

Chapter 5

MAGNETIZED PLASMAS

5.1 Introduction

We are now in a position to study the behaviour of plasma in a magnetic field. In the first instance we will re-examine particle diffusion and mobility with magnetic field included. It will be shown that the importance of magnetic effects depends on the ratio of the collision and cyclotron frequencies. When the collision frequency is high, the magnetic field is not felt by the plasma. When the collision frequency is low, and since a fluid element is composed of many individual particles, we expect the fluid to exhibit drifts if the guiding centres drift. However, there is also a drift associated with the pressure gradient that is not found in the single particle picture.

Later in this chapter, we combine the electron and ion fluid equations into a pair of equations – force balance and Ohm’s law – that describe the plasma as a single conducting fluid. These equations will be used to treat a number of important problems and will be useful as a platform for the study of low frequency wave behaviour in a plasma.

5.2 Diamagnetic current

In the previous chapter it was noted that plasma particles are diamagnetic - i.e. they produce a magnetic flux which opposes the ambient magnetic field. The amount of expelled flux depends on the particle thermal energy. When taken over the volume of the plasma, this diamagnetism must give rise to a nett current flowing in the plasma – the *diamagnetic current*. To see how this arises, consider the equation of motion and ignore collisions:

$$mn \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p. \quad (5.1)$$

Again, we use the notation \mathbf{u} to designate mean fluid velocities. The ratio of the first and third terms is

$$\begin{aligned} \frac{mn\partial\mathbf{u}/\partial t}{qn\mathbf{u}\times\mathbf{B}} &\sim \frac{mni\omega u_{\perp}}{qn u_{\perp} B} \\ &\sim \frac{\omega}{\omega_c}. \end{aligned}$$

We assume $\omega \ll \omega_c$ and neglect also the convective term on the left side (we show why later). Now

$$\begin{aligned} (\mathbf{u}\times\mathbf{B})\times\mathbf{B} &= \mathbf{B}(\mathbf{B}\cdot\mathbf{u}) - \mathbf{u}(\mathbf{B}\cdot\mathbf{B}) \\ &= B^2 u_{\parallel} \hat{\mathbf{k}} - \mathbf{u}_{\perp} B^2 - B^2 u_{\parallel} \hat{\mathbf{k}} \\ &= -\mathbf{u}_{\perp} B^2 \end{aligned}$$

so that taking the cross product of Eq. (5.1) with \mathbf{B} gives

$$0 = qn(\mathbf{E}\times\mathbf{B} - \mathbf{u}_{\perp} B^2) - \nabla p \times \mathbf{B}.$$

Solving for the velocity gives

$$\begin{aligned} \mathbf{u}_{\perp} &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{-\nabla p \times \mathbf{B}}{qnB^2} \\ &= \mathbf{u}_E + \mathbf{u}_D. \end{aligned} \tag{5.2}$$

The first term on the right is immediately recognizable as the $\mathbf{E}\times\mathbf{B}$ drift. The second term is the so-called *diamagnetic drift* – a fluid effect. Note that \mathbf{u}_D depends on the particle charge and so gives rise to a current flow:

$$\begin{aligned} \mathbf{j}_D &= ne(\mathbf{u}_{Di} - \mathbf{u}_{De}) \\ &= (k_B T_i + k_B T_e) \frac{\mathbf{B}\times\nabla n}{B^2}. \end{aligned} \tag{5.3}$$

This current flows in such a way as to cancel the imposed field (see Fig. 5.1). For a plasma cylinder we have

$$u_D = \frac{-\nabla p}{qnB} = \mp \frac{\gamma k_B T_e}{eB} \frac{\nabla n}{n} \tag{5.4}$$

Aside Justification for the neglect of the second convective term in the equation of motion:

If $\mathbf{E} = 0$ and ∇ points in the $-\hat{\mathbf{r}}$ direction and \mathbf{u}_D is in the $\hat{\boldsymbol{\theta}}$ direction then $\mathbf{u}\cdot\nabla = 0$. (\mathbf{u}_E is also in the $\hat{\boldsymbol{\theta}}$ direction if $\mathbf{E} = -\nabla\phi$ is in the radial direction).

