(c) Assume that the ion oscillations are so slow that the electrons remain in a Maxwell-Boltzmann distribution. If $e\phi/k_\mathrm{B}T_\mathrm{e} \ll 1$, show that the perturbed charge density of the electrons is given by $-(n_0e^2/k_\mathrm{B}T_\mathrm{e})\phi$.

(d) Use Poisson's equation to deduce the following dispersion relation:

$$k^2 = (n_0e^2/m_\mathrm{i}\varepsilon_0\omega^2)k^2 - n_0e^2/k_\mathrm{B}T_\mathrm{e}\varepsilon_0$$

(e) Recast the dispersion relation in the following form:

$$\omega^2 = \omega_\mathrm{pi}^2/(1 + 1/k^2\lambda_\mathrm{D}^2).$$

Discuss the low and high-k limits and compare with the Bohm-Gross dispersion relation for electron plasma waves.

Question 3 (10 marks)

(a) Using the steady-state force balance equation (ignore the convective derivative) show that the particle flux $\Gamma = n\boldsymbol{u}$ for electrons and singly charged ions in a fully ionized unmagnetized plasma is given by:

$$\Gamma_j = n\boldsymbol{u}_j = \pm\mu_j n\boldsymbol{E} - D_j\nabla n$$

with mobility $\mu =\mid q \mid /m\nu$ where ν is the electron-ion collision frequency and diffusion coefficient $D = k_\mathrm{B}T/m\nu$.

(b) Show that the diffusion coefficient can be expressed as $D \sim \lambda_\mathrm{mfp}^2/\tau$ where λ_mfp is the mean free path between collisions and τ is the collision time.

(c) Show that the plasma resistivity is given approximately by $\eta = m_\mathrm{e}\nu/ne^2$.

(d) In the presence of a magnetic field, the mean perpedicular velocity of particles across the field is given by

$$\boldsymbol{u}_\perp = \pm\mu_\perp\boldsymbol{E} - D_\perp\frac{\nabla n}{n} + \frac{\boldsymbol{u}_E + \boldsymbol{u}_D}{1 + \nu^2/\omega_\mathrm{c}^2}$$

with $\boldsymbol{u}_E = \boldsymbol{E}\times\boldsymbol{B}/B^2$, $\boldsymbol{u}_D = -\nabla p\times\boldsymbol{B}/qnB^2$ and where $\mu_\perp = \mu/(1 + \omega_\mathrm{c}^2\tau^2)$ and $D_\perp = D/(1 + \omega_\mathrm{c}^2\tau^2)$. Discuss the scaling with ν of each of the four terms in the expression for $\boldsymbol{u}_\perp$.

Question 4 (10 marks)

(a) Show that the drift speed of a charge q in a toroidal magnetic field can be written as

$$v_T = 2k_\mathrm{B}T/qBR$$

where R is the radius of curvature of the field. (Hint: Consider both gradient and curvature drifts)

(b) Compute the value of v_T for a plasma at a temperature of 10 keV, a magnetic field strength of 2 T and a major radius $R = 1$ m.

(c) Compute the time required by a charge to drift across a toroidal container of minor radius 1 m.

(d) Suppose an electric field is applied perpendicular to the plane of the torus. Describe what happens.

Question 5 (10 marks)

(a) Show that the MHD force balance equation $\nabla p = \boldsymbol{j}\times\boldsymbol{B}$ requires both $\boldsymbol{j}$ and $\boldsymbol{B}$ to lie on surfaces of constant pressure.

(b) Using Ampere's law and MHD force balance, show that

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=\frac{1}{\mu_0}(\boldsymbol{B}.\nabla)\boldsymbol{B}$$

and discuss the meaning of the various terms.

(c) A straight current carrying plasma cylinder (linear pinch) is subject to a range of instabilities (sausage, kink etc.). These can be suppressed by providing an axial magnetic field B_z that stiffens the plasma through the additional magnetic pressure $B_z^2/2\mu_0$ and tension against bending. Consider a local constriction $\mathrm{d}r$ in the radius r of the plasma column. Assuming that the longitudinal magnetic flux Φ through the cross-section of the cylinder remains constant during the compression ($\mathrm{d}\Phi = 0$), show that the axial magnetic field strength is increased by an amount $\mathrm{d}B_z = -2B_z\mathrm{d}r/r$.

(d) Show that the internal magnetic pressure increases by an amount $\mathrm{d}p_z = B_z\mathrm{d}B_z/\mu_0 = -(2B_z^2/\mu_0)\mathrm{d}r/r$. [The last step uses the result obtained in (c)].

(e) By Ampere's law we have for the azimuthal field component $rB_\theta(r) =$ constant. Show that the change in azimuthal field strength due the compression $\mathrm{d}r$ is $\mathrm{d}B_\theta = -B_\theta\mathrm{d}r/r$ and that the associated increase in external azimuthal magnetic pressure is $\mathrm{d}p_\theta = -(B_\theta^2/\mu_0)\mathrm{d}r/r$.

(f) Show that the plasma column is stable against sausage distortion provided $B_z^2 > B_\theta^2/2$.

Question 6 (10 marks)

(a) Plot the wave phase velocity as a function of frequency for plasma waves propagating perpendicular to the magnetic field $\boldsymbol{B}$, identifying cutoffs and resonances for both ordinary and extraordinary modes.

(b) Using the matrix form of the wave dispersion relation

$$\begin{pmatrix} S-n_z^2 & -\mathrm{i}D & n_xn_z \\ \mathrm{i}D & S-n_x^2-n_z^2 & 0 \\ n_xn_z & 0 & P-n_x^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}=0$$

show that the polarization state for the extraordinary wave is given by

$$E_x/E_y = \mathrm{i}D/S.$$

Using a diagram, show the relative orientations of $\boldsymbol{B}$, $\boldsymbol{k}$ and $\boldsymbol{E}$ for this wave.

THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination 2000

PHYSICS C17
PLASMA PHYSICS

Writing period 2 hours duration
Study period 15 minutes duration
Permitted materials: Calculators

Attempt four questions. All are of equal value.
Show all working and state and justify relevant assumptions.

Question 1 (10 marks)
Attempt **three** of the following. Answers for each should require at most half a page.

(a) Discuss the relationship between the Boltzmann equation, the electron and ion equations of motion and the single fluid force balance euqation.

(b) Describe electric breakdown with reference to the parameter E/p and the role of secondary emission.

(c) Discuss the physical meaning of the Boltzmann relation. Use diagrams to aid your explanation.

