Chapter 10

Past exam papers

Plasma Physics C17 1993: Final Examination

Attempt **four** questions. All six are of equal value. The best four marks will be considered, but candidates are discouraged from answering all six questions because it is unlikely that there will be sufficient time.

Show all working and state and justify relevant assumptions briefly.

Question 1 (10 marks): (answer **both** parts, illustrate with appropriate equations)

a List three quantitative criteria for a plasma and explain each in a few lines.

b Describe **three** out of four of the following phenomena, and their relation to adiabatic invariants.

i adiabatic compression

ii Fermi acceleration

iii ion cyclotron heating

iv transit time magnetic pumping

Question 2 (10 marks): Discuss **one of** the following: answers are not restricted to material from the specialist lectures

a Plasma fusion and magnetic confinement devices

b Extraterrestrial plasma and plasma phenomena

c Plasma diagnostics using laser radiation

d Describe the process of electrical breakdown between electrodes in gas at pressures near 1 Torr, including relevant equations. Explain why secondary emission is important, and at which electrode.

Question 3 (10 marks): Derive an expression for the Debye length in planar (1-D slab) geometry taking into account both T_{e} and T_{i}. Assume time scales long enough so that both species have equilibrium (Maxwellian) distributions. Discuss the validity of your treatment of the ions.

Question 4 (10 marks):

(5/10) aUsing the single fluid MHD equations and Fick's law ($\Gamma = -D\nabla n$), obtain the coefficient of diffusion perpendicular to magnetic field lines.

(2/10) b Explain how and why this diffusion depends on plasma resistivity.

(3/10) c In a few sentences, explain neoclassical diffusion qualitatively with the aid of a few sketches.

Question 5 (10 marks): Consider a high frequency plane **transverse electromagnetic** wave in an unmagnetized plasma. ($B_0 = 0$)

a From the two fluid electron equation, show that

$$\boldsymbol{j}_1 = \frac{ie^2 n_0 \boldsymbol{E}_1}{m\omega}$$

b and continue, by considering Maxwell's equations, to derive the dispersion relation.

c Calculate the group velocity and sketch both the group and phase velocities on graphs with labels and numerical scales for $n_\mathrm{e} = 1 \times 10^{18} \pm 3$.

Question 6 (10 marks): Consider the plasma sheath region near a wall in planar geometry.

a Write down Poisson's equation including both electron and ion terms, explaining and justifying your assumptions.

b Justify under what conditions the electron contribution in (a) can be ignored, and solve the equation for those conditions to obtain a relation between V (or Φ) the sheath width d, and J.

Plasma Physics C17 1994: Final Examination

Attempt **four** questions. All are of equal value. Candidates are **discouraged** from answering all six questions because it is unlikely that there will be sufficient time.

Show all working and state and justify relevant assumptions briefly.

Question 1 (10 marks): (answer **both** parts, illustrate with appropriate equations)

a List three quantitative criteria for a plasma and explain each in a few lines.

b Describe **three** out of five of the following phenomena.

i Debye shielding.

ii Boltzmann's relation for electrons.

iii Energy transfer from a plasma to a conducting wall.

iv Mechanisms for plasma generation, confinement, and loss.

v Discuss an example of a plasma heating scheme that relies on conservation of an adiabatic invariant, and one that relies on the breaking of an adiabatic invariant.

Question 2 (10 marks): Discuss **one of** the following: answers are not restricted to material from the specialist lectures

a Plasma fusion and magnetic confinement devices

b Low temperature plasma, and its use in materials processing.

c Plasma diagnostics - measurements of density, temperature etc.

d Discuss Coulomb collisions, explaining the basic properties of the collisions, the range of the interaction, the effect on plasma resistivity, runaway electrons, indicating scaling (e.g. with n, T etc.) where appropriate.

Question 3 (10 marks):

a Show that the electrical resistivity of a fully ionized plasma can be expressed in the form

$$\eta = \frac{\nu_{\mathrm{ei}} m_{\mathrm{e}}}{n_{\mathrm{e}} e^2}$$

where ν_{ei} is the electron-ion collision frequency. **Do not** attempt to find an expression for ν_{ei} or derive Coulomb scattering!

b Explain why η is *almost independent* of n_e even though n_e appears in the equation for η. Why "almost independent"?

c Explain why η decreases as the electron temperature T_{e} increases.

Question 4 (10 marks):

a Sketch the motion of ions and electrons in a magnetic field $\boldsymbol{B}$, directed out of the page, when $\boldsymbol{B}$ increases in a direction vertically up the page as shown below. Label the sketches with appropriate dimensions.

b An electron moves from point P to point Q in a magnetic mirror. At point P, the magnetic field is 0.5 Tesla, the perpendicular energy ($\frac{1}{2}mv_\perp^2 = 200\,\mathrm{eV}$), and the parallel energy ($\frac{1}{2}mv_\parallel^2 = 600\,\mathrm{eV}$). What is the magnetic field at point Q if the electron is reflected at this point?

c Obtain and expression for the curvature drift of an electron travelling at velocity $v_\parallel$ along a circular magnetic field line of radius R_c. How are curvature drift and grad-B drift related?

Question 5 (10 marks): For the electromagnetic mode with $\boldsymbol{E}_1 \perp \boldsymbol{B}_0$ and $\boldsymbol{k} \parallel \boldsymbol{B}_0$ it can be shown that

$$E_x(\omega^2 - c^2k^2 - \alpha) + E_y i\alpha\omega_c/\omega = 0$$

$$E_y(\omega^2 - c^2k^2 - \alpha) - E_x i\alpha\omega_c/\omega = 0$$

(Equn 6.24 in notes), where

$$\alpha = \frac{\omega_{\mathrm{p}}^2}{1 - \omega_{\mathrm{c}}^2/\omega^2}$$

(3/10) a Continue to obtain the dispersion relation for this wave (in the form given in the formula handout for the exam)

(4/10) b show (briefly) that the modes are right and left hand circularly polarized, and identify which is which.

(3/10) c Define and obtain the cutoff frequencies.

Question 6 (10 marks):

aUsing the single fluid MHD equations and Fick's law ($\Gamma = -D\nabla n$), obtain the coefficient of diffusion perpendicular to magnetic field lines.

b Use the single fluid MHD Equation of motion, and the mass continuity equation to calculate the phase velocity of an ion-acoustic wave in an unmagnetized plasma with $T_{\mathrm{e}} >> T_{\mathrm{i}}$.

spare,10) The plasma potential is usually a few KT_e above the potential of its (conducting) container. Explain, and, justifying your assumptions, obtain an approximate relation for the difference in potential $\Phi_p - \Phi_w$. (DON'T derive the Langmuir-Child law.)

Plasma Physics C17 1996: Final Examination

Attempt four questions. All are of equal value. Candidates are discouraged from answering all six questions as it is unlikely there will be sufficient time.

Show all working and state and justify relevant assumptions.

Question 1 (10 marks)

a Describe oscillation at the plasma frequency and the Debye length and the relation between them.

b Discuss distribution functions, the Boltzmann equation and the relationship between their respective zeroth, first and second order velocity moments.

c Using the equilibrium fluid equation of motion (set the convective derivative to zero) and in the absence of collisions, show that the particle number density distribution for Maxwellian electrons at temperature T_e is described by the Boltzmann relation

$$n_\mathrm{e} = n_0 \exp\left(e\phi/kT_\mathrm{e}\right)$$

where $\boldsymbol{E} = -\nabla\phi$ is the plasma electric field. What is the physical interpretation of this formula. Use pictures to illustrate.

Question 2 (10 marks)

a Show that the resistivity of a very weakly ionized plasma can be expressed in the form

$$\eta = 1/\sigma = \frac{\nu_\mathrm{en} m_\mathrm{e}}{n_\mathrm{e} e^2}$$

where ν_en is the electron-neutral collision frequency.

b Assuming that the electron-neutral collision cross section is independent of particle velocity, calculate the scaling of the resistivity with T_e and n_e for constant neutral density and compare and contrast with those scalings for a fully ionized plasma.

Question 3 (10 marks)

a By considering the circular motion of an ion in a magnetic field B as a current loop of magnetic moment $\mu = IA$ where I is the circulating ion current and A is the area of the orbit, show that the diamagnetic flux associated with the particle motion is proportional to the particle perpendicular kinetic energy. Discuss how can this effect be used to estimate the plasma internal perpendicular thermal energy.

b Draw orbits for electrons and ions in orthogonal electric and magnetic fields for both weak and strong electric fields. Explain why there is no net current associated with the particle drifts.

c The grad B drift is in opposite directions for electrons and ions. Show, with the aid of diagrams, how this drift renders impossible plasma confinement in a purely toroidal magnetic field.