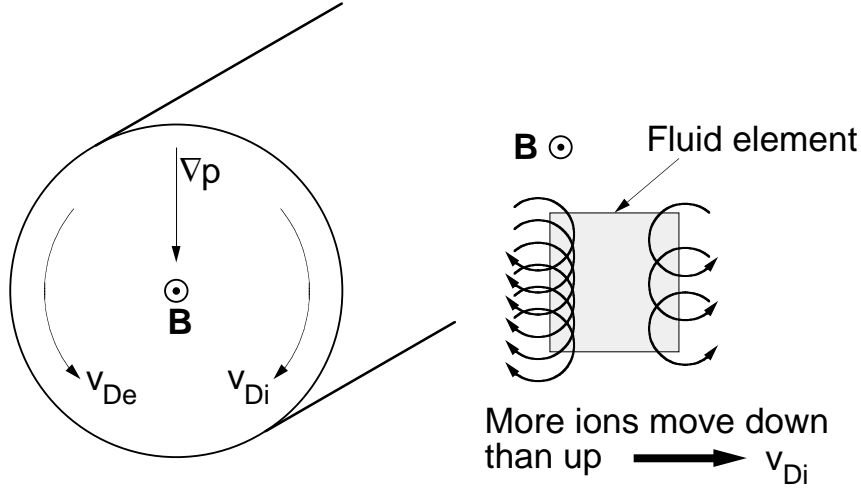


Figure 5.1: Left: Diamagnetic current flow in a plasma cylinder. Right: more ions moving downwards than upwards gives rise to a fluid drift perpendicular to both the density gradient and B . However, the guiding centres remain stationary.

5.3 Particle Transport in a Weakly Ionized Magnetoplasma

We once again use the fluid equation of motion for both electrons and singly charged ions, but retain collisions:

$$mn \frac{\partial \mathbf{u}_{\perp}}{\partial t} = qn(\mathbf{E} + \mathbf{u}_{\perp} \times \mathbf{B}) - k_B T \nabla n + P. \quad (5.5)$$

$P = -mn\mathbf{u}_{\perp}\nu$ is the rate of change of momentum due to neutral collisions (we ignore motion of neutrals). We assume steady state ($\partial/\partial t = 0$) and that the plasma is isothermal $\gamma = 1$. The x and y components of Eq. (5.5) give the coupled equations

$$\begin{aligned} mn\nu u_x &= qnE_x - k_B T \frac{\partial n}{\partial x} + qn u_y B \\ mn\nu u_y &= qnE_y - k_B T \frac{\partial n}{\partial y} - qn u_x B \end{aligned}$$

which become

$$\begin{aligned} u_x &= \pm \mu E_x - \frac{D}{n} \frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu} u_y \\ u_y &= \pm \mu E_y - \frac{D}{n} \frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu} u_x \end{aligned} \quad (5.6)$$

where the mobility and diffusion coefficients are defined by Eqs. (3.4) and (3.5). The plus and minus signs hold respectively for ions and electrons (we have not bothered to distinguish notationally other species specific quantities such as mass etc.). The above equations can be decoupled by substituting for u_x and solving for u_y (and vice versa) to find

$$\begin{aligned} u_x(1 + \omega_c^2 \tau^2) &= \pm \mu E_x - \frac{D}{n} \frac{\partial n}{\partial x} + \omega_c^2 \tau^2 \frac{E_y}{B} \mp \omega_c^2 \tau^2 \frac{k_B T}{eB} \frac{1}{n} \frac{\partial n}{\partial y} \\ u_y(1 + \omega_c^2 \tau^2) &= \pm \mu E_y - \frac{D}{n} \frac{\partial n}{\partial y} - \omega_c^2 \tau^2 \frac{E_x}{B} \pm \omega_c^2 \tau^2 \frac{k_B T}{eB} \frac{1}{n} \frac{\partial n}{\partial x} \end{aligned} \quad (5.7)$$