(d) Discuss the origin of plasma diamagnetism and its implications for magnetic plasma confinement.

(e) Draw a Langmuir probe I-V characteristic indicating the saturation currents, plasma potential and floating potential. How can the characteristic be used to estimate temperature?

(f) Describe Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2 (10 marks)

(a) Consider two infinite, perfectly conducting plates A_1 and A_2 occupying the planes $y = 0$ and $y = d$ respectively. An electron enters the space between the plates through a small hole in plate A_1 with initial velocity v towards plate A_2. A potential difference V between the plates is such as to decelerate the electron. What is the minimum potential difference to prevent the electron from reaching plate A_2.

(b) Suppose the region between the plates is permeated by a uniform magnetic field B parallel to the plate surfaces (imagine it as pointing into the page). A proton appears at the surface of plate A_1 with zero initial velocity. As before, the potential V between the plates is such as to accelerate the proton towards plate A_2. What is the minimum value of the magnetic field B necessary to prevent the proton from reaching plate B? Sketch what you think the proton trajectory might look like. (HINT: Energy considerations may be useful).

Question 3 (10 marks)

(a) Using the equilibrium force balance equation for electrons (assume ions are relatively immobile) show that the conductivity of an unmagnetized plasma is given by

$$\sigma_0 = \frac{ne^2}{m_e \nu} \tag{10.4}$$

(b) What is the dependence of the conductivity on electron temperature and density in the fully ionized case?

(c) When the plasma is magnetized, the Ohm's law for a given plasma species (electrons or ions) becomes $\boldsymbol{j} = \sigma_0(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B})$ where $\boldsymbol{j} = nq\boldsymbol{u}$ is the current density. Show that the familiar $\boldsymbol{E}\times\boldsymbol{B}$ drift is recovered when the collision frequency becomes very small.

(d) If $\boldsymbol{E}$ is at an angle to $\boldsymbol{B}$, there will be current flow components both parallel and perpendicular to $\boldsymbol{B}$, If $\boldsymbol{u}_i$ is different from $\boldsymbol{u}_e$, there is also a nett Hall current $\boldsymbol{j}_\perp = en(\boldsymbol{u}_{i\perp} - \boldsymbol{u}_{e\perp})$ that flows in the direction $\boldsymbol{E}\times\boldsymbol{B}$. To conveniently describe all these currents, the Ohm's law can equivalently be expressed by the tensor relation $\boldsymbol{j} = \overleftrightarrow{\sigma}\, \boldsymbol{E}$ with conductivity tensor given by

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \tag{10.5}$$

where

$$\begin{aligned} \sigma_\perp &= \sigma_0 \frac{\nu^2}{\nu^2 + \omega_c^2} \\ \sigma_H &= \sigma_0 \frac{\mp \nu \omega_c}{\nu^2 + \omega_c^2} \\ \sigma_\parallel &= \sigma_0 = \frac{ne^2}{m\nu} \end{aligned}$$

Explain the collision frequency dependence of the perpendicular and Hall conductivities.

Question 4 (10 marks) *Answer* **either** *part (a) or part (b)*

(a) Consider the two sets of long and straight current carrying conductors shown in confurations A and B of Figure 1.

(i) Sketch the magnetic field line configuration for each case.

(ii) Describe the particle guiding centre drifts in each case, with particular emphasis on the conservation of the first adiabatic moment.

(iii) Charge separation will occur due to the magnetic field inhomogeneity. This in turn establishes an electric field. Comment on the confining properties (or otherwise) of this electric field.

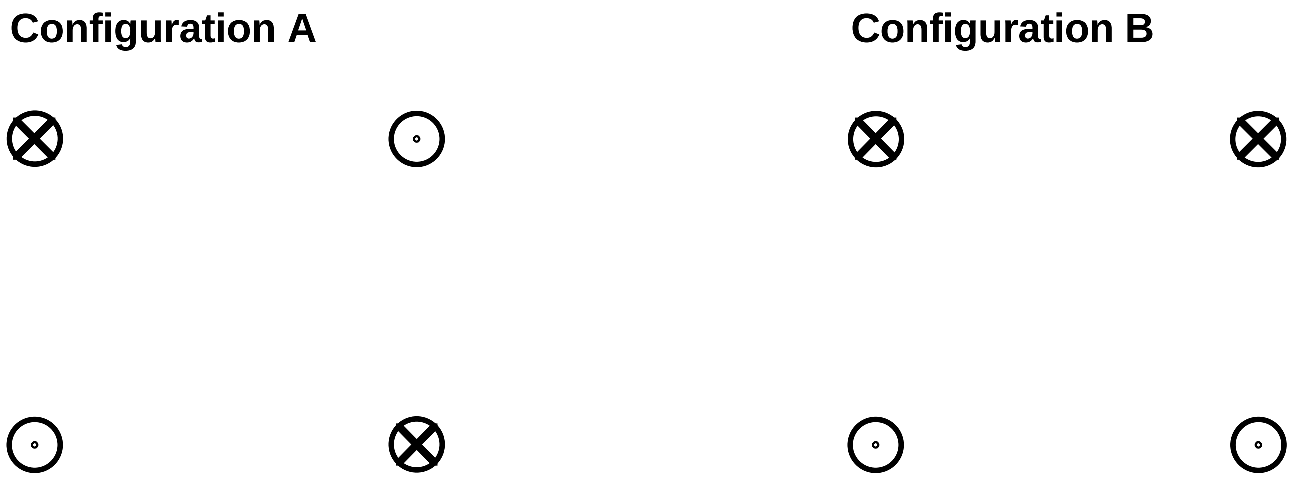

Figure 10.1: Conductors marked with a cross carry current into the page (z direction), while the dots indicate current out of the page.

(b) In a small experimental plasma device, a toroidal B-field is produced by uniformly winding 120 turns of conductor around a toroidal vacuum vessel and passing a current of 250A through it. The major radius of the torus is 0.6m.
A plasma is produced in hydrogen by a radiofrequency heating field. The electrons and ions have Maxwellian velocity distribution functions at temperatures 80eV and 10eV respectively. The plasma density at the centre of the vessel is $10^{16}\,\mathrm{m}^{-3}$.

(i) Use Ampere's law around a toroidal loop linking the winding to calculate the vacuum field on the axis of the torus.

(ii) What is the field on axis in the presence of the plasma?

(iii) Calculate the total drift for both ions and electrons at the centre of the vessel and show the drifts on a sketch.

(iv) Explain how these drifts are compensated when a toroidal current is induced to flow.