Question 4 (10 marks)

a Consider the following simplified steady state equation of motion for each species in a fluid plasma

$$0 = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p$$

where the electric and magentic fields are uniform but the number density and pressure have a gradient. Taking the cross product of this equation with $\boldsymbol{B}$ show that, besides the $\boldsymbol{E}\times\boldsymbol{B}$ drift, there is also a diamagnetic drift given by

$$\boldsymbol{u}_D = (1/n)\nabla p\times\boldsymbol{B}/(qB^2).$$

b Provide physical arguments to justify the reason for this drift. Explain if there is any motion of the particle guiding centres associated with this fluid drift and why it does not appear in the particle orbit theory.

Question 5 (10 marks)

a Using the single fluid equilibrium MHD equations and Fick's law ($\Gamma = -D\nabla n$ where $\Gamma = n\boldsymbol{u}$ is the particle flux and D the diffusion coefficient), obtain the coefficient of diffusion perpendicular to a magnetic field.

b Explain how and why this diffusion depends on plasma resistivity.

c Consider the non-equilibrium case. Ignoring gravity and Hall currents, combine the single fluid equation of motion and the Ohm's law to obtain

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} = \sigma(\boldsymbol{E}\times\boldsymbol{B}) + \sigma(\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B} - \nabla p$$

By considering $\boldsymbol{E} = 0$ and $p =$ constant, solve this equation to show that the fluid velocity perpendicular to $\boldsymbol{B}$ is given by

$$\boldsymbol{u}_\perp = \boldsymbol{u}_\perp(0)\exp\left(-t/\tau\right)$$

where τ, the characteristic time for damping of the fluid flow across the field lines is given by $\tau = \rho/(\sigma B^2)$. Comment on the scaling of τ with B and σ.

Question 6 (10 marks)

a Plot the wave phase velocity as a function of frequency for plasma waves propagating along the direction of the magnetic field $\boldsymbol{B}$, identifying cutoffs and resonances for both electromagnetic and electrostatic wave modes.

b The dispersion relation for an em wave propagating in an unmagnetized plasma is

$$v_\phi^2 = c^2/n^2 = \omega^2/k^2 = c^2/(1 - \omega_{\mathrm{pe}}^2/\omega^2).$$

For $\omega^2 \gg \omega_{\mathrm{pe}}^2$ show that the phase shift suffered by such a wave (compared with vacuum) on propagation through a plasma of length L is given by

$$\phi = -\frac{\omega}{2n_{\mathrm{cr}}c}\int_0^L n_{\mathrm{e}}\mathrm{d}\ell$$

where n_{cr} is the cutoff plasma density (at which $\omega = \omega_{\mathrm{pe}}$).

Plasma Physics C17 1997: Final Examination

Attempt four questions. All are of equal value. Candidates are discouraged from answering all six questions as it is unlikely there will be sufficient time.
Show all working and state and justify relevant assumptions.

Question 1 (10 marks)

a Assume a perturbation to charge neutrality in an unmagnetized plasma such that, under the action of the restoring Coulomb force, an oscillating electric field $E = E_0 \exp(-i\omega t)$ is established. By considering the force felt by an electron in such a field, show that an oscillating electron current $j = n_e e^2 E_0/(i\omega m_e)$ results. Using Maxwell's equation

$$\nabla \times \boldsymbol{b} = \mu_0 \left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

where $\boldsymbol{b}$ is the associated magnetic perturbation, and assuming the perturbation not to propagate (set the left side to zero), obtain an alternative expression for j. By equating these expressions, obtain a formula for the oscillation frequency ω. What is this frequency?

b Describe Debye shielding and the relationship between the plasma frequency and Debye length.

c Briefly discuss distribution functions and the Boltzmann equation. What is the relationship between their respective zeroth and first order velocity moments.

Question 2 (10 marks)
Discuss **two** of the following

a Electric breakdown. Discuss to the significance of the parameter E/p and the role of secondary emission.

b Boltzmann's relation for electrons

c Ambipolar diffusion in an unmagnetized plasma

d Faraday rotation of an electromagnetic wave traversing a magnetized plasma.

e "Frozen-in" magnetic fields and resistive diffusion.

Question 3 (10 marks)

a The Ohm's law for an unmagnetized plasma in steady state is given by $\boldsymbol{E} = \eta \boldsymbol{j}$ where η is the resistivity. Show that the resistivity of a fully ionized plasma can be expressed in the form

$$\eta = 1/\sigma = \frac{\nu_{\rm ei} m_{\rm e}}{n_{\rm e} e^2}$$

where $\nu_{\rm ei}$ is the electron-ion collision frequency. Do **not** derive the expression for the Coulomb collision frequency. With reference to the formula sheet, explain why η is "almost" independent of density and why it decreases with increasing temperature.

b In the time-varying case, and including magnetic effects, the equations of motion for the ions and electrons can be combined to give the single fluid force balance and Ohm's Laws:

$$\begin{aligned} \rho \frac{\partial \boldsymbol{u}}{\partial t} &= \boldsymbol{j} \times \boldsymbol{B} \\ \boldsymbol{E} - \eta \boldsymbol{j} &= -\boldsymbol{u} \times \boldsymbol{B} + \frac{m_{\rm e}}{n_{\rm e} e^2} \frac{\partial \boldsymbol{j}}{\partial t} + \frac{1}{n_{\rm e} e} \boldsymbol{j} \times \boldsymbol{B} \end{aligned} \tag{10.1}$$

In these expressions, we have ignored the plasma kinetic pressure gradient terms. Assuming a time variation of the form $\partial/\partial t = -i\omega$ compare the magnitudes of the various terms on the right side of the Ohm's law as a function of frequency. Which terms dominate for $\omega \ll \omega_{\rm ci}$. What about for $\omega \gg \omega_{\rm ce}$? (HINT: you will need to use the force equation for $\boldsymbol{u}$).

c Combine the steady state Ohm's law (neglecting Hall current)

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j}$$

and the force balance equation to obtain the the fluid flow velocity perpendicular to the magnetic field

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\eta_\perp}{B^2} \nabla p.$$

Use Fick's diffusion law $\boldsymbol{\Gamma}_\perp = D_\perp \nabla n$ to obtain an expression for the classical perpendicular diffusion coefficient for a fully ionized plasma and discuss its scaling with temperature. If the Hall current were retained in the Ohm's law, which additional component of the fluid flow would have been obtained?

Question 4 (10 marks)

a Draw orbits for electrons and ions in orthogonal electric and magnetic fields for both weak and strong electric fields. Explain why there is no net current associated with the particle drifts.

b The grad B drift is in opposite directions for electrons and ions. Show, with the aid of diagrams, how this drift renders impossible plasma confinement in a purely toroidal magnetic field.

c For slow time variations $\omega \ll \omega_{\mathrm{ci}}$, the polarization drift velocity for ions and electrons respectively is given by

$$\boldsymbol{v}_p = \pm \frac{1}{\omega_c B} \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}$$

where ω_c is the associated cycoltron frequency.

(i) Explain the origin of this effect

(ii) Calculate the current which flows as a result of a time varying electric field.

(iii) Identifying this polarization current as equivalent to the electric displacement current density for solid dielectrics

$$\boldsymbol{j}_D = \varepsilon_0 \varepsilon_r \frac{\partial \boldsymbol{E}}{\partial t}$$

where ε_r is the relative permittivity, express the relative permittivity for the plasma in terms of the Alfven wave speed.

Question 5 (10 marks)

Consider a particle that is gyrating in a circular orbit in a substantially uniform magnetic field.

a Obtain an expression for the radius of the Larmor orbit of the particle in terms of the orbital magnetic moment $\mu = mv_\perp^2/(2B)$.

b Calculate the magnetic flux linked by this orbit as B is slowly changed? Comment on its dependence on B.

c Evaluate the volume of magnetic field that has the same energy as the kinetic energy of the particle. Consider the cylinder that has this volume and has the same radius as the orbit of the particle. What is the height of this cylinder? Do you recognize this expression?

d Suppose that a mirror field increases slowly with time. What will happen to a particle that is confined between the magnetic mirrors?

e Why is the adiabatic invariant broken for time variations comparable with or faster than the cyclotron frequency?