where $\tau = 1/\nu$ is the collision time. The final two terms in each expression are proportional to the E/B drift and the diamagnetic drift perpendicular to \mathbf{B}

$$\begin{aligned} u_{Ex} &= \frac{E_y}{B} & u_{Ey} &= -\frac{E_x}{B} \\ u_{Dx} &= \mp \frac{k_B T}{eB} \frac{1}{n} \frac{\partial n}{\partial y} & u_{Dy} &= \pm \frac{k_B T}{eB} \frac{1}{n} \frac{\partial n}{\partial x}. \end{aligned} \quad (5.8)$$

We simplify further by defining perpendicular mobility and diffusion coefficients:

$$\mu_{\perp} = \frac{\mu}{1 + \omega_c^2 \tau^2} \quad (5.9)$$

$$D_{\perp} = \frac{D}{1 + \omega_c^2 \tau^2} \quad (5.10)$$

and Eq. (5.7) becomes

$$\mathbf{u}_{\perp} = \pm \mu_{\perp} \mathbf{E} - D_{\perp} \frac{\nabla n}{n} + \frac{\mathbf{u}_E + \mathbf{u}_D}{1 + \nu^2 / \omega_c^2}. \quad (5.11)$$

The expression for the species flow speed perpendicular to the field is composed of two parts:

1. \mathbf{u}_E and \mathbf{u}_D drifts *perpendicular* to $\nabla \phi$ and ∇p (and \mathbf{B}) are slowed down by collisions with neutrals (this can be species dependent and so lead to currents).
2. Mobility and diffusion drifts *parallel* to $\nabla \phi$ and ∇p are reduced by factor $(1 + \omega_c^2 \tau^2)$

When $\omega_c^2 \tau^2 \ll 1$ the magnetic field has a weak effect and vice versa when $\omega_c^2 \tau^2 \gg 1$. In other words, the magnetic field can significantly retard diffusion processes, the diffusion coefficient becoming

$$D_{\perp} \approx \left(\frac{k_B T}{m\nu} \right) \frac{1}{\omega_c^2 \tau^2} = \frac{k_B T \nu}{m \omega_c^2} \quad (5.12)$$

Note that in the magnetized case, the role of collisions is reversed compared with the electrostatic case. Thus

$$\begin{aligned} \parallel \mathbf{B} \quad D &\sim \nu^{-1} && \text{(collisions retard the motion)} \\ \perp \mathbf{B} \quad D_{\perp} &\sim \nu && \text{(collisions needed for cross field migration.)} \end{aligned} \quad (5.13)$$

Also note the mass dependence ($\nu \sim m^{-1/2}$):

$$\begin{aligned} \parallel \mathbf{B} \quad D &\sim m^{-1/2} && \text{(electrons travel faster than ions)} \\ \perp \mathbf{B} \quad D_{\perp} &\sim m^{1/2} && \text{(ions have larger Larmor radius)} \end{aligned} \quad (5.14)$$

It is instructive to look at the scaling of the diffusion coefficients in another way:

$$D = \frac{k_B T}{m\nu} \sim v_{\text{th}}^2 \tau \sim \lambda_{\text{mfp}}^2 / \tau \quad (5.15)$$

$$D_{\perp} = \frac{k_B T \nu}{m\omega_c^2} \sim v_{\text{th}}^2 \frac{r_L^2}{v_{\text{th}}^2} \nu \sim r_L^2 / \tau. \quad (5.16)$$

In the parallel case, the step length for diffusion is the mean free path between collisions. In the magnetized case, the step length is, not surprisingly, the Larmor radius. It is interesting that the fluid theory “knows” about r_L - a single particle quantity.