(v) The toroidal current produces a poloidal field. The combined fields result in helical magnetic field lines that encircle the torus axis. For particles not on the torus axis and which have a high parallel to perpendicular velocity ratio the projected guiding centre motion executes a rotation in the poloidal plane (a vertical cross-section of the torus) as it moves helically along a field line. What happens to particles that have a high perpendicular to parallel velocity ratio?

Question 5 (10 marks) There is a standard way to check the relative importance of terms in the single fluid MHD equations. For space derivatives we choose a scale length L such that we can write $\partial u/\partial x \sim u/L$. Similarly we choose a time scale τ such that $\partial u/\partial t \sim u/\tau$. So $\nabla\times\boldsymbol{E} = \partial\boldsymbol{B}/\partial t$ becomes $E/L \sim B/\tau$. Introduce velocity $V = L/\tau$ so that $E \sim BV$.

(a) Examine the single fluid momentum equation.

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} = \boldsymbol{j}\times\boldsymbol{B} - \nabla p \tag{10.6}$$

Show that the terms are in the ratio

$$nm_{\mathrm{i}}\frac{V}{\tau} \;:\; jB \;:\; \frac{nm_{\mathrm{e}}v_{\mathrm{the}}^2}{L} \quad \text{or} \quad 1 \;:\; \frac{jB\tau}{nm_{\mathrm{i}}V} \;:\; \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\frac{v_{\mathrm{the}}^2}{V^2} \tag{10.7}$$

When the plasma is cold, show that this suggests $V \sim jB\tau/nm_{\mathrm{i}}$

(b) Examine the generalized Ohm's law:

$$\frac{m_{\mathrm{e}}}{ne^2}\frac{\partial\boldsymbol{j}}{\partial t} = \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} - \frac{1}{ne}\boldsymbol{j}\times\boldsymbol{B} + \frac{1}{ne}\nabla p_{\mathrm{e}} - \eta\boldsymbol{j} \tag{10.8}$$

Show that the terms are in the ratio

$$\frac{1}{\omega_{\mathrm{ce}}\omega_{\mathrm{ci}}\tau^2} \;:\; 1 \;:\; 1 \;:\; \frac{1}{\omega_{\mathrm{ci}}\tau} \;:\; \frac{1}{\omega_{\mathrm{ce}}\tau}\frac{v_{\mathrm{the}}^2}{V^2} \;:\; \frac{\nu_{\mathrm{ei}}}{\omega_{\mathrm{ce}}\omega_{\mathrm{ci}}\tau} \tag{10.9}$$

(c) Which terms of the Ohm's law can be neglected if

(i) $\tau \gg 1/\omega_{\mathrm{ci}}$

(ii) $\tau \approx 1/\omega_{\mathrm{ci}}$

(iii) $\tau \approx 1/\omega_{\mathrm{ce}}$

(iv) $\tau \ll 1/\omega_{\mathrm{ce}}$

When can the resistive term $\eta\boldsymbol{j}$ be dropped?

Question 6 (10 marks)

Electromagnetic wave propagation in an unmagnetized plasma. Consider an electromagnetic wave propagating in an unbounded, unmagnetized uniform plasma of equilibrium density n_0. We assume the bulk plasma velocity is zero ($\boldsymbol{v}_0 = 0$) but allow small drifts v_1 to be induced by the one-dimensional harmonic electric field perturbation $E = E_1 \exp[\mathrm{i}(kx - \omega t)]$ that is transverse to the wave propagation direction.

(a) Assuming the plasma is also cold ($\nabla p = 0$) and collisionless, show that the momentum equations for electrons and ions give

$$\begin{aligned} n_0 m_\mathrm{i}(-\mathrm{i}\omega v_{i1}) &= n_0 e E_1 \\ n_0 m_\mathrm{e}(-\mathrm{i}\omega v_{e1}) &= -n_0 e E_1 \end{aligned}$$

(b) The ion motions are small and can be neglected (why?). Show that the resulting current density flowing in the plasma due to the imposed oscillating wave electric field is given by

$$j_1 = e n_0 (v_{i1} - v_{e1}) \approx \mathrm{i}\frac{n_0 e^2}{m_\mathrm{e}\omega} E_1. \qquad (10.10)$$

(c) Associated with the fluctuating current is a small magnetic field oscillation which is given by Ampere's law. Use the differential forms of Faraday's law and Ampere's law (Maxwell's equations) to obtain the first order equations $kE_1 = \omega B_1$ and $\mathrm{i}kB_1 = \mu_0 j_1 - \mathrm{i}\omega\mu_0\varepsilon_0 E_1$ linking B_1, E_1 and j_1.

(d) Use these relations to eliminate B_1 and j_1 to obtain the dispersion relation for plane electromagnetic waves propagating in a plasma:

$$k^2 = \frac{\omega^2 - \omega_\mathrm{pe}^2}{c^2} \qquad (10.11)$$

(d) Sketch the dispersion relation and comment on the physical significance of the dispersion near the region $\omega = \omega_\mathrm{pe}$.

THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination 2001

PHYSICS C17
PLASMA PHYSICS

Writing period 2 hours duration
Study period 15 minutes duration
Permitted materials: Calculators

Attempt four questions. All are of equal value.
Show all working and state and justify relevant assumptions.

Question 1
Attempt **three** of the following. Answers for each should require at most half a page.

(a) Discuss the relationship between moments of the particle distribution function f and moments of the Boltzmann equation. Plot $f(v)$ for a one dimensional drifting Maxwellian distribution, indicating pictorially the meaning of the three lowest order velocity moments.

(b) Describe electric breakdown with reference to the parameter E/p and the role of secondary emission.

(c) Discuss the physical meaning of the Boltzmann relation. Use diagrams to aid your explanation.

(d) Discuss the origin of plasma diamagnetism and its implications for magnetic plasma confinement.

(e) Elaborate the role of Coulomb collisions for diffusion in a magnetized plasma.

(f) Discuss magnetic mirrors with reference to the adiabatic invariance of the orbital magnetic moment μ.

(g) Describe Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2
Consider an axisymmetric cylindrical plasma with $\boldsymbol{E} = E\hat{\boldsymbol{r}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and $\nabla p_{\mathrm{i}} = \nabla p_{\mathrm{e}} = \hat{\boldsymbol{r}}\partial p/\partial r$. If we negelct $(\boldsymbol{v}.\nabla)\boldsymbol{v}$, the steady state two-fluid momentum-balance equations can be written in the form

$$\begin{aligned} en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{i}}\times\boldsymbol{B}) - \nabla p_{\mathrm{i}} - e^2n^2\eta(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) &= 0 \\ -en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{e}}\times\boldsymbol{B}) - \nabla p_{\mathrm{e}} + e^2n^2\eta(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) &= 0 \end{aligned}$$

(a) From the $\hat{\boldsymbol{\theta}}$ components of these equations, show that $u_{\mathrm{i}r} = u_{\mathrm{e}r}$.