Question 6 (10 marks)

a Plot the wave phase velocity as a function of frequency for electromagnetic plasma waves propagating along the direction of the magnetic field $\boldsymbol{B}$, identifying cutoffs and resonances.

b Radio signals from pulsars pass through the interstellar medium that contains free electrons. Assume that the dispersion relation for an em wave propagating in the interstellar plasma is

$$v_\phi^2 = \omega^2/k^2 = c^2/(1 - \omega_{\rm pe}^2/\omega^2).$$

(i) What is the plasma frequency if the mean interstellar electron density is $n_{\rm e} = 10^{4.5}\ \mathrm{m}^{-3}$?

(ii) Show that the wave group velocity is given by $v_g = c^2/v_\phi$.

(iii) Assuming $\omega_{\rm pe}^2 \ll \omega^2$ show that the arrival time $t(\nu)$ of a signal will be a function of frequency of the form

$$t(\nu) = D\nu^{-2} + \text{constant}$$

where ν is the frequency in Hz and the "dispersion coefficient" D is expressible as

$$D = C \int n_{\rm e} \mathrm{d}\ell$$

where the integral represents the integral of the electron density along the propagation path of the radio signal.

(iv) Find the coefficient C

(v) For a particular pulsar it is found that the signal at 100 MHz arrives 2 seconds later than the signal at 200 MHz. What is the value of D for that pulsar? Given $n_{\rm e}$ as in part (i), what is the distance to the pulsar?

(vi) What complicating factors are neglected in deriving the above simple expression for the time delay as a function of frequency?

Plasma Physics C17 1998: Final Examination

Attempt four questions. All are of equal value. Only the first four answers will be marked. Nominal time allowed is 2 hours.
Show all working and state and justify relevant assumptions.

Question 1 (10 marks)

Attempt **three** of the following. Answers for each should require at most half a page.

(a) Show that the pressure for a Maxwellian electron gas is given by $p = nk_{\mathrm{B}}T_{\mathrm{e}}$.

(b) Describe electric breakdown with reference to the parameter E/p and the role of secondary emission.

(c) Discuss the physics of Landau damping of electron plasma waves.

(d) Discuss ambipolar diffusion in an unmagnetized plasma

(e) In terms of the magnetic Reynold's number, explain "Frozen-in" magnetic fields and resistive diffusion.

(f) Describe the Boltzmann equation and its relation to distribution functions and their moments.

(g) Describe Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2 (10 marks)

(a) The distance between electrons in a plasma is of order $n_{\mathrm{e}}^{-1/3}$. Show that the potential energy of electrons that are this close is much less than their kinetic energy provided $n_{\mathrm{e}}\lambda_{\mathrm{D}}^3 >> 1$. What is the significance of this condition?

(b) Use the parallel component (parallel to $\boldsymbol{B}$) of the equilibrium equation of motion for electrons in the absence of collisions to show that the number density for Maxwellian electrons at temperature T_{e} in an electric potential ϕ is given by the Boltzmann relation:

$$n_{\mathrm{e}} = n_0 \exp(e\phi/k_{\mathrm{B}}T_{\mathrm{e}}).$$

(c) Suppose a small varying electric potential $\phi = \phi_1 \sin kx$ is created in an initially uniform neutral plasma ($e\phi_1 \ll k_{\mathrm{B}}T_{\mathrm{e}}$). Show that the electrons will come to equilibrium with $n_{\mathrm{e}}(x) = n_0 + n_{\mathrm{e1}} \sin kx$ where $n_{\mathrm{e1}}/n_0 = e\phi_1/k_{\mathrm{B}}T_{\mathrm{e}} \ll 1$. Using Poisson's equation, show that the ion density will be given by $n_{\mathrm{i}}(x) = n_0 + n_{\mathrm{i1}} \sin kx$ with $(n_{\mathrm{i1}} - n_{\mathrm{e1}})/n_{\mathrm{e1}} = k^2\lambda_{\mathrm{D}}^2$. Explain this result.

Question 3 (10 marks)

(a) Using the collisionless equation of motion for a speces of charge q in a plasma with uniform electric and magnetic fields, show that the particles have drift velocity

$$\boldsymbol{v}_D = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} - \frac{\nabla p\times\boldsymbol{B}}{qnB^2}.$$

(b) With the aid of diagrams, show the origin of the E/B drift and explain why it is independent of species charge and mass.

(c) Show that for a neutral plasma consisting of electrons and singly charged ions, the diamagnetic drift results in a current flow $j_D = (\boldsymbol{B}\times\nabla p)/B^2$. Explain how this current, which is not due to a guiding centre drift can arise.

Question 4 (10 marks) Consider a plasma cylinder of radius a with uniform axial vacuum magnetic field B_0. Assume the plasma has a parabolic radial pressure profile $p = p_0(1 - r^2/a^2)$.

(a) What is maximum value of p_0?

(b) Using this value of p_0 and Ampere's law, obtain an expression for the magnetic field $B(r)$ and plot it on a graph for $r < a$ and $r > a$.

(c) What is the diamagnetic current density $j_D(r)$?

(d) Obtain an expression for the associated ∇B and curvature drifts.

(e) Show that $\mid v_{\nabla B}(r) \mid / \mid v_D(r) \mid$ is in the ratio of the kinetic and magnetic pressures.

Question 5 (10 marks) The dispersion relation for low frequency magnetohydrodynamic waves in a magnetized plasma was derived in lectures as

$$-\omega^2\boldsymbol{u}_1 + (V_S^2 + V_A^2)(\boldsymbol{k}.\boldsymbol{u}_1)\boldsymbol{k} + (\boldsymbol{k}.\boldsymbol{V}_A)[(\boldsymbol{k}.\boldsymbol{V}_A)\boldsymbol{u}_1 - (\boldsymbol{V}_A.\boldsymbol{u}_1)\boldsymbol{k} - (\boldsymbol{k}.\boldsymbol{u}_1)\boldsymbol{V}_A] = 0$$

where $\boldsymbol{u}_1$ is the perturbed fluid velocity, $\boldsymbol{k}$ is the propagataion wavevector and $\boldsymbol{V}_A = \boldsymbol{B}_0/(\mu_0\rho_0)^{1/2}$ is a velocity vector in the direction of the magnetic field with magnitude equal to the Alfvén speed and V_S is the sound speed.

(a) Derive the following dispersion relations:

$$v_\phi = \frac{\omega}{k} = (V_S^2 + V_A^2)^{1/2} \qquad \text{for } \boldsymbol{k}.\boldsymbol{V}_A = 0 \qquad (10.2)$$

$$
\begin{aligned}
v_\phi &= V_S \qquad\qquad \text{for } \boldsymbol{k} \parallel \boldsymbol{V}_A \text{ and } \boldsymbol{u}_1 \parallel \boldsymbol{V}_A \\
v_\phi &= V_A \qquad\qquad \text{for } \boldsymbol{k} \parallel \boldsymbol{V}_A \text{ and } \boldsymbol{u}_1 . \boldsymbol{V}_A = 0
\end{aligned}
\tag{10.3}
$$

and identify the wave modes.

(b) Using

$$
\begin{aligned}
\frac{\partial \boldsymbol{B}_1}{\partial t} - \nabla\times(\boldsymbol{u}_1\times\boldsymbol{B}_0) &= 0 \\
\boldsymbol{E}_1 + \boldsymbol{u}_1\times\boldsymbol{B} &= 0
\end{aligned}
$$

and assuming plane wave propagation so that $\frac{\partial}{\partial t} \to -\mathrm{i}\omega$ and $\nabla\times \to \mathrm{i}\boldsymbol{k}\times$, make a sketch showing the relation between the perturbed quantities $\boldsymbol{u}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{k}$ and $\boldsymbol{B}_0$ for wave propagation perpendicular to $\boldsymbol{B}_0$. What is the nature of this wave?

Question 6 (10 marks)

(a) Plot the wave phase velocity as a function of frequency for plasma waves propagating along the direction of the magnetic field $\boldsymbol{B}$, identifying cutoffs and resonances for both electromagnetic and electrostatic wave modes.

(b) Starting with the dispersion relation for L and R waves in the form $n^2 = S \pm D$, show that the phase velocity for both waves is given by

$$
v_\phi^2 = \frac{V_A^2}{1 + V_A^2/c^2}
$$

in the low frequency limit. HINT: You must consider both ions and electrons.

THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination 1999

PHYSICS C17
PLASMA PHYSICS

Writing period 2 hours duration
Study period 15 minutes duration
Permitted materials: Calculators

Attempt four questions. All are of equal value.
Show all working and state and justify relevant assumptions.

Question 1 (10 marks)
Attempt **three** of the following. Answers for each should require at most half a page.