Ambipolar diffusion across B

This is not a trivial problem due to the anisotropy imposed by the magnetic field. With reference to Fig. 5.2, it would be expected that $\Gamma_{e\perp} < \Gamma_{i\perp}$ because of the smaller Larmor radius of the electrons, and hence the more rapid diffusion loss of ions. We would then expect an electric field to be established which accelerates electrons and retards ions (opposite the electrostatic case). However, in linear systems, the fluxes can be compensated by $\Gamma_{e\parallel} > \Gamma_{i\parallel}$: conducting endplates would short-circuit the ambipolar electric field. A given situation must be assessed using the continuity equation $\nabla \cdot \mathbf{\Gamma}_i = \nabla \cdot \mathbf{\Gamma}_e$ (steady state).

5.4 Conductivity in a Weakly Ionized Magnetoplasma

Let us consider a plasma in equilibrium and ignore pressure gradients. It is instructive to compare the treatment given here with that presented for the electrostatic case in Sec. 3.2. The perpendicular equations (5.7) become

$$\begin{aligned} u_x(1 + \omega_c^2 \tau^2) &= \pm \mu E_x + \omega_c^2 \tau^2 \frac{E_y}{B} \\ u_y(1 + \omega_c^2 \tau^2) &= \pm \mu E_y - \omega_c^2 \tau^2 \frac{E_x}{B}. \end{aligned} \quad (5.17)$$

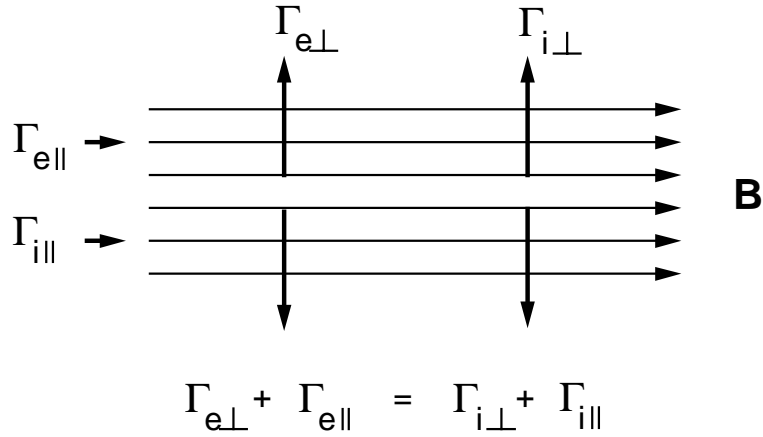


Figure 5.2: Schematic showing the parallel and perpendicular electron and ion fluxes for a magnetized plasma

The associated equation of motion is

$$0 = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - mn\nu\mathbf{u} \quad (5.18)$$

which can be rewritten

$$\mathbf{j} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (5.19)$$

where $\mathbf{j} = nq\mathbf{u}$ is the species current density (\mathbf{j}_e or \mathbf{j}_i) and

$$\sigma_0 = \frac{ne^2}{m\nu} \quad (5.20)$$

is the dc conductivity along \mathbf{B} .

Observe that when $\nu \rightarrow 0$, $\sigma_0 \rightarrow \infty$ implying that $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ and the usual $\mathbf{E} \times \mathbf{B}$ drift is recovered. If \mathbf{u}_i is different from \mathbf{u}_e due to larger collisional retardation of motion of the ions compared with electrons, the fluids no longer move together and σ_0 is finite. There is then a resulting net current flow

$$\mathbf{j}_\perp = en(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) \quad (5.21)$$

which is perpendicular to both \mathbf{E} and \mathbf{B} and is known as the *Hall current*. Since $\mathbf{u}_{i\perp} < \mathbf{u}_{e\perp}$, this current flows in the $-\mathbf{E} \times \mathbf{B}$ direction – i.e. opposite the fluid drift direction.