(b) From the $\hat{\theta}$ components, show that $u_{j\theta} = u_E + u_{Dj}$ ($j = \text{i, e}$).

(c) Find an expression for $u_{\text{i}r}$ showing that it does not depend on E.

Question 3

The induced emf at the terminals of a wire loop that encircles a plasma measures the rate of change of magnetic flux expelled by the plasma. You are given the following parameters:
Vacuum magnetic field strength - 1 Tesla
Number of turns on the diamagnetic loop - $N = 75$
Radius of the loop - $a_L = 0.075$m
Plasma radius - $a = .05$m.
Given the observed diamagnetic flux loop signal shown below, calculate the plasma pressure as a function of time. If the temperature of the plasma is constant at 1 keV, what is the plasma density as a function of time? (HINT: use Faraday's law relating the emf to the time derivative of the magnetic flux)

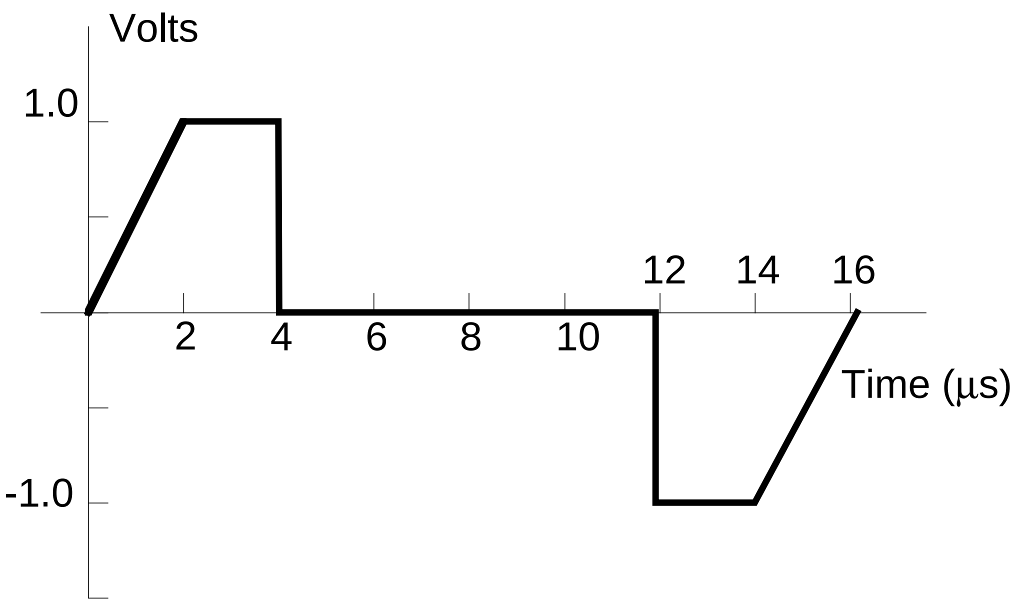

Figure 10.2: Magnetic flux loop signal as a function of time.

Question 4

An infinite straight wire carries a constant current I in the $+z$ direction. At $t = 0$ an electron of small gyroradius is at $z = 0$ and $r = r_0$ with $v_{\perp 0} = v_{\parallel 0}$ ($\perp$ and $\parallel$ refer to the direction relative to the magnetic field.)

(a) Calculate the magnitude and direction of the resulting guiding centre drift velocity.

(b) Suppose the current increases slowly in time in such a way that a constant electric field is induced in the $\pm z$ direction. Indicate on a diagram the relative directions of $\boldsymbol{I}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{v}_E$.

(c) Do $v_\perp$ and $v_\parallel$ increase, decrease or remain the same as the current increases? Explain your answer.

Question 5

Magnetic pumping is a means of heating plasmas that is based on the constancy of the magnetic moment μ. Consider a magnetized plasma for which the magnetic field strength is modulated in time according to

$$B = B_0(1 + \epsilon \cos \omega t) \tag{10.12}$$

where $\omega \ll \omega_\mathrm{c}$ and $\epsilon \ll 1$. If $U_\perp = mv_\perp^2/2 = (mv_x^2 + mv_y^2)/2$ is the particle perpendicular kinetic energy (electrons or ions) *show* that the kinetic energy is also modulated as

$$\frac{\mathrm{d}U_\perp}{\mathrm{d}t} = \frac{U_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t}.$$

We now allow for a collisional relaxation between the perpendicular ($U_\perp$) and parallel ($U_\parallel$) kinetic energies modelled according to the coupled equations

$$\begin{aligned} \frac{\mathrm{d}U_\perp}{\mathrm{d}t} &= \frac{U_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \nu\left(\frac{U_\perp}{2} - U_\parallel\right) \\ \frac{\mathrm{d}U_\parallel}{\mathrm{d}t} &= \nu\left(\frac{U_\perp}{2} - U_\parallel\right) \end{aligned}$$

where ν is the collision frequency. By suitably combining these equations, one can calculate the increment ΔU in total kinetic energy during a period $\Delta t = 2\pi/\omega$ to obtain a nett heating rate

$$\frac{\Delta U}{\Delta t} = \frac{\epsilon^2}{6}\frac{\omega^2 \nu}{9\nu^2/4 + \omega^2}U \equiv \alpha U. \tag{10.13}$$

This heating scheme is called collisional magnetic pumping. *Comment* on the physical reasons for the ν-dependence of α in the cases $\omega \gg \nu$ and $\omega \ll \nu$.

Assuming that the plasma is fully ionized (Coulomb collisions), and in the case $\omega \gg \nu$, *show* that the heating rate $\Delta U/\Delta t$ decreases as the temperature increases. What would happen if the magnetic field were oscillating at frequency $\omega = \omega_\mathrm{c}$?

Question 6

On a graph of wave frequency ω versus wavenumber k show the dispersion relations for the ion and electron acoustic waves, and a transverse electromagnetic wave ($\omega > \omega_\mathrm{pe}$) propagating in an unmagnetized plasma. (HINT: Draw the ion and electron plasma frequencies and lines corresponding to the electron sound speed, the ion sound speed and the speed of light.)