(a) Discuss the relationship between moments of the particle distribution function f and moments of the Boltzmann equation. Draw a contour plot of $f(v_x, v_y)$ for an anisotropic electron velocity distribution and for a beam of electrons propagating in the x direction.

(b) Describe electric breakdown with reference to the parameter E/p and the role of secondary emission.

(c) Describe the physics of Landau damping of electron plasma waves.

(d) Discuss the role of Coulomb collisions for diffusion in a magnetized plasma.

(e) Discuss magnetic mirrors with reference to the adiabatic invariance of the orbital magnetic moment μ.

(g) Describe Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2 (10 marks)
Low frequency ion oscillations: Let n_0 be the equilibrium number density of singly charged ions and electrons and assume a one-dimensional harmonic perturbation of the form $\tilde{\phi} = \phi \exp[\mathrm{i}(kx - \omega t)]$. We assume the plasma is collisionless and that the ion temperature is small and can be neglected.

(a) Show that the perturbed ion velocity is given by

$$v_{\mathrm{i}} = (ek/m_{\mathrm{i}}\omega)\phi$$

where ϕ is the electric potential perturbation.

(b) From the equation of continuity, show that the perturbation *charge* density of the ions is obtained as $(n_0 e^2 k^2/m_{\mathrm{i}}\omega^2)\phi$.

(c) Assume that the ion oscillations are so slow that the electrons remain in a Maxwell-Boltzmann distribution. If $e\phi/k_{\mathrm{B}}T_{\mathrm{e}} \ll 1$, show that the perturbed charge density of the electrons is given by $-(n_0 e^2/k_{\mathrm{B}}T_{\mathrm{e}})\phi$.

(d) Use Poisson's equation to deduce the following dispersion relation:

$$k^2 = (n_0 e^2/m_{\mathrm{i}}\varepsilon_0\omega^2)k^2 - n_0 e^2/k_{\mathrm{B}}T_{\mathrm{e}}\varepsilon_0$$

(e) Recast the dispersion relation in the following form:

$$\omega^2 = \omega_{\mathrm{pi}}^2/(1 + 1/k^2\lambda_{\mathrm{D}}^2).$$

Discuss the low and high-k limits and compare with the Bohm-Gross dispersion relation for electron plasma waves.

Question 3 (10 marks)

(a) Using the steady-state force balance equation (ignore the convective derivative) show that the particle flux $\Gamma = n\boldsymbol{u}$ for electrons and singly charged ions in a fully ionized unmagnetized plasma is given by:

$$\Gamma_j = n\boldsymbol{u}_j = \pm\mu_j n\boldsymbol{E} - D_j\nabla n$$

with mobility $\mu = \mid q \mid /m\nu$ where ν is the electron-ion collision frequency and diffusion coefficient $D = k_{\mathrm{B}}T/m\nu$.

(b) Show that the diffusion coefficient can be expressed as $D \sim \lambda_{\mathrm{mfp}}^2/\tau$ where λ_{mfp} is the mean free path between collisions and τ is the collision time.

(c) Show that the plasma resistivity is given approximately by $\eta = m_{\mathrm{e}}\nu/ne^2$.

(d) In the presence of a magnetic field, the mean perpedicular velocity of particles across the field is given by

$$\boldsymbol{u}_\perp = \pm\mu_\perp\boldsymbol{E} - D_\perp\frac{\nabla n}{n} + \frac{\boldsymbol{u}_E + \boldsymbol{u}_D}{1 + \nu^2/\omega_{\mathrm{c}}^2}$$

with $\boldsymbol{u}_E = \boldsymbol{E}\times\boldsymbol{B}/B^2$, $\boldsymbol{u}_D = -\nabla p\times\boldsymbol{B}/qnB^2$ and where $\mu_\perp = \mu/(1+\omega_{\mathrm{c}}^2\tau^2)$ and $D_\perp = D/(1+\omega_{\mathrm{c}}^2\tau^2)$. Discuss the scaling with ν of each of the four terms in the expression for $\boldsymbol{u}_\perp$.

Question 4 (10 marks)

(a) Show that the drift speed of a charge q in a toroidal magnetic field can be written as

$$v_T = 2k_\mathrm{B}T/qBR$$

where R is the radius of curvature of the field. (Hint: Consider both gradient and curvature drifts)

(b) Compute the value of v_T for a plasma at a temperature of 10 keV, a magnetic field strength of 2 T and a major radius $R = 1$ m.

(c) Compute the time required by a charge to drift across a toroidal container of minor radius 1 m.

(d) Suppose an electric field is applied perpendicular to the plane of the torus. Describe what happens.

Question 5 (10 marks)

(a) Show that the MHD force balance equation $\nabla p = \boldsymbol{j}\times\boldsymbol{B}$ requires both $\boldsymbol{j}$ and $\boldsymbol{B}$ to lie on surfaces of constant pressure.

(b) Using Ampere's law and MHD force balance, show that

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\boldsymbol{B}.\nabla)\boldsymbol{B}$$

and discuss the meaning of the various terms.

(c) A straight current carrying plasma cylinder (linear pinch) is subject to a range of instabilities (sausage, kink etc.). These can be suppressed by providing an axial magnetic field B_z that stiffens the plasma through the additional magnetic pressure $B_z^2/2\mu_0$ and tension against bending. Consider a local constriction $\mathrm{d}r$ in the radius r of the plasma column. Assuming that the longitudinal magnetic flux Φ through the cross-section of the cylinder remains constant during the compression ($\mathrm{d}\Phi = 0$), show that the axial magnetic field strength is increased by an amount $\mathrm{d}B_z = -2B_z\mathrm{d}r/r$.

(d) Show that the internal magnetic pressure increases by an amount $\mathrm{d}p_z = B_z\mathrm{d}B_z/\mu_0 = -(2B_z^2/\mu_0)\mathrm{d}r/r$. [The last step uses the result obtained in (c)].

(e) By Ampere's law we have for the azimuthal field component $rB_\theta(r) =$ constant. Show that the change in azimuthal field strength due the compression $\mathrm{d}r$ is $\mathrm{d}B_\theta = -B_\theta\mathrm{d}r/r$ and that the associated increase in external azimuthal magnetic pressure is $\mathrm{d}p_\theta = -(B_\theta^2/\mu_0)\mathrm{d}r/r$.

(f) Show that the plasma column is stable against sausage distortion provided $B_z^2 > B_\theta^2/2$.

Question 6 (10 marks)

(a) Plot the wave phase velocity as a function of frequency for plasma waves propagating perpendicular to the magnetic field $\boldsymbol{B}$, identifying cutoffs and resonances for both ordinary and extraordinary modes.

(b) Using the matrix form of the wave dispersion relation

$$\begin{pmatrix} S - n_z^2 & -\mathrm{i}D & n_x n_z \\ \mathrm{i}D & S - n_x^2 - n_z^2 & 0 \\ n_x n_z & 0 & P - n_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

show that the polarization state for the extraordinary wave is given by

$$E_x/E_y = \mathrm{i}D/S.$$

Using a diagram, show the relative orientations of $\boldsymbol{B}$, $\boldsymbol{k}$ and $\boldsymbol{E}$ for this wave.

THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination 2000

PHYSICS C17
PLASMA PHYSICS

Writing period 2 hours duration
Study period 15 minutes duration
Permitted materials: Calculators

Attempt four questions. All are of equal value.
Show all working and state and justify relevant assumptions.

Question 1 (10 marks)
Attempt **three** of the following. Answers for each should require at most half a page.

(a) Discuss the relationship between the Boltzmann equation, the electron and ion equations of motion and the single fluid force balance euqation.

(b) Describe electric breakdown with reference to the parameter E/p and the role of secondary emission.

(c) Discuss the physical meaning of the Boltzmann relation. Use diagrams to aid your explanation.

(d) Discuss the origin of plasma diamagnetism and its implications for magnetic plasma confinement.

(e) Draw a Langmuir probe I-V characteristic indicating the saturation currents, plasma potential and floating potential. How can the characteristic be used to estimate temperature?

(f) Describe Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2 (10 marks)

(a) Consider two infinite, perfectly conducting plates A_1 and A_2 occupying the planes $y = 0$ and $y = d$ respectively. An electron enters the space between the plates through a small hole in plate A_1 with initial velocity v towards plate A_2. A potential difference V between the plates is such as to decelerate the electron. What is the minimum potential difference to prevent the electron from reaching plate A_2.

(b) Suppose the region between the plates is permeated by a uniform magnetic field B parallel to the plate surfaces (imagine it as pointing into the page). A proton appears at the surface of plate A_1 with zero initial velocity. As before, the potential V between the plates is such as to accelerate the proton towards plate A_2. What is the minimum value of the magnetic field B necessary to prevent the proton from reaching plate B? Sketch what you think the proton trajectory might look like. (HINT: Energy considerations may be useful).

