_0^2 \left[ e^{i\theta} \left( e^{i\varphi} \Theta^1 \Theta^0 \right) + \text{c.c.} \right] + \cdots.$$

The exterior constant $\Lambda$ (see eq. (37)) shrinks to $\mathcal{O}(\varepsilon^2)$. To show this requires some algebra which we present, for completeness' sake, in appendix 2.

Thus the NLS envelope (the coefficient of $e^{i\theta}$) is given by

$$q(x', t') \propto e^{i\varphi} \Theta^1 \Theta^0$$

as required [11, 15].

6. Summary and conclusions

The NLS limit of the KdV equation has been considered in detail. By an extension of techniques developed in ref. [3] explicit connections between all of the major elements of the theory of periodic KdV
wavetrains and that of its envelope, described by the NLS, have been derived. The principle results include:

1) A derivation that the NLS squared eigenfunction, \( f \) (defined in the text), is the average of the KdV squared eigenfunction, \( R \). The NLS squared eigenfunctions \( g \) and \( h \) are also related to suitable averages of \( R \). This gives an appealing physical interpretation of the NLS auxiliary variables as appropriate averages of the KdV variables.

2) The derivation of the infinite hierarchy of NLS conservation laws from the KdV hierarchy.

3) The derivation of the NLS \( (N-1) \)-band solutions from the KdV \( N \)-band solutions.

4) An elucidation of the physical origin of the NLS 'reality' constraint. If the original KdV wavetrain is real (as it generally is for physical applications) then the phase constants of the NLS \( \Theta \)-functions related to them all satisfy the necessary constraint. This may be thought of as an independent means of solving of the NLS reality constraint, distinct from those presented in refs. (7), (10), and (11).

Thus it is possible to extract each element of the periodic NLS theory as a singular limit of the KdV theory.

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Appendix 1

*Proof of the assertion that the change of variables given by eq. (5) removes any first order time dependence in the arguments of the NLS \( \Theta \)-functions.*

The techniques developed in this section are also useful for computing the higher order corrections to any of the results stated in the text.

Consider eq. (39) from the text. The wavenumbers \( k_j \) and the elements of the \( \mathbf{C} \)-matrix can be expanded as a Taylor series in \( \epsilon \):

\[
k_j = k_j^{(0)} + \epsilon k_j^{(1)} + \mathcal{O}(\epsilon^2)
\]

\[
= 2\pi ik_j^{(0)}C_{Nj}^{(0)} + 2\pi ik_j^{(1)}C_{Nj}^{(1)} + \mathcal{O}(\epsilon^2).
\]

The superscripts denote the order of the expansion. Therefore,

\[
k_j^{(1)} = 2\pi ik_j^{(1)}.
\]  \( (A1.1) \)
The frequencies, $\Omega_j$, can also be Taylor-expanded (see eq. (40)), with the result

$$\Omega_j^{(1)} = 2\pi i k_0 \left[ 4C_{N-1,j}^{(0)} + k_0^2 C_{Nj}^{(1)} + 2 \sum_{n=1}^{2N} \lambda_n C_{Nj}^{(0)} \right].$$

(A1.2)

Thus, the first order space–time dependence is given by

$$\epsilon \left( k_j^{(1)} x + \Omega_j^{(1)} t \right) = 2\pi i k_0 \left( C_{Nj}^{(1)} x + \left[ 4C_{N-1,j}^{(0)} + k_0^2 C_{Nj}^{(1)} + 2 \sum_{n=1}^{2N} \lambda_n C_{Nj}^{(0)} \right] t \right).$$

The goal of the following calculation is to show that, by using the definition of $x'$ (eq. (5)), it is possible to rewrite the expression in curly brackets as $\alpha_j x'$ for some suitable $\alpha_j$. This can be done if and only if the expression in square brackets obeys

$$4C_{N-1,j}^{(0)} + k_0^2 C_{Nj}^{(1)} + 2\lambda_n C_{Nj}^{(0)} = 3k_0^2 C_{Nj}^{(1)}.$$  

(A1.3)

To show that this identity is satisfied requires an evaluation of $C_{Nj}^{(1)}$. We start by evaluating the first order terms in the $A$-matrix. Taylor-expanding the differentials, $dU_j$, in both the interior and the exterior regions we find:

**Exterior region:**

$$dU_N = \frac{k_0}{2} \frac{d(E^2)}{\sqrt{E^2}} \left[ 1 + \frac{\epsilon}{2} \sum_{n=1}^{2N} \lambda_n \left( \frac{1}{E^2 - \frac{k_0^2}{4}} + O(\epsilon^2) \right) \right],$$

(A1.4a)

$$dU_{N-1} = \epsilon \frac{k_0}{2} \frac{d(E^2)}{\sqrt{E^2}} \frac{1}{\left( E^2 - \frac{k_0^2}{4} \right)^2} + O(\epsilon^2),$$