Consider the case of electron plasma oscillations in a uniform plasma of density n_0 in the presence of a uniform steady magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{k}}$. We take the background electric field to be zero ($\boldsymbol{E}_0 = 0$) and assume the plasma is at rest $\boldsymbol{u} = 0$. We shall consider longitudinal electron oscillations having $\boldsymbol{k} \parallel \boldsymbol{E}_1$ where we take the oscillating electric field perturbation associated with the electron wave $\boldsymbol{E}_1 \equiv E\hat{\boldsymbol{\imath}}$ to be parallel to the x-axis.

Replacing time derivatives by $-\mathrm{i}\omega$ and spatial gradients by $\mathrm{i}\boldsymbol{k}$, and ignoring pressure gradients and the convective term $(\boldsymbol{u}.\nabla)\boldsymbol{u}$, show that for small amplitude perturbations, the electron motion is governed by the linearized mass and momentum conservation equations and Maxwell's equation:

$$-\mathrm{i}\omega n_1 + n_0 \mathrm{i}k u_x = 0 \quad (10.14)$$
$$-\mathrm{i}\omega \boldsymbol{u} = -e(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}_0) \quad (10.15)$$
$$\varepsilon_0 \mathrm{i}kE = -en_1. \quad (10.16)$$

Use Eq. (10.15) to show that the x component of the electron motion is given by

$$u_x = \frac{eE/\mathrm{i}\omega m}{1 - \omega_\mathrm{c}^2/\omega^2} \quad (10.17)$$

Substituting for u_x from the continuity equation and eliminating the density perturbation using Eq. (10.16), obtain the dispersion relation for the longitudinal electron plasma oscillation transverse to $\boldsymbol{B}$:

$$\omega^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2. \quad (10.18)$$

Why is the oscillation frequency greater than ω_p? By expressing the ratio u_x/u_y in terms of ω and ω_c show that the electron trajectory is an ellipse elongated in the x direction.

THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination 2002

PHYSICS C17
PLASMA PHYSICS

Writing period 2 hours duration
Study period 15 minutes duration
Permitted materials: Calculators

Attempt four questions. All are of equal value.
Show all working and state and justify relevant assumptions.

Question 1

Discuss, using diagrams where appropriate, **three** of the following issues. Answers for each should require at most half a page.

(a) Ambipolar diffusion in unmagnetized plasma.

(b) Electric breakdown. Why is E/p important and what is the role of secondary emission?

(c) The physics underlying the Boltzmann relation.

(d) MHD waves that propagate perpendicular and parallel to B.

(e) The resistivity of weakly and fully ionized unmagnetized plasmas.

(f) Magnetic mirrors and the role of the invariance of the orbital magnetic moment μ.

(g) Debye shielding and the relationship between the plasma frequency and Debye length.

Question 2

(a) Explain using a diagram why the orbit of a particle gyrating in a magnetic field is diamagnetic.

(b) Given that the magnetic moment of a gyrating particle is $\mu = W_\perp/B$ where $W_\perp$ is the kinetic energy of the motion perpendicular to the magnetic field of strength B, find an expression for the magnetic moment per unit volume M in a plasma with particle density n and temperature T immersed in a uniform magnetic field.

(c) Supposing the field inside the plasma to be reduced compared with that outside the plasma B by $\mu_0 M \ll B$, calculate the difference in magnetic pressure $B^2/2\mu_0$ inside the plasma and confirm that the total pressure is constant.

(d) Describe in general terms what happens to maintain pressure balance as the plasma temperature is increased, keeping the total number of particles constant.

Question 3

(a) With the aid of diagrams, explain why magnetic plasma confinement is not possible in a purely toroidal magnetic field.

(b) The earth's magnetic field may be approximated as a magnetic dipole out to a few earth radii ($R_E = 6370$km). The magnetic field for a dipole can be written approximately as

$$B_r = \frac{\mu_0}{4\pi}\frac{2M\cos\theta}{r^3} \qquad (10.19)$$

$$B_\theta = \frac{\mu_0}{4\pi}\frac{M\sin\theta}{r^3} \qquad (10.20)$$

where θ is the polar angle from the direction of the dipole moment vector, B_r is the magnetic field radial component and B_θ is the component orthogonal to B_r. Using the fact that, at one of the magnetic poles ($r = R_E$) , the field has a magnitude of 0.5 Gauss, calculate the earth's dipole moment M.

(c) Assuming an electron is constrained to move in the earth's magnetic equatorial plane ($v_\parallel = 0$), calculate the guiding centre drift velocity, and determine the time it takes to drift once around the earth at a radial distance r_0. What is the direction of drift.

(d) Let there be an isotropic population of 1 eV protons and 30 keV electrons each with density $n = 10^7\,\mathrm{m}^{-3}$ at $r = 5R_E$ in the equatorial plane. Compute the ring current density in $\mathrm{A/m}^2$ associated with the drift obtained in (c).

(e) Now assume that the perpendicular kinetic energy equals the parallel kinetic energy at the magnetic equatorial plane. Qualitatively describe the motion of the electron guiding centre.

Question 4

(a) Using the steady-state force balance equation (ignore the convective derivative) show that the particle flux $\Gamma = n\boldsymbol{u}$ for electrons and singly charged ions in an unmagnetized plasma is given by:

$$\Gamma_j = n\boldsymbol{u}_j = \pm\mu_j n\boldsymbol{E} - D_j\nabla n$$

with mobility $\mu = \mid q \mid /m\nu$ where ν is the collision frequency and diffusion coefficient $D = k_\mathrm{B}T/m\nu$.

(b) Show that the diffusion coefficient can be expressed as $D \sim \lambda_\mathrm{mfp}^2/\tau$ where λ_mfp is the mean free path between collisions and τ is the collision time.

(c) Show that the plasma resistivity is given approximately by $\eta = m_\mathrm{e}\nu/ne^2$.

(d) In a weakly ionized magnetoplasma, the mean perpendicular velocity of particles across the field is given by

$$\boldsymbol{u}_\perp = \pm\mu_\perp \boldsymbol{E} - D_\perp \frac{\nabla n}{n} + \frac{\boldsymbol{u}_E + \boldsymbol{u}_D}{1 + \nu^2/\omega_c^2}$$

with $\boldsymbol{u}_E = \boldsymbol{E}\times\boldsymbol{B}/B^2$, $\boldsymbol{u}_D = -\nabla p\times\boldsymbol{B}/qnB^2$ and where $\mu_\perp = \mu/(1+\omega_c^2\tau^2)$ and $D_\perp = D/(1+\omega_c^2\tau^2)$. Discuss the physical origin of each of these terms and their behaviour in the limit of weak and strong magnetic fields.