Question 3 (10 marks)

(a) Using the equilibrium force balance equation for electrons (assume ions are relatively immobile) show that the conductivity of an unmagnetized plasma is given by

$$\sigma_0 = \frac{ne^2}{m_e \nu} \tag{10.4}$$

(b) What is the dependence of the conductivity on electron temperature and density in the fully ionized case?

(c) When the plasma is magnetized, the Ohm's law for a given plasma species (electrons or ions) becomes $\boldsymbol{j} = \sigma_0(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})$ where $\boldsymbol{j} = nq\boldsymbol{u}$ is the current density. Show that the familiar $\boldsymbol{E}\times\boldsymbol{B}$ drift is recovered when the collision frequency becomes very small.

(d) If $\boldsymbol{E}$ is at an angle to $\boldsymbol{B}$, there will be current flow components both parallel and perpendicular to $\boldsymbol{B}$, If $\boldsymbol{u}_i$ is different from $\boldsymbol{u}_e$, there is also a nett Hall current $\boldsymbol{j}_\perp = en(\boldsymbol{u}_{i\perp} - \boldsymbol{u}_{e\perp})$ that flows in the direction $\boldsymbol{E}\times\boldsymbol{B}$. To conveniently describe all these currents, the Ohm's law can equivalently be expressed by the tensor relation $\boldsymbol{j} = \overleftrightarrow{\sigma}\,\boldsymbol{E}$ with conductivity tensor given by

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \tag{10.5}$$

where

$$\sigma_\perp = \sigma_0 \frac{\nu^2}{\nu^2 + \omega_c^2}$$
$$\sigma_H = \sigma_0 \frac{\mp \nu \omega_c}{\nu^2 + \omega_c^2}$$
$$\sigma_\parallel = \sigma_0 = \frac{ne^2}{m\nu}$$

Explain the collision frequency dependence of the perpendicular and Hall conductivities.

Question 4 (10 marks) *Answer* **either** *part (a) or part (b)*

(a) Consider the two sets of long and straight current carrying conductors shown in confurations A and B of Figure 1.

(i) Sketch the magnetic field line configuration for each case.

(ii) Describe the particle guiding centre drifts in each case, with particular emphasis on the conservation of the first adiabatic moment.

(iii) Charge separation will occur due to the magnetic field inhomogeneity. This in turn establishes an electric field. Comment on the confining properties (or otherwise) of this electric field.

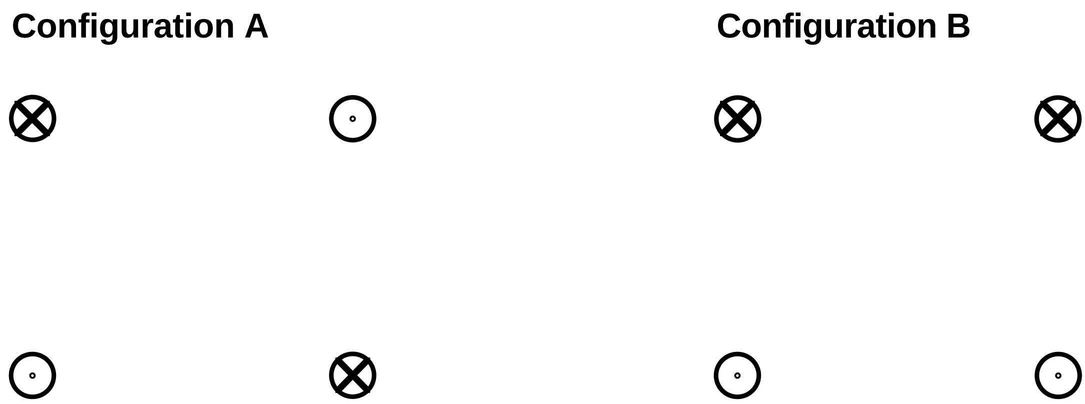

Figure 10.1: Conductors marked with a cross carry current into the page (z direction), while the dots indicate current out of the page.

(b) In a small experimental plasma device, a toroidal B-field is produced by uniformly winding 120 turns of conductor around a toroidal vacuum vessel and passing a current of 250A through it. The major radius of the torus is 0.6m. A plasma is produced in hydrogen by a radiofrequency heating field. The electrons and ions have Maxwellian velocity distribution functions at temperatures 80eV and 10eV respectively. The plasma density at the centre of the vessel is $10^{16}\,\mathrm{m}^{-3}$.

(i) Use Ampere's law around a toroidal loop linking the winding to calculate the vacuum field on the axis of the torus.

(ii) What is the field on axis in the presence of the plasma?

(iii) Calculate the total drift for both ions and electrons at the centre of the vessel and show the drifts on a sketch.

(iv) Explain how these drifts are compensated when a toroidal current is induced to flow.

(v) The toroidal current produces a poloidal field. The combined fields result in helical magnetic field lines that encircle the torus axis. For particles not on the torus axis and which have a high parallel to perpendicular velocity ratio the projected guiding centre motion executes a rotation in the poloidal plane (a vertical cross-section of the torus) as it moves helically along a field line. What happens to particles that have a high perpendicular to parallel velocity ratio?

Question 5 (10 marks) There is a standard way to check the relative importance of terms in the single fluid MHD equations. For space derivatives we choose a scale length L such that we can write $\partial u/\partial x \sim u/L$. Similarly we choose a time scale τ such that $\partial u/\partial t \sim u/\tau$. So $\nabla\times\boldsymbol{E} = \partial\boldsymbol{B}/\partial t$ becomes $E/L \sim B/\tau$. Introduce velocity $V = L/\tau$ so that $E \sim BV$.

(a) Examine the single fluid momentum equation.

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} = \boldsymbol{j}\times\boldsymbol{B} - \nabla p \tag{10.6}$$

Show that the terms are in the ratio

$$nm_\mathrm{i}\frac{V}{\tau} \,:\, jB \,:\, \frac{nm_\mathrm{e}v_\mathrm{the}^2}{L} \quad \text{or} \quad 1 \,:\, \frac{jB\tau}{nm_\mathrm{i}V} \,:\, \frac{m_\mathrm{e}}{m_\mathrm{i}}\frac{v_\mathrm{the}^2}{V^2} \tag{10.7}$$

When the plasma is cold, show that this suggests $V \sim jB\tau/nm_\mathrm{i}$

(b) Examine the generalized Ohm's law:

$$\frac{m_\mathrm{e}}{ne^2}\frac{\partial\boldsymbol{j}}{\partial t} = \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} - \frac{1}{ne}\boldsymbol{j}\times\boldsymbol{B} + \frac{1}{ne}\nabla p_\mathrm{e} - \eta\boldsymbol{j} \tag{10.8}$$

Show that the terms are in the ratio

$$\frac{1}{\omega_\mathrm{ce}\omega_\mathrm{ci}\tau^2} \,:\, 1 \,:\, 1 \,:\, \frac{1}{\omega_\mathrm{ci}\tau} \,:\, \frac{1}{\omega_\mathrm{ce}\tau}\frac{v_\mathrm{the}^2}{V^2} \,:\, \frac{\nu_\mathrm{ei}}{\omega_\mathrm{ce}\omega_\mathrm{ci}\tau} \tag{10.9}$$

(c) Which terms of the Ohm's law can be neglected if

(i) $\tau \gg 1/\omega_{\rm ci}$

(ii) $\tau \approx 1/\omega_{\rm ci}$

(iii) $\tau \approx 1/\omega_{\rm ce}$

(iv) $\tau \ll 1/\omega_{\rm ce}$

When can the resistive term $\eta \boldsymbol{j}$ be dropped?

Question 6 (10 marks)

Electromagnetic wave propagation in an unmagnetized plasma. Consider an electromagnetic wave propagating in an unbounded, unmagnetized uniform plasma of equilibrium density n_0. We assume the bulk plasma velocity is zero ($\boldsymbol{v}_0 = 0$) but allow small drifts v_1 to be induced by the one-dimensional harmonic electric field perturbation $E = E_1 \exp\left[{\rm i}(kx - \omega t)\right]$ that is transverse to the wave propagation direction.