(A1.4b)

$$dU_j = O(\epsilon^2), \quad j = 1, 2, \ldots, N - 2.$$  

(A1.4c)

**Interior region:**

$$dU_j = \left( 1 - \frac{2\epsilon\lambda}{k_0^2} \right) \frac{-1}{\sqrt{\prod_{n=1}^{2N} (\lambda - \lambda_n)}} + O(\epsilon^2),$$

which can be written as

$$dU_j = dU_j^{(0)} - \frac{2\epsilon}{k_0^2} dU_{j+1}^{(0)} + O(\epsilon^2), \quad j = 1, 2, \ldots, N - 1.$$  

(A1.5)

The first order correction to $dU_N$ ($dU_N^{(1)}$) in the interior region is a new singular differential, and therefore cannot be written in terms of differentials which have been encountered previously.
Using eqs. (A1.4) and (A1.5) it is possible to evaluate the first order terms in the $A$-matrix. Since $a_i$-cycle lies entirely in the exterior region, eqs. (A1.4a–c) must be used to find the first order correction to the first row of the $A$-matrix:

$$A_{1j} = A_{1j}^{(0)} + \varepsilon A_{1j}^{(1)} + \mathcal{O}(\varepsilon^2)$$

$$= \int_{a_i} dU_j = \int_{a_i} dU_j^{(0)} + \varepsilon \int_{a_i} dU_j^{(1)} + \mathcal{O}(\varepsilon^2).$$

Term by term integration is justified since the Taylor series of $dU_j$ converges within a given scaling region. This leads to

$$A_{1N}^{(0)} = -2\pi i, \quad A_{1N}^{(1)} = \frac{2\pi i}{k_0^2} \sum_{n=1}^{2N} \lambda_n,$$

(A1.6a)

$$A_{1,N-1}^{(1)} = \frac{4\pi i}{k_0^2} = -\frac{2}{k_0^2} A_{1N}^{(0)},$$

(A1.6b)

$$A_{1j}^{(1)} = 0, \quad j = 1, 2, \ldots, N - 2.$$  

(A1.6c)

Since all of the other $a$-cycles lie entirely in the interior region, eq. (A1.5) must be used to compute all but the first row of $A$, with the result

$$A_{mj} = A_{mj}^{(0)} - \frac{2\varepsilon}{k_0^2} A_{m,j+1}^{(0)} + \mathcal{O}(\varepsilon^2), \quad m = 2, 3, \ldots, N, \quad j = 1, 2, \ldots, N - 1,$$

(A1.7a)

$$A_{mN} = A_{mN}^{(0)} + \varepsilon v_m + \mathcal{O}(\varepsilon^2), \quad m = 2, 3, \ldots, N.$$  

(A1.7b)

Notice that $m$ runs from 2 to $N$. Notice, also, that the first order terms in eq. (A1.7a) have precisely the same form as that of eq. (A1.6b). The $N - 1$ constants $v_m$ ($m = 2, 3, \ldots, N$) are defined by eq. (A1.7b). They should be thought of as components of an $N$-dimensional vector, $v$, where the first entry, $v_1$ is given by eq. (A1.6a). Eqs. (A1.6) and (A1.7) can be summarized as

$$A_{mj}^{(1)} = -\frac{2}{k_0^2} A_{m,j+1}^{(0)}, \quad m = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, N - 1,$$

(A1.8a)

$$A_{mN}^{(1)} = v_m, \quad m = 1, 2, \ldots, N.$$  

(A1.8b)

The first order matrix of $a$-periods can thus be written as a 'shift' operator acting on the zeroth order
matrix: $A^{(1)} = A^{(0)}S$ where the 'shift' matrix, $S$, has the form

$$
S = -\left(\frac{2}{k_0^2}\right) \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
$$

The vector $w$ satisfies

$$
\left(-\frac{2}{k_0^2}\right) A^{(0)} w = v,
$$

which can be written as

$$
w = \left(-\frac{k_0^2}{2}\right) C^{(0)} v.
$$

We shall only need $w_N$:

$$
w_N = \left(-\frac{k_0^2}{2}\right) \sum_{m=1}^{N} C^{(0)}_{nm} v_m = \frac{1}{2} \sum_{n=1}^{2N} \lambda_n.
$$