Let us rewrite Eq. (5.19) in a tensor form that relates \mathbf{j} directly to the electric field:

$$\mathbf{j} = \overleftrightarrow{\sigma} \mathbf{E} \quad (5.22)$$

(we have already seen this equation in relation to single particle motions). Using Eq. (5.17) we obtain

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sigma_0 \begin{pmatrix} \frac{\nu^2}{\nu^2 + \omega_c^2} & \frac{-\nu\omega_c}{\nu^2 + \omega_c^2} & 0 \\ \frac{\nu\omega_c}{\nu^2 + \omega_c^2} & \frac{\nu^2}{\nu^2 + \omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \quad (5.23)$$

allowing us to define a conductivity tensor

$$\vec{\sigma} = \begin{pmatrix} \sigma_{\perp} & -\sigma_H & 0 \\ \sigma_H & \sigma_{\perp} & 0 \\ 0 & 0 & \sigma_{\parallel} \end{pmatrix} \quad (5.24)$$

with

$$\sigma_{\perp} = \sigma_0 \frac{\nu^2}{\nu^2 + \omega_c^2} \quad \text{perpendicular conductivity} \quad (5.25)$$

$$\sigma_H = \sigma_0 \frac{\mp \nu\omega_c}{\nu^2 + \omega_c^2} \quad \text{Hall conductivity} \quad (5.26)$$

$$\sigma_{\parallel} = \sigma_0 = \frac{ne^2}{m\nu} \quad \text{Longitudinal conductivity} \quad (5.27)$$

Note that the Hall term disappears when $\nu \rightarrow 0$. The directions of current flow in a weakly ionized magnetoplasma and the collision-frequency dependence of the conductivity terms are shown in Fig. 5.3.

5.4.1 Conductivity in time-varying fields

The result Eq. (5.24) applies for dc or steady state conditions. In the presence of a time varying electric field (for example and electromagnetic wave), the equation of motion becomes

$$-i\omega nm\mathbf{u} = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - mn\nu\mathbf{u} \quad (5.28)$$

or

$$0 = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - mn\mathbf{u}(\nu - i\omega). \quad (5.29)$$

This is identical to Eq. (5.18) apart from the substitution $\nu \rightarrow (\nu - i\omega)$. The conductivity tensor for oscillating electric fields is therefore of the same form as Eq. (5.24) but with this substitution carried through.

When collisions with neutrals can be neglected, we take $\nu \rightarrow 0$ and the ac conductivity reduces to the expression obtained in the single particle picture Eq. (4.53) but with the understanding that the current in the fluid picture is averaged over the distribution of particle velocities.