(a) Assuming the plasma is also cold ($\nabla p = 0$) and collisionless, show that the momentum equations for electrons and ions give

$$\begin{aligned} n_0 m_{\rm i}(-{\rm i}\omega v_{i1}) &= n_0 e E_1 \\ n_0 m_{\rm e}(-{\rm i}\omega v_{e1}) &= -n_0 e E_1 \end{aligned}$$

(b) The ion motions are small and can be neglected (why?). Show that the resulting current density flowing in the plasma due to the imposed oscillating wave electric field is given by

$$j_1 = e n_0 (v_{i1} - v_{e1}) \approx {\rm i}\frac{n_0 e^2}{m_{\rm e}\omega} E_1. \tag{10.10}$$

(c) Associated with the fluctuating current is a small magnetic field oscillation which is given by Ampere's law. Use the differential forms of Faraday's law and Ampere's law (Maxwell's equations) to obtain the first order equations $kE_1 = \omega B_1$ and ${\rm i}kB_1 = \mu_0 j_1 - {\rm i}\omega\mu_0\varepsilon_0 E_1$ linking B_1, E_1 and j_1.

(d) Use these relations to eliminate B_1 and j_1 to obtain the dispersion relation for plane electromagnetic waves propagating in a plasma:

$$k^2 = \frac{\omega^2 - \omega_{\rm pe}^2}{c^2} \tag{10.11}$$

(d) Sketch the dispersion relation and comment on the physical significance of the dispersion near the region $\omega = \omega_{\rm pe}$.

THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination 2001

PHYSICS C17
PLASMA PHYSICS

Writing period 2 hours duration
Study period 15 minutes duration
Permitted materials: Calculators

Attempt four questions. All are of equal value.
Show all working and state and justify relevant assumptions.

Question 1

Attempt **three** of the following. Answers for each should require at most half a page.

(a) Discuss the relationship between moments of the particle distribution function f and moments of the Boltzmann equation. Plot $f(v)$ for a one dimensional drifting Maxwellian distribution, indicating pictorially the meaning of the three lowest order velocity moments.

(b) Describe electric breakdown with reference to the parameter E/p and the role of secondary emission.

(c) Discuss the physical meaning of the Boltzmann relation. Use diagrams to aid your explanation.

(d) Discuss the origin of plasma diamagnetism and its implications for magnetic plasma confinement.

(e) Elaborate the role of Coulomb collisions for diffusion in a magnetized plasma.

(f) Discuss magnetic mirrors with reference to the adiabatic invariance of the orbital magnetic moment μ.

(g) Describe Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2

Consider an axisymmetric cylindrical plasma with $\boldsymbol{E} = E\hat{\boldsymbol{r}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and $\nabla p_{\mathrm{i}} = \nabla p_{\mathrm{e}} = \hat{\boldsymbol{r}}\partial p/\partial r$. If we negelct $(\boldsymbol{v}.\nabla)\boldsymbol{v}$, the steady state two-fluid momentum-balance equations can be written in the form

$$\begin{aligned} en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{i}}\times\boldsymbol{B}) - \nabla p_{\mathrm{i}} - e^2n^2\eta(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) &= 0 \\ -en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{e}}\times\boldsymbol{B}) - \nabla p_{\mathrm{e}} + e^2n^2\eta(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) &= 0 \end{aligned}$$

(a) From the $\hat{\boldsymbol{\theta}}$ components of these equations, show that $u_{\mathrm{i}r} = u_{\mathrm{e}r}$.

(b) From the $\hat{\boldsymbol{r}}$ components, show that $u_{j\theta} = u_E + u_{Dj}$ ($j =$ i, e).

(c) Find an expression for $u_{\mathrm{i}r}$ showing that it does not depend on E.

Question 3

The induced emf at the terminals of a wire loop that encircles a plasma measures the rate of change of magnetic flux expelled by the plasma. You are given the following parameters:
Vacuum magnetic field strength - 1 Tesla
Number of turns on the diamagnetic loop - $N = 75$
Radius of the loop - $a_L = 0.075$m
Plasma radius - $a = .05$m.
Given the observed diamagnetic flux loop signal shown below, calculate the plasma pressure as a function of time. If the temperature of the plasma is constant at 1 keV, what is the plasma density as a function of time? (HINT: use Faraday's law relating the emf to the time derivative of the magnetic flux)

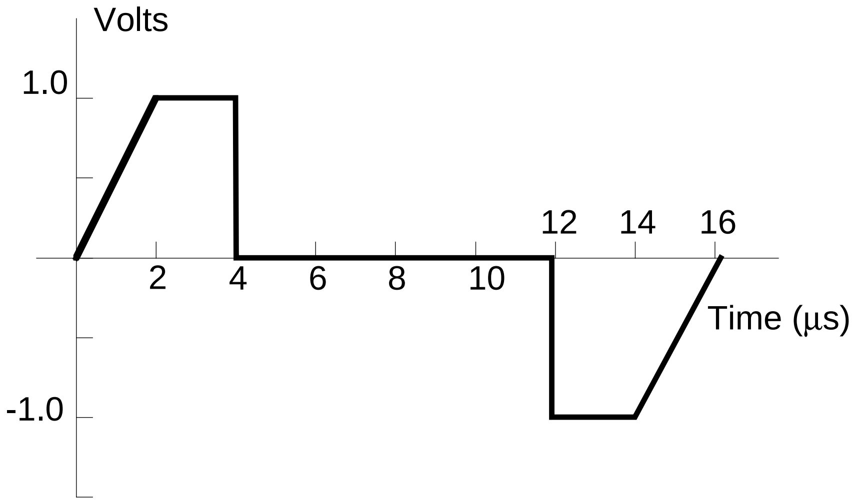

Figure 10.2: Magnetic flux loop signal as a function of time.

Question 4

An infinite straight wire carries a constant current I in the $+z$ direction. At $t = 0$ an electron of small gyroradius is at $z = 0$ and $r = r_0$ with $v_{\perp 0} = v_{\parallel 0}$ ($\perp$ and $\parallel$ refer to the direction relative to the magnetic field.)

(a) Calculate the magnitude and direction of the resulting guiding centre drift velocity.

(b) Suppose the current increases slowly in time in such a way that a constant electric field is induced in the $\pm z$ direction. Indicate on a diagram the relative directions of $\boldsymbol{I}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{v}_E$.

(c) Do $v_\perp$ and $v_\parallel$ increase, decrease or remain the same as the current increases? Explain your answer.

Question 5

Magnetic pumping is a means of heating plasmas that is based on the constancy of the magnetic moment μ. Consider a magnetized plasma for which the magnetic field strength is modulated in time according to

$$B = B_0(1 + \epsilon \cos \omega t) \tag{10.12}$$

where $\omega \ll \omega_{\mathrm{c}}$ and $\epsilon \ll 1$. If $U_\perp = mv_\perp^2/2 = (mv_x^2 + mv_y^2)/2$ is the particle perpendicular kinetic energy (electrons or ions) *show* that the kinetic energy is also modulated as

$$\frac{\mathrm{d}U_\perp}{\mathrm{d}t} = \frac{U_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t}.$$

We now allow for a collisional relaxation between the perpendicular ($U_\perp$) and parallel ($U_\parallel$) kinetic energies modelled according to the coupled equations

$$\begin{aligned} \frac{\mathrm{d}U_\perp}{\mathrm{d}t} &= \frac{U_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \nu\left(\frac{U_\perp}{2} - U_\parallel\right) \\ \frac{\mathrm{d}U_\parallel}{\mathrm{d}t} &= \nu\left(\frac{U_\perp}{2} - U_\parallel\right) \end{aligned}$$

where ν is the collision frequency. By suitably combining these equations, one can calculate the increment ΔU in total kinetic energy during a period $\Delta t = 2\pi/\omega$ to obtain a nett heating rate

$$\frac{\Delta U}{\Delta t} = \frac{\epsilon^2}{6}\frac{\omega^2 \nu}{9\nu^2/4 + \omega^2} U \equiv \alpha U. \tag{10.13}$$

This heating scheme is called collisional magnetic pumping. *Comment* on the physical reasons for the ν-dependence of α in the cases $\omega \gg \nu$ and $\omega \ll \nu$.

Assuming that the plasma is fully ionized (Coulomb collisions), and in the case $\omega \gg \nu$, *show* that the heating rate $\Delta U/\Delta t$ decreases as the temperature increases. What would happen if the magnetic field were oscillating at frequency $\omega = \omega_c$?

Question 6

On a graph of wave frequency ω versus wavenumber k show the dispersion relations for the ion and electron acoustic waves, and a transverse electromagnetic wave ($\omega > \omega_{pe}$) propagating in an unmagnetized plasma. (HINT: Draw the ion and electron plasma frequencies and lines corresponding to the electron sound speed, the ion sound speed and the speed of light.)