Question 5 (10 marks)

The dispersion relation for low frequency magnetohydrodynamic waves in a magnetized plasma was derived in lectures as

$$-\omega^2\boldsymbol{u}_1 + (V_S^2 + V_A^2)(\boldsymbol{k}.\boldsymbol{u}_1)\boldsymbol{k} + (\boldsymbol{k}.\boldsymbol{V}_A)[(\boldsymbol{k}.\boldsymbol{V}_A)\boldsymbol{u}_1 - (\boldsymbol{V}_A.\boldsymbol{u}_1)\boldsymbol{k} - (\boldsymbol{k}.\boldsymbol{u}_1)\boldsymbol{V}_A] = 0$$

where $\boldsymbol{u}_1$ is the perturbed fluid velocity, $\boldsymbol{k}$ is the propagataion wavevector and $\boldsymbol{V}_A = \boldsymbol{B}_0/(\mu_0\rho_0)^{1/2}$ is a velocity vector in the direction of the magnetic field with magnitude equal to the Alfvén speed and V_S is the sound speed.

(a) Deduce the dispersion relations for waves propagating parallel to the magnetic field and identify the wave modes.

(b) Using

$$\begin{aligned} \frac{\partial \boldsymbol{B}_1}{\partial t} - \nabla\times(\boldsymbol{u}_1\times\boldsymbol{B}_0) &= 0 \\ \boldsymbol{E}_1 + \boldsymbol{u}_1\times\boldsymbol{B} &= 0 \end{aligned}$$

and assuming plane wave propagation so that $\frac{\partial}{\partial t} \to -\mathrm{i}\omega$ and $\nabla\times \to \mathrm{i}\boldsymbol{k}\times$, make a sketch showing the relation between the perturbed quantities $\boldsymbol{u}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{k}$ and $\boldsymbol{B}_0$ for the transverse wave propagating along $\boldsymbol{B}_0$.

Question 6 (10 marks)

Consider an electromagnetic wave propagating in an unbounded, em unmagnetized uniform plasma of equilibrium density n_0. We assume the bulk plasma velocity is zero ($\boldsymbol{v}_0 = 0$) but allow small drifts v_1 to be induced by the *one-dimensional* harmonic electric field perturbation $E = E_1 \exp[\mathrm{i}(kx - \omega t)]$ that is transverse to the wave propagation direction.

(a) Assuming the plasma is also cold ($\nabla p = 0$) and collisionless, show that the momentum equations for electrons and ions give

$$\begin{aligned} n_0 m_\mathrm{i}(-\mathrm{i}\omega v_{i1}) &= n_0 e E_1 \\ n_0 m_\mathrm{e}(-\mathrm{i}\omega v_{e1}) &= -n_0 e E_1 \end{aligned}$$

(b) The ion motions are small and can be neglected. Show that the resulting current density flowing in the plasma due to the imposed oscillating wave electric field is given by

$$j_1 = en_0(v_{i1} - v_{e1}) \approx \mathrm{i}\frac{n_0 e^2}{m_\mathrm{e}\omega}E_1. \tag{10.21}$$

(c) Associated with the fluctuating current is a small magnetic field oscillation which is given by Ampere's law. Use the differential forms of Faraday's law and Ampere's law (Maxwell's equations) to obtain the first order equations $kE_1 = \omega B_1$ and $\mathrm{i}kB_1 = \mu_0 j_1 - \mathrm{i}\omega\mu_0\varepsilon_0 E_1$ linking B_1, E_1 and j_1.

(d) Use these relations to eliminate B_1 and j_1 to obtain the dispersion relation for plane electromagnetic waves propagating in a plasma:

$$k^2 = \frac{\omega^2 - \omega_{\mathrm{pe}}^2}{c^2} \tag{10.22}$$

(d) Sketch the dispersion relation and comment on the physical significance of the dispersion near the region $\omega = \omega_{\mathrm{pe}}$.

APPENDIX: A Glossary of Useful Formulae

Chapter 1: Basic plasma phenomena

$$\omega_{\rm pe} = \sqrt{\frac{e^2 n_{\rm e}}{\varepsilon_0 m_{\rm e}}}$$

$$\lambda_{\rm D} = \sqrt{\frac{\varepsilon_0 k_{\rm B} T_{\rm e}}{n_{\rm e} e^2}}$$

$$f_{pe} \simeq 9\sqrt{n_{\rm e}}(\,{\rm Hz})$$

$$\frac{n_i}{n} \simeq 2.4 \times \times 10^{21} \frac{T_{\rm e}^{\frac{3}{2}}}{n_i} \exp \frac{-U_i}{k_{\rm B} T}$$

Chapter 2: Kinetic theory

$$\frac{\partial f}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla}_r.f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).\boldsymbol{\nabla}_v.f = \left(\frac{\partial f}{\partial t}\right)_{\rm coll}$$

$$n_{\rm e} = n_{\rm e0} \exp\left(\frac{e\phi}{k_{\rm B} T_{\rm e}}\right)$$

$$\Gamma = n\bar{\boldsymbol{v}}$$

$$\lambda_{\rm mfp} = \frac{1}{n\sigma}$$

$$\boldsymbol{j} = qn\bar{\boldsymbol{v}}$$

$$p = \frac{2}{3} n \bar{U}_r$$

$$\tau = \frac{\lambda_{\rm mfp}}{v}$$

$$\nu = n\sigma v$$

$$f_M(v) = A\exp(\frac{-mv^2}{2k_{\rm B}T}) = A\exp\left(-v^2/v_{\rm th}^2\right)$$

$$b_0 = \frac{2qq_0}{4\pi\varepsilon_0 m v^2}$$

$$\bar{U}_r(\text{Maxwellian}) \equiv E_{\rm Av} = \frac{1}{2} k_{\rm B} T(1 - {\rm D})$$

$$\ln\Lambda = \ln\langle\frac{\lambda_{\rm D}}{b_0}\rangle$$

$$1\,{\rm eV} \simeq 11,600\,{\rm K}$$

$$\sigma^{\rm ei}_{\rm coulomb} \simeq \frac{Z^2 e^4 \ln\Lambda}{2\pi\varepsilon_0^2 m_{\rm e}^2 v_e^4}$$