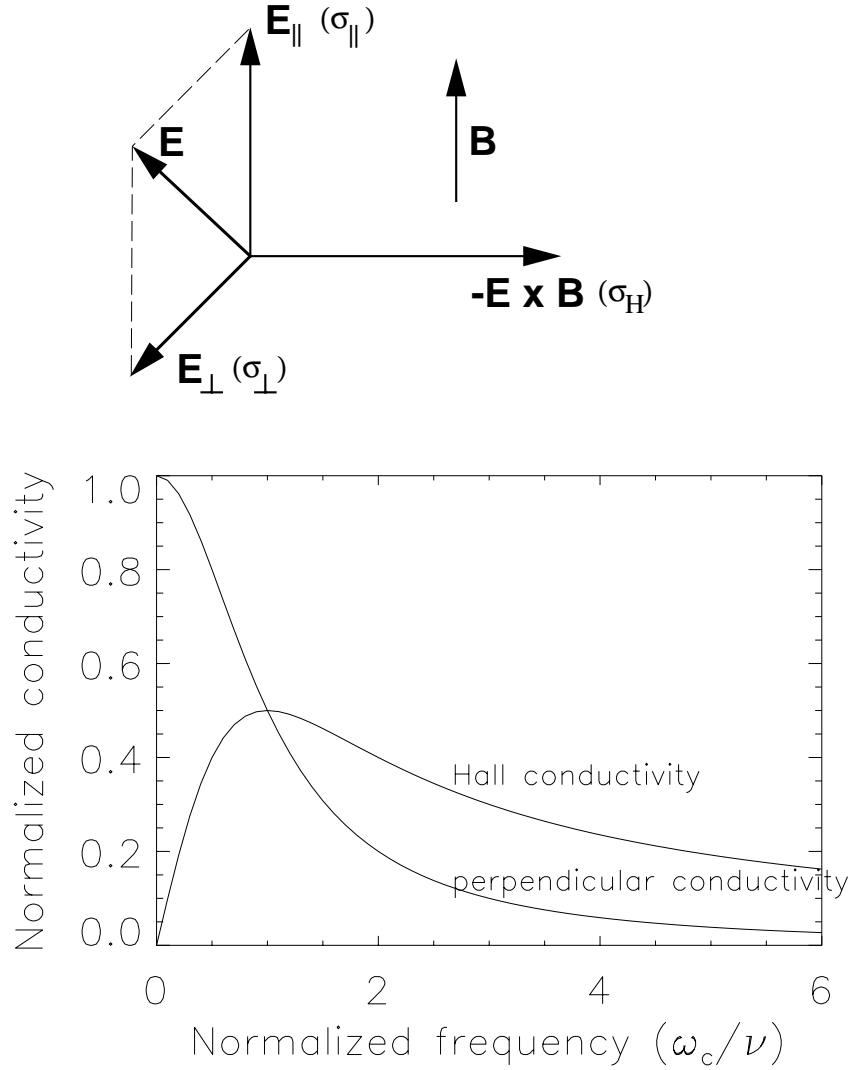


Figure 5.3: Top: The directions of current flow and their associated conductivities in a weakly-ionized magnetoplasma. Bottom: The collision frequency dependence of the perpendicular conductivity.

Again, the dielectric tensor for the plasma can be expressed as

$$\vec{\epsilon} = \epsilon_0 \left(\vec{I} + \frac{i}{\epsilon_0 \omega} \vec{\sigma} \right) \quad (5.30)$$

or

$$\vec{\epsilon} = \epsilon_0 \begin{pmatrix} \epsilon_1 & -\epsilon_2 & 0 \\ \epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \quad (5.31)$$

with

$$\epsilon_1 = 1 + \frac{i}{\omega \epsilon_0} \sigma_{\perp} \quad (5.32)$$

$$\epsilon_2 = \frac{i}{\omega \epsilon_0} \sigma_H \quad (5.33)$$

$$\epsilon_3 = 1 + \frac{i}{\omega \epsilon_0} \sigma_0. \quad (5.34)$$

Particle fluxes are thus produced by either, or both, electromagnetic fields and density gradients and are governed by the associated mobilities and diffusion coefficients.

5.5 Single Fluid Equations

When the plasma is essentially fully ionized, some considerable simplifications in the description of the physics can be obtained by combining the fluid equations for the electrons and ions into a set of equations that describe the single fluid. At frequencies $\omega \ll \omega_{ci}$ this gives rise to the study of *magnetohydrodynamics*. The plasma fluid is then like, say, liquid mercury with a corresponding mass density ρ and conductivity σ .

At low frequencies the role of the electrons is mainly shielding (quasineutrality), while m_e is small so that inertial effects can be neglected. In deriving the *single fluid equations* we thus make the following assumptions and simplifications:

1. Let $m_e/m_i \rightarrow 0$
2. Assume $n_e = n_i$ (quasineutrality). A consequence of this is that we can ignore displacement currents $\epsilon_0 \partial \mathbf{E} / \partial t \rightarrow 0$ in Maxwell's equations (low frequency).
3. Ignore $(\mathbf{u} \cdot \nabla) \mathbf{u}$ (\mathbf{u} is small \Rightarrow quadratic)
4. For generality, allow for non-electromagnetic force $m\mathbf{g}$.