Consider the case of electron plasma oscillations in a uniform plasma of density n_0 in the presence of a uniform steady magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{k}}$. We take the background electric field to be zero ($\boldsymbol{E}_0 = 0$) and assume the plasma is at rest $\boldsymbol{u} = 0$. We shall consider longitudinal electron oscillations having $\boldsymbol{k} \parallel \boldsymbol{E}_1$ where we take the oscillating electric field perturbation associated with the electron wave $\boldsymbol{E}_1 \equiv E\hat{\boldsymbol{\imath}}$ to be parallel to the x-axis.

Replacing time derivatives by $-\mathrm{i}\omega$ and spatial gradients by $\mathrm{i}\boldsymbol{k}$, and ignoring pressure gradients and the convective term $(\boldsymbol{u}.\nabla)\boldsymbol{u}$, show that for small amplitude perturbations, the electron motion is governed by the linearized mass and momentum conservation equations and Maxwell's equation:

$$\begin{aligned} -\mathrm{i}\omega n_1 + n_0 \mathrm{i} k u_x &= 0 && (10.14)\\ -\mathrm{i}\omega \boldsymbol{u} &= -e(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}_0) && (10.15)\\ \varepsilon_0 \mathrm{i} k E &= -e n_1. && (10.16) \end{aligned}$$

Use Eq. (10.15) to show that the x component of the electron motion is given by

$$u_x = \frac{eE/\mathrm{i}\omega m}{1-\omega_c^2/\omega^2} \tag{10.17}$$

Substituting for u_x from the continuity equation and eliminating the density perturbation using Eq. (10.16), obtain the dispersion relation for the longitudinal electron plasma oscillation transverse to $\boldsymbol{B}$:

$$\omega^2 = \omega_p^2 + \omega_c^2. \tag{10.18}$$

Why is the oscillation frequency greater than ω_p? By expressing the ratio u_x/u_y in terms of ω and ω_c show that the electron trajectory is an ellipse elongated in the x direction.

APPENDIX: A Glossary of Useful Formulae

Chapter 1: Basic plasma phenomena

$$\omega_{\rm pe} = \sqrt{\frac{e^2 n_{\rm e}}{\varepsilon_0 m_{\rm e}}} \qquad \lambda_{\rm D} = \sqrt{\frac{\varepsilon_0 k_{\rm B} T_{\rm e}}{n_{\rm e} e^2}}$$

$$f_{pe} \simeq 9\sqrt{n_{\rm e}}(\,{\rm Hz}) \qquad \frac{n_i}{n} \simeq 2.4 \times \times 10^{21} \frac{T_{\rm e}^{\frac{3}{2}}}{n_i} \exp \frac{-U_i}{k_{\rm B} T}$$

Chapter 2: Kinetic theory

$$\frac{\partial f}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}_r.f + \frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}).\boldsymbol{\nabla}_v.f = \left(\frac{\partial f}{\partial t}\right)_{\rm coll} \qquad n_{\rm e} = n_{\rm e0} \exp\left(\frac{e\phi}{k_{\rm B} T_{\rm e}}\right)$$

$$\Gamma = n\bar{\boldsymbol{v}}$$

$$\lambda_{\rm mfp} = \frac{1}{n\sigma}$$

$$\boldsymbol{j} = qn\bar{\boldsymbol{v}}$$

$$\tau = \frac{\lambda_{\rm mfp}}{v}$$

$$p = \frac{2}{3} n \bar{U}_r$$

$$\nu = n\sigma v$$

$$f_M(v) = A\exp(\frac{-mv^2}{2k_{\rm B}T}) = A\exp\left(-v^2/v_{\rm th}^2\right) \qquad b_0 = \frac{2qq_0}{4\pi\varepsilon_0 m v^2}$$

$$\bar{U}_r(\text{Maxwellian}) \equiv E_{\rm Av} = \frac{1}{2} k_{\rm B} T(1-{\rm D}) \qquad \ln\Lambda = \ln\langle\frac{\lambda_{\rm D}}{b_0}\rangle$$

$$1\,{\rm eV} \simeq 11,600\,{\rm K}$$

$$\sigma^{\rm ei}_{\rm coulomb} \simeq \frac{Z^2 e^4 \ln\Lambda}{2\pi\varepsilon_0^2 m_{\rm e}^2 v_e^4}$$

$$v_{\rm rms} = \sqrt{\frac{3k_{\rm B}T}{m}}$$

$$\delta E_{\rm ei} \sim \frac{4E_e m_{\rm e}}{m_{\rm i}}$$

$$v_{\rm th} = \sqrt{\frac{2k_{\rm B}T}{m}}$$

$$P_{\rm ei} = -\frac{m_{\rm e} n_{\rm e}(\boldsymbol{u}_{\rm e} - \boldsymbol{u}_{\rm i})}{\tau_{\rm ei}}$$

$$p_j = n_j k_{\rm B} T_j$$

Chapter 2: Fluid and Maxwell's equations

$$\begin{aligned}
\sigma &= n_{\rm i} q_{\rm i} + n_{\rm e} q_{\rm e} \\
\boldsymbol{j} &= n_{\rm i} q_{\rm i} \boldsymbol{u}_{\rm i} + n_{\rm e} q_{\rm e} \boldsymbol{u}_{\rm e} \\
\frac{\partial n_j}{\partial t} + \nabla.(n_j \boldsymbol{u}_j) &= 0 \\
m_j n_j \left[\frac{\partial \boldsymbol{u}_j}{\partial t} + (\boldsymbol{u}_j.\nabla)\boldsymbol{u}_j\right] &= q_j n_j(\boldsymbol{E} + \boldsymbol{u}_j \times \boldsymbol{B}) - \nabla p_j + P_{\rm coll}
\end{aligned}$$

$$\begin{aligned}
p_j &= C_j n_j^{\gamma_j} \\
\nabla.\boldsymbol{E} &= \frac{\sigma}{\varepsilon_0} \\
\nabla\times\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla.\boldsymbol{B} &= 0 \\
\nabla\times\boldsymbol{B} &= \mu_0 \boldsymbol{j} + \mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t}
\end{aligned}$$

Chapter 3: Gaseous Electronics

$$\Gamma_j = n\boldsymbol{u}_j = \pm\mu_j n\boldsymbol{E} - D_j\nabla n$$

$$\mu = \frac{|q|}{m\nu}$$

$$D = \frac{k_{\mathrm{B}}T}{m\nu}$$

$$\boldsymbol{E} = \eta\boldsymbol{j}$$

$$\eta = \frac{\nu_{\mathrm{ei}}m_{\mathrm{e}}}{n_{\mathrm{e}}e^2}$$

$$\eta \simeq \frac{Ze^2\sqrt{m_{\mathrm{e}}}\ln\Lambda}{6\sqrt{3}\pi\varepsilon_0^2(k_{\mathrm{B}}T_{\mathrm{e}})^{3/2}}$$

$$\eta_\parallel = \frac{5.2\times10^{-5}Z\ln\Lambda}{T_{\mathrm{e(eV)}}^{\frac{3}{2}}}$$

$$I = \frac{I_0 e^{\alpha x}}{(1-\gamma e^{\alpha x})}$$

$$J = \frac{4}{9}\sqrt{\frac{2e}{m_{\mathrm{i}}}}\frac{\varepsilon_0|\phi_w|^{3/2}}{d^2}$$

$$u_{\mathrm{Bohm}} = \sqrt{\frac{k_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{i}}}}$$

$$\frac{e\phi_w}{k_{\mathrm{B}}T_{\mathrm{e}}} \approx \frac{1}{2}\ln\left(\frac{2\pi m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)$$

$$I_{\mathrm{si}} \simeq \frac{1}{2}n_0 eA\sqrt{\frac{k_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{i}}}}$$