$$v_{\rm rms} = \sqrt{\frac{3k_{\rm B}T}{m}}$$

$$\delta E_{\rm ei} \sim \frac{4E_e m_{\rm e}}{m_{\rm i}}$$

$$v_{\rm th} = \sqrt{\frac{2k_{\rm B}T}{m}}$$

$$P_{\rm ei} = -\frac{m_{\rm e} n_{\rm e}(\boldsymbol{u}_{\rm e} - \boldsymbol{u}_{\rm i})}{\tau_{\rm ei}}$$

$$p_j = n_j k_{\rm B} T_j$$

Chapter 2: Fluid and Maxwell's equations

$$\begin{aligned}
\sigma &= n_{\rm i} q_{\rm i} + n_{\rm e} q_{\rm e} \\
\boldsymbol{j} &= n_{\rm i} q_{\rm i} \boldsymbol{u}_{\rm i} + n_{\rm e} q_{\rm e} \boldsymbol{u}_{\rm e} \\
\frac{\partial n_j}{\partial t} + \nabla.(n_j \boldsymbol{u}_j) &= 0 \\
m_j n_j \left[\frac{\partial \boldsymbol{u}_j}{\partial t} + (\boldsymbol{u}_j.\nabla)\boldsymbol{u}_j\right] &= q_j n_j(\boldsymbol{E} + \boldsymbol{u}_j \times \boldsymbol{B}) - \nabla p_j + P_{\rm coll} \\
p_j &= C_j n_j^{\gamma_j} \\
\nabla.\boldsymbol{E} &= \frac{\sigma}{\varepsilon_0} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}
\end{aligned}$$

$$\nabla.\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$$

Chapter 3: Gaseous Electronics

$$\Gamma_j = n\boldsymbol{u}_j = \pm\mu_j n\boldsymbol{E} - D_j\nabla n$$

$$\mu = \frac{|q|}{m\nu}$$

$$D = \frac{k_{\mathrm{B}}T}{m\nu}$$

$$\boldsymbol{E} = \eta\boldsymbol{j}$$

$$\eta = \frac{\nu_{\mathrm{ei}}m_{\mathrm{e}}}{n_{\mathrm{e}}e^2}$$

$$\eta \simeq \frac{Ze^2\sqrt{m_{\mathrm{e}}}\ln\Lambda}{6\sqrt{3}\pi\varepsilon_0^2(k_{\mathrm{B}}T_{\mathrm{e}})^{3/2}}$$

$$\eta_{\parallel} = \frac{5.2\times10^{-5}Z\ln\Lambda}{T_{\mathrm{e(eV)}}^{\frac{3}{2}}}$$

$$I = \frac{I_0e^{\alpha x}}{(1-\gamma e^{\alpha x})}$$

$$J = \frac{4}{9}\sqrt{\frac{2e}{m_{\mathrm{i}}}}\frac{\varepsilon_0|\phi_w|^{3/2}}{d^2}$$

$$u_{\mathrm{Bohm}} = \sqrt{\frac{k_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{i}}}}$$

$$\frac{e\phi_w}{k_{\mathrm{B}}T_{\mathrm{e}}} \approx \frac{1}{2}\ln\left(\frac{2\pi m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)$$

$$I_{\mathrm{si}} \simeq \frac{1}{2}n_0eA\sqrt{\frac{k_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{i}}}}$$

Chapter 4: Single Particle Motions

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$$

$$\omega_{\mathrm{c}} \equiv \frac{|q|B}{m}$$

$$r_{\mathrm{L}} = \frac{v_\perp}{\omega_{\mathrm{c}}}$$

$$\mu = \frac{mv_\perp^2}{2B}$$

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$$

$$\boldsymbol{v}_F = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}$$

$$\boldsymbol{v}_R = \frac{mv_\parallel^2}{qB^2}\frac{\boldsymbol{R}_{\mathrm{c}}\times\boldsymbol{B}}{R^2}$$

$$\boldsymbol{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_{\mathrm{L}}\frac{\boldsymbol{B}\times\nabla B}{B^2}$$

$$\boldsymbol{v}_P = \frac{1}{\omega_{\mathrm{c}}}\frac{\dot{\boldsymbol{E}}}{B}$$

$$\boldsymbol{j} = \overleftrightarrow{\sigma}\,\boldsymbol{E}$$

$$F_\parallel = -\mu\nabla_\parallel B$$

$$v_\parallel = \left[\frac{2}{m}(K-\mu B)\right]^{1/2}$$

$$\frac{B_m}{B_0} = \frac{1}{\sin^2\theta_m}$$

$$q(r) = \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \frac{rB_0}{RB_\theta} = \epsilon\frac{B_0}{B_\theta}$$

$$\overleftrightarrow{\sigma}_{\mathrm{e}} = \frac{\mathrm{i}ne^2}{m_{\mathrm{e}}\omega}\begin{pmatrix} \frac{\omega^2}{\omega^2-\omega_{\mathrm{ce}}^2} & \frac{-\mathrm{i}\omega_{\mathrm{ce}}\omega}{\omega^2-\omega_{\mathrm{ce}}^2} & 0 \\ \frac{\mathrm{i}\omega_{\mathrm{ce}}\omega}{\omega^2-\omega_{\mathrm{ce}}^2} & \frac{\omega^2}{\omega^2-\omega_{\mathrm{ce}}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overleftrightarrow{\varepsilon} = \varepsilon_0\left(\overleftrightarrow{I} + \frac{\mathrm{i}}{\varepsilon_0\omega}\overleftrightarrow{\sigma}\right)$$

Chapter 5: Magnetized Plasmas

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} + \frac{-\nabla p\times\boldsymbol{B}}{qnB^2}$$

$$\boldsymbol{j}_D = (k_{\mathrm{B}}T_{\mathrm{i}}+k_{\mathrm{B}}T_{\mathrm{e}})\frac{\boldsymbol{B}\times\nabla n}{B^2}$$

$$\boldsymbol{u}_\perp = \pm\mu_\perp \boldsymbol{E} - D_\perp \frac{\nabla n}{n} + \frac{\boldsymbol{u}_E + \boldsymbol{u}_D}{1+\nu^2/\omega_c^2}$$

$$\mu_\perp = \frac{\mu}{1+\omega_c^2\tau^2}$$

$$D_\perp = \frac{D}{1+\omega_c^2\tau^2}$$

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}$$

$$\begin{aligned} \sigma_\perp &= \sigma_0 \frac{\nu^2}{\nu^2+\omega_c^2} \\ \sigma_H &= \sigma_0 \frac{\mp\nu\omega_c}{\nu^2+\omega_c^2} \\ \sigma_\parallel &= \sigma_0 = \frac{ne^2}{m\nu} \end{aligned}$$

$$D_\perp = \frac{\eta_\perp \sum n_s k_B T_s}{B^2}$$

Chapter 5: Single Fluid Equations

$$\begin{aligned} \rho\frac{\partial \boldsymbol{u}}{\partial t} &= \boldsymbol{j}\times\boldsymbol{B} - \nabla p + \rho\boldsymbol{g} \\ \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} &= \eta\boldsymbol{j} \\ \frac{\partial\rho}{\partial t} + \nabla\boldsymbol{.}(\rho\boldsymbol{u}) &= 0 \\ \frac{\partial\sigma}{\partial t} + \nabla\boldsymbol{.}\boldsymbol{j} &= 0 \\ \nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\times\boldsymbol{B} &= \mu_0\boldsymbol{j} \\ p &= Cn^\gamma \end{aligned}$$