We make the following definitions

$$\rho = n_i m_i + n_e m_e = n(m_i + m_e) \quad (5.35)$$

$$\begin{aligned} \mathbf{u} &= \frac{1}{\rho} (n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e) \\ &\approx \frac{m_i \mathbf{u}_i + m_e \mathbf{u}_e}{m_i + m_e} \end{aligned} \quad (5.36)$$

$$\mathbf{j} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e) = ne(\mathbf{u}_i - \mathbf{u}_e) \quad (5.37)$$

and note that \mathbf{j} need not necessarily be parallel to the \mathbf{u} as was the case for the individual fluids. The electron and ion equations of motion are

$$nm_e \frac{\partial \mathbf{u}_e}{\partial t} = -en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e + nm_e \mathbf{g} + \mathbf{P}_{ei} \quad (5.38)$$

$$nm_i \frac{\partial \mathbf{u}_i}{\partial t} = en(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i + nm_i \mathbf{g} + \mathbf{P}_{ie} \quad (5.39)$$

where for the electron-ion collision term we have $\mathbf{P}_{ie} = -\mathbf{P}_{ei}$. We now form the first of the single fluid equations by adding the ion and electron equations:

$$n \frac{\partial}{\partial t} (m_i \mathbf{u}_i + m_e \mathbf{u}_e) = en(\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B} - \nabla p + n(m_e + m_i) \mathbf{g} \quad (5.40)$$

where $p = p_i + p_e$ and the \mathbf{E} field and collisions cancel. Using our definitions (5.35)-(5.37) we obtain the single fluid equation of motion

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g}. \quad (5.41)$$

There are no electric field effects because the single fluid is neutral!

There are two independent equations for the ions and electrons so that we can use a different linear combination of the two to obtain another single fluid equation. We form $m_e \times (5.39) - m_i \times (5.38)$ to give:

$$\begin{aligned} nm_e m_i \frac{\partial}{\partial t} (\mathbf{u}_i - \mathbf{u}_e) &= ne(m_i + m_e) \mathbf{E} + ne(m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} \\ &\quad - m_e \nabla p_i + m_i \nabla p_e - (m_e + m_i) \mathbf{P}_{ei}. \end{aligned} \quad (5.42)$$

We now simplify each of the terms of this equation in turn:

$$\begin{aligned} nm_e m_i \frac{\partial}{\partial t} (\mathbf{u}_i - \mathbf{u}_e) &= nm_e m_i \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{ne} \right) \\ ne(m_i + m_e) \mathbf{E} &= e\rho \mathbf{E} \\ m_e \mathbf{u}_i + m_i \mathbf{u}_e &= m_e \mathbf{u}_i - m_e \mathbf{u}_e + m_e \mathbf{u}_e + m_e \mathbf{u}_e - m_i \mathbf{u}_i + m_i \mathbf{u}_i \\ &= m_e (\mathbf{u}_i - \mathbf{u}_e) + m_i (\mathbf{u}_e - \mathbf{u}_i) + m_e \mathbf{u}_e + m_i \mathbf{u}_i \\ &= -(m_i - m_e) \frac{\mathbf{j}}{ne} + \frac{\rho}{n} \mathbf{u} \\ \Rightarrow ne(m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} &= e\rho \mathbf{u} - (m_i - m_e) \mathbf{j} \times \mathbf{B} \\ m_e \nabla p_i + m_i \nabla p_e &\approx m_i \nabla p_e \\ (m_e + m_i) \mathbf{P}_{ei} &= -(m_e + m_i) \eta en \mathbf{j} \\ &= e\rho \eta \mathbf{j} \end{aligned}$$

where we have used

$$\begin{aligned} \mathbf{P}_{ei} &= nm_e (\mathbf{u}_i - \mathbf{u}_e) \nu_{ei} \\ &= (m_e \nu_{ei} / e) \mathbf{j} \\ &= ne \left(\frac{m_e \nu_{ei}}{ne^2} \right) \mathbf{j} \\ &= ne \eta \mathbf{j}. \end{aligned} \quad (5.43)$$

Combining the above simplifications and dividing through by $e\rho$ we obtain the intermediate result