Using the results above it is possible to get a compact representation of the first order correction to the $C$-matrix (and therefore the wavenumbers and frequencies) as follows:

$$
A = A^{(0)} + \varepsilon A^{(1)} + O(\varepsilon^2) = A^{(0)} + \varepsilon A^{(0)} S + O(\varepsilon^2)
$$

$$
= A^{(0)} (1 + \varepsilon S) + O(\varepsilon^2).
$$

Here, $1$ is the $N \times N$ identity matrix. Inverting this to first order in $\varepsilon$ gives

$$
C = (1 - \varepsilon S) C^{(0)} + O(\varepsilon^2)
$$

and thus,

$$
C^{(1)} = -SC^{(0)}.
$$

This is the needed result. Using the above stated form of $S$ we can now write

$$
C_{Nj}^{(1)} = \frac{1}{k_0^2} \left[ C_{N-1,j}^{(0)} + \sum_{n} \lambda_n C^{(0)}_{nj} \right].
$$

The reader should compare this with eq. (A1.3). This result implies that the change of variables from $(x, t)$
to \((x', t')\) given by eq. (5) in the text removes any time dependence at first order in the arguments of the NLS \(\Theta\)-functions.

Appendix 2

Proof that the exterior constant \(A\), defined by eq. (37), is of \(\mathcal{O}(\epsilon^3)\), as required by eq. (6).

Consider the integrals

\[
\sum_{n=1}^{N} \int_{a_n} E^2 \, d\omega_n.
\]

Since the differentials are normalized, this can be rewritten as

\[
\sum_{n=1}^{N} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, d\omega_n + N \frac{k_0^2}{4}.
\]

Now focus on the integrals

\[
\sum_{n=1}^{N} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, d\omega_k = \sum_{n=1}^{N} \sum_{m=1}^{N} C_{mn} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, dU_m.
\]

From the definition of \(dU_m\) (eq. (38)) it is possible to see that, for \(m\) less than \(N\),

\[
\left( E^2 - \frac{k_0^2}{4} \right) \, dU_m = \epsilon \, dU_{m+1}, \quad m = 1, 2, \ldots, N - 1,
\]

and thus,

\[
\sum_{n=1}^{N} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, d\omega_n = \epsilon \sum_{n=1}^{N} \sum_{m=1}^{N-1} C_{mn} \int_{a_n} dU_{m+1} + \sum_{n=1}^{N} C_{NN} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, dU_N \\
= \epsilon \sum_{m=1}^{N-1} \sum_{n=1}^{N} C_{mn} A_{nm+1} + \sum_{n=1}^{N} C_{NN} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, dU_N.
\]

Since the \(C\)-matrix is the inverse of the \(A\)-matrix, the summation over \(n\) in the first expression gives \(\delta_{m+1 m} = 0\). Therefore,

\[
\sum_{n=1}^{N} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, d\omega_n = \sum_{n=1}^{N} C_{NN} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) \, dU_N.
\]

The relation above is true for general \(N\)-band solutions of the KdV equation. Now we use the fact that we are interested in small amplitude narrow banded solutions. The integrals for \(n \neq 1\) are within the interior region. In this region, \(dU_N\) is \(\mathcal{O}(1)\), but the other factor in the integrand is of \(\mathcal{O}(\epsilon)\). The constants, \(C_{NN}\), are
all of $\mathcal{O}(\varepsilon)$ or smaller unless $n = 1$:

$$
\sum_{n=1}^{N} \int_{a_n} \left( E^2 - \frac{k_0^2}{4} \right) d\omega_n = C_{N1} \int_{a_1} \left( E^2 - \frac{k_0^2}{4} \right) dU_N + \mathcal{O}(\varepsilon^2).
$$

Since the cycle $a_1$ lies entirely in the exterior region, expand the integrand there as a Taylor series in $\varepsilon$ to find

$$
\left( E^2 - \frac{k_0^2}{4} \right) dU_N = \frac{k_0}{2} \left( 1 + \left[ \frac{\varepsilon}{2} \sum_{m=1}^{2N} \lambda_m \right] \frac{1}{\left( E^2 - \frac{k_0^2}{4} \right)} \right) \frac{d(E^2)}{\sqrt{E^2}} + \mathcal{O}(\varepsilon^2).
$$

The $a_1$-cycle can be deformed so that it orbits the resonance in a clockwise manner, leading to

$$
\int_{a_1} \left( E^2 - \frac{k_0^2}{4} \right) dU_N = -2\pi i \frac{\varepsilon}{2} \sum_{m=1}^{2N} \lambda_m + \mathcal{O}(\varepsilon^2).
$$

Using the fact that $C_{N1} = -1/2\pi i + \mathcal{O}(\varepsilon)$ we get, finally,

$$
2 \sum_{n=1}^{N} \int_{a_n} E^2 d\omega_n = 2N \frac{k_0^2}{4} + \varepsilon \sum_{m=1}^{2N} \lambda_m + \mathcal{O}(\varepsilon^2).
$$

The proof is completed by inserting this expansion into eq. (37) and using eqs. (27). The $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ terms in $\Lambda$ cancel identically, implying that the uniform background of the KdV solution is $\mathcal{O}(\varepsilon^2)$, as required by eq. (6).

References