Chapter 6: Magnetohydrodynamics

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\boldsymbol{B}\boldsymbol{.}\nabla)\boldsymbol{B}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \frac{\eta}{\mu_0}\nabla^2 B + \nabla\times(\boldsymbol{u}\times\boldsymbol{B})$$

$$R_M = \frac{\mu_0 vL}{\eta}$$

Chapter 7, 8, 9: Waves

$$\boldsymbol{v}_g = \frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{k}}$$

$$V_A = \left(\frac{B^2}{\mu_0\rho}\right)^{/2}$$

$$V_S = \left(\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}\right)^{1/2}$$

$$v_\phi = \frac{\omega}{k} = \frac{c}{(1-\omega_{pe}^2/\omega^2)^{1/2}}$$

$$\omega^2 = \omega_{pe}^2 + \frac{3k_B T}{m}k^2$$

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{E}) + \overleftrightarrow{K}\boldsymbol{.}\boldsymbol{E} = 0$$

$$\boldsymbol{n} = \frac{c}{\omega}\boldsymbol{k}$$

$$n = |\boldsymbol{n}| = ck/\omega = c/v_\phi$$

$$\overleftrightarrow{K} = \overleftrightarrow{\epsilon}/\varepsilon_0 = \begin{pmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$S = 1 - \sum_{\mathrm{i,e}} \frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_{\mathrm{c}}^2}$$

$$D = \sum_{\mathrm{i,e}} \pm \frac{\omega_{\mathrm{p}}^2 \omega_{\mathrm{c}}}{\omega(\omega^2 - \omega_{\mathrm{c}}^2)}$$

$$P = 1 - \sum_{\mathrm{i,e}} \frac{\omega_{\mathrm{p}}^2}{\omega^2}$$

$$\begin{aligned} R &= S + D && \text{Right} \\ L &= S - D && \text{Left} \\ S &= (R+L)/2 && \text{Sum} \\ D &= (R-L)/2 && \text{Diff} \\ P & && \text{Plasma} \end{aligned}$$

$$\tan^2\theta = \frac{P(n^2 - L)(n^2 - R)}{(n^2 - P)(RL - n^2 S)}$$

$$\left(\frac{c^2}{v_\phi^2}\right)_{\substack{L\\R}} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega(\omega \pm \omega_{\mathrm{ce}})} - \frac{\omega_{\mathrm{pi}}^2}{\omega(\omega \mp \omega_{\mathrm{ci}})}$$

$$\omega_{0L} = \left[-\omega_{\mathrm{ce}} + (\omega_{\mathrm{ce}}^2 + 4\omega_{\mathrm{pe}}^2)^{1/2}\right]/2$$

$$\omega_{0R} = \left[\omega_{\mathrm{ce}} + (\omega_{\mathrm{ce}}^2 + 4\omega_{\mathrm{pe}}^2)^{1/2}\right]/2$$

$$n^2 = \frac{c^2}{v_\phi^2} = \frac{(\omega^2 - \omega_{0L}^2)(\omega^2 - \omega_{0R}^2)}{\omega^2(\omega^2 - \omega_{UH}^2)}$$

$$\omega_{UH} = (\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{ce}}^2)^{1/2}$$

$$\omega_{LH} \approx (\omega_{\mathrm{ci}}\omega_{\mathrm{ce}})^{1/2}$$

$$n^2 = \frac{c^2}{v_\phi^2} = \frac{(\omega^2 - \omega_{0L}^2)(\omega^2 - \omega_{0R}^2)}{\omega^2(\omega^2 - \omega_{UH}^2)}$$

Useful Mathematical Identities

$$\boldsymbol{A}.(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}.(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}.(\boldsymbol{A}\times\boldsymbol{B})$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = \boldsymbol{B}(\boldsymbol{C}.\boldsymbol{A}) - \boldsymbol{A}(\boldsymbol{C}.\boldsymbol{B})$$

$$\nabla.(\phi\boldsymbol{A}) = \boldsymbol{A}.\nabla\phi + \phi\nabla.\boldsymbol{A}$$

$$\nabla\times(\phi\boldsymbol{A}) = \nabla\phi\times\boldsymbol{A} + \phi\nabla\times\boldsymbol{A}$$

$$\begin{aligned} \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) &= \nabla(\boldsymbol{A}.\boldsymbol{B}) - (\boldsymbol{A}.\nabla)\boldsymbol{B} \\ &- (\boldsymbol{B}.\nabla)\boldsymbol{A} - \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \end{aligned}$$

$$(\boldsymbol{A}.\nabla)\boldsymbol{A} = \nabla(\frac{1}{2}A^2) - \boldsymbol{A}\times(\nabla\times\boldsymbol{A})$$

$$\nabla.(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}.(\nabla\times\boldsymbol{A}) - \boldsymbol{A}.(\nabla\times\boldsymbol{B})$$

$$\begin{aligned} \nabla\times(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{A}(\nabla.\boldsymbol{B}) - \boldsymbol{B}(\nabla.\boldsymbol{A}) \\ &+ (\boldsymbol{B}.\nabla)\boldsymbol{A} - (\boldsymbol{A}.\nabla)\boldsymbol{B} \end{aligned}$$

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla.\boldsymbol{A}) - (\nabla.\nabla)\boldsymbol{A}$$

$$\nabla\times\nabla\phi = 0$$

$$\nabla.(\nabla\times\boldsymbol{A}) = 0$$

$$\int_{-\infty}^{\infty} v^2 \exp(-av^2)\mathrm{d}v = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

Cylindrical coordinates

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\phi}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla.\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned} \nabla\times\boldsymbol{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) \\ &+ \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{z}} \end{aligned}$$

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