Chapter 4: Single Particle Motions

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$$

$$\omega_{\mathrm{c}} \equiv \frac{|q|B}{m}$$

$$r_{\mathrm{L}} = \frac{v_\perp}{\omega_{\mathrm{c}}}$$

$$\mu = \frac{mv_\perp^2}{2B}$$

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$$

$$\boldsymbol{v}_F = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}$$

$$\boldsymbol{v}_R = \frac{mv_\parallel^2}{qB^2}\frac{\boldsymbol{R}_c\times\boldsymbol{B}}{R^2}$$

$$\boldsymbol{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_{\mathrm{L}}\frac{\boldsymbol{B}\times\nabla B}{B^2}$$

$$\boldsymbol{v}_P = \frac{1}{\omega_{\mathrm{c}}}\frac{\dot{\boldsymbol{E}}}{B}$$

$$\boldsymbol{j} = \overset{\leftrightarrow}{\sigma}\,\boldsymbol{E}$$

$$F_\parallel = -\mu\nabla_\parallel B$$

$$v_\parallel = \left[\frac{2}{m}(K-\mu B)\right]^{1/2}$$

$$\frac{B_m}{B_0} = \frac{1}{\sin^2\theta_m}$$

$$q(r) = \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \frac{rB_0}{RB_\theta} = \epsilon\frac{B_0}{B_\theta}$$

$$\overleftrightarrow{\sigma}_{\rm e} = \frac{{\rm i}ne^2}{m_{\rm e}\omega}\begin{pmatrix} \dfrac{\omega^2}{\omega^2-\omega_{\rm ce}^2} & \dfrac{-{\rm i}\omega_{\rm ce}\omega}{\omega^2-\omega_{\rm ce}^2} & 0 \\ \dfrac{{\rm i}\omega_{\rm ce}\omega}{\omega^2-\omega_{\rm ce}^2} & \dfrac{\omega^2}{\omega^2-\omega_{\rm ce}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overleftrightarrow{\varepsilon} = \varepsilon_0\left(\overleftrightarrow{I} + \frac{\rm i}{\varepsilon_0\omega}\overleftrightarrow{\sigma}\right)$$

Chapter 5: Magnetized Plasmas

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} + \frac{-\nabla p\times\boldsymbol{B}}{qnB^2}$$

$$\boldsymbol{j}_D = (k_{\rm B}T_{\rm i} + k_{\rm B}T_{\rm e})\frac{\boldsymbol{B}\times\nabla n}{B^2}$$

$$\boldsymbol{u}_\perp = \pm\mu_\perp\boldsymbol{E} - D_\perp\frac{\nabla n}{n} + \frac{\boldsymbol{u}_E + \boldsymbol{u}_D}{1+\nu^2/\omega_{\rm c}^2}$$

$$\mu_\perp = \frac{\mu}{1+\omega_{\rm c}^2\tau^2}$$

$$D_\perp = \frac{D}{1+\omega_{\rm c}^2\tau^2}$$

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}$$

$$\begin{aligned} \sigma_\perp &= \sigma_0\frac{\nu^2}{\nu^2+\omega_{\rm c}^2} \\ \sigma_H &= \sigma_0\frac{\mp\nu\omega_{\rm c}}{\nu^2+\omega_{\rm c}^2} \\ \sigma_\parallel &= \sigma_0 = \frac{ne^2}{m\nu} \end{aligned}$$

$$D_\perp = \frac{\eta_\perp\sum n_s k_{\rm B}T_s}{B^2}$$

Chapter 5: Single Fluid Equations

$$\begin{aligned} \rho\frac{\partial\boldsymbol{u}}{\partial t} &= \boldsymbol{j}\times\boldsymbol{B} - \nabla p + \rho\boldsymbol{g} \\ \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} &= \eta\boldsymbol{j} \\ \frac{\partial\rho}{\partial t} + \nabla.(\rho\boldsymbol{u}) &= 0 \\ \frac{\partial\sigma}{\partial t} + \nabla.\boldsymbol{j} &= 0 \\ \nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\times\boldsymbol{B} &= \mu_0\boldsymbol{j} \\ p &= Cn^\gamma \end{aligned}$$

Chapter 6: Magnetohydrodynamics

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\boldsymbol{B}.\nabla)\boldsymbol{B}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \frac{\eta}{\mu_0}\nabla^2 B + \nabla\times(\boldsymbol{u}\times\boldsymbol{B})$$

$$R_M = \frac{\mu_0 v L}{\eta}$$

Chapter 7, 8, 9: Waves

$$\boldsymbol{v}_g = \frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{k}}$$

$$V_A = \left(\frac{B^2}{\mu_0 \rho}\right)^{/2}$$

$$V_S = \left(\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}\right)^{1/2}$$

$$v_\phi = \frac{\omega}{k} = \frac{c}{(1 - \omega_{pe}^2/\omega^2)^{1/2}}$$

$$\omega^2 = \omega_{pe}^2 + \frac{3 k_B T}{m} k^2$$

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{E}) + \overleftrightarrow{K} . \boldsymbol{E} = 0$$

$$\boldsymbol{n} = \frac{c}{\omega}\boldsymbol{k}$$

$$n = |\boldsymbol{n}| = ck/\omega = c/v_\phi$$

$$\overleftrightarrow{K} = \overleftrightarrow{\epsilon}/\varepsilon_0 = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$\begin{aligned} S &= 1 - \sum_{i,e} \frac{\omega_p^2}{\omega^2 - \omega_c^2} \\ D &= \sum_{i,e} \pm \frac{\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)} \\ P &= 1 - \sum_{i,e} \frac{\omega_p^2}{\omega^2} \end{aligned}$$

$$\begin{aligned} R &= S + D \quad \text{Right} \\ L &= S - D \quad \text{Left} \\ S &= (R + L)/2 \quad \text{Sum} \\ D &= (R - L)/2 \quad \text{Diff} \\ P & \quad \text{Plasma} \end{aligned}$$

$$\tan^2\theta = \frac{P(n^2 - L)(n^2 - R)}{(n^2 - P)(RL - n^2 S)}$$

$$\left(\frac{c^2}{v_\phi^2}\right)_{\substack{L \\ R}} = 1 - \frac{\omega_{pe}^2}{\omega(\omega \pm \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega \mp \omega_{ci})}$$

$$\omega_{0L} = \left[-\omega_{ce} + (\omega_{ce}^2 + 4\omega_{pe}^2)^{1/2}\right]/2$$

$$\omega_{0R} = \left[\omega_{ce} + (\omega_{ce}^2 + 4\omega_{pe}^2)^{1/2}\right]/2$$

$$n^2 = \frac{c^2}{v_\phi^2} = \frac{(\omega^2 - \omega_{0L}^2)(\omega^2 - \omega_{0R}^2)}{\omega^2(\omega^2 - \omega_{UH}^2)}$$

$$\omega_{UH} = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2}$$

$$\omega_{LH} \approx (\omega_{ci}\omega_{ce})^{1/2}$$

$$n^2 = \frac{c^2}{v_\phi^2} = \frac{(\omega^2 - \omega_{0L}^2)(\omega^2 - \omega_{0R}^2)}{\omega^2(\omega^2 - \omega_{UH}^2)}$$

Useful Mathematical Identities

$$\boldsymbol{A}.(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}.(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}.(\boldsymbol{A}\times\boldsymbol{B})$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = \boldsymbol{B}(\boldsymbol{C}.\boldsymbol{A}) - \boldsymbol{A}(\boldsymbol{C}.\boldsymbol{B})$$

$$\nabla.(\phi\boldsymbol{A}) = \boldsymbol{A}.\nabla\phi + \phi\nabla.\boldsymbol{A}$$

$$\nabla\times(\phi\boldsymbol{A}) = \nabla\phi\times\boldsymbol{A} + \phi\nabla\times\boldsymbol{A}$$

$$(\boldsymbol{A}.\nabla)\boldsymbol{A} = \nabla(\frac{1}{2}A^2) - \boldsymbol{A}\times(\nabla\times\boldsymbol{A})$$

$$\nabla.(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}.(\nabla\times\boldsymbol{A}) - \boldsymbol{A}.(\nabla\times\boldsymbol{B})$$

$$\begin{aligned} \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) &= \nabla(\boldsymbol{A}.\boldsymbol{B}) - (\boldsymbol{A}.\nabla)\boldsymbol{B} \\ &- (\boldsymbol{B}.\nabla)\boldsymbol{A} - \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \end{aligned}$$

$$\begin{aligned} \nabla\times(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{A}(\nabla.\boldsymbol{B}) - \boldsymbol{B}(\nabla.\boldsymbol{A}) \\ &+ (\boldsymbol{B}.\nabla)\boldsymbol{A} - (\boldsymbol{A}.\nabla)\boldsymbol{B} \end{aligned}$$

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\boldsymbol{.}\boldsymbol{A}) - (\nabla\boldsymbol{.}\nabla)\boldsymbol{A}$$

$$\nabla\times\nabla\phi = 0$$

$$\nabla\boldsymbol{.}(\nabla\times\boldsymbol{A}) = 0$$

$$\int_{-\infty}^{\infty} v^2 \exp(-av^2)\mathrm{d}v = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

Cylindrical coordinates

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\phi}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla\boldsymbol{.}\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned}\nabla\times\boldsymbol{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} \\ &+ \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{z}}\end{aligned}$$