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} - \eta \mathbf{j} = \frac{1}{e\rho} \left[\frac{nm_e m_i}{e} \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{n} \right) + (m_i - m_e) \mathbf{j} \times \mathbf{B} + m_e \nabla p_i \right] \quad (5.44)$$

Comparing the magnitudes of the first two terms on the right side:

$$\left(\frac{m_e m_i \omega j}{e} \right) / (m_i j B) \sim \frac{\omega}{\omega_{ce}} \quad (5.45)$$

which implies that we can neglect the first term for low-frequency analyses (unless $\mathbf{j} \parallel \mathbf{B}$ in which case the comparison is invalid). The final result is the *generalized Ohm's law*

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{en} (\mathbf{j} \times \mathbf{B} - \nabla p_e). \quad (5.46)$$

It is often valid to ignore the final two terms in Eq. (5.46) as well. In slowly time varying situations, and ignoring gravity, the force equation Eq. (5.41) yields

$$\nabla p = \mathbf{j} \times \mathbf{B}. \quad (5.47)$$

This is valid provided

$$\left(\rho \frac{\partial \mathbf{u}}{\partial t} \right) / (\mathbf{j} \times \mathbf{B}) \sim \frac{nm_i \omega u}{en u B} \sim \frac{\omega}{\omega_{ci}} \ll 1. \quad (5.48)$$

Using Eq. (5.47), the right side of Eq. (5.46) can be written

$$\frac{1}{en} (\mathbf{j} \times \mathbf{B} - \nabla p_e) = \frac{\nabla p_i}{en} \quad (5.49)$$

and can be ignored for low ion temperature or when the plasma fluid velocity is large [left side of Eq. (5.46)] and the magnetic field strong:

$$\left(\frac{\nabla p_i}{en} \right) / (\mathbf{u} \times \mathbf{B}) \sim \frac{nm_i v_{thi}^2}{ne L u B} = \frac{v_{thi}^2}{u L \omega_{ci}} \sim \frac{r_L}{L} \frac{v_{thi}}{u}.$$

Finally, equations of continuity for mass and charge are obtained from the sum and difference of the ion and electron continuity equations. The full set of *single fluid equations* becomes

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g} \quad (5.50)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \quad (5.51)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (5.52)$$

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (5.53)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (5.54)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (5.55)$$

$$p = C n^\gamma \quad (5.56)$$

5.6 Diffusion in Fully Ionized Plasma

We assume equilibrium ($\partial/\partial t = 0$) and neglect gravity to obtain

$$\nabla p = \mathbf{j} \times \mathbf{B} \quad (5.57)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \quad (5.58)$$

$$E_{\parallel} = \eta_{\parallel} j_{\parallel} \quad \text{parallel component.} \quad (5.59)$$

The perpendicular component of the diffusive flow is, as usual, found by taking the cross product with \mathbf{B} :

$$\mathbf{E} \times \mathbf{B} + (\mathbf{u}_{\perp} \times \mathbf{B}) \times \mathbf{B} = \eta_{\perp} \mathbf{j} \times \mathbf{B} = \eta_{\perp} \nabla p \quad (5.60)$$

to give

$$\mathbf{u}_{\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\eta_{\perp}}{B^2} \nabla p \quad (5.61)$$

where the first term is the bulk plasma drift \mathbf{u}_E and the second term is the diffusion velocity of the single plasma fluid under the action of a pressure gradient.

For a cylindrical plasma with electric field and pressure gradient in the radial direction

$$\begin{aligned} u_{\theta} &= -\frac{E_r}{B} \\ u_r &= -\frac{\eta_{\perp}}{B^2} \frac{\partial p}{\partial r} \end{aligned} \quad (5.62)$$

and the radial fluid flux is

$$\begin{aligned} \Gamma_{\perp} &= n \mathbf{u}_{\perp} \\ &= -\eta_{\perp} \frac{n(k_B T_e + k_B T_i)}{B^2} \nabla n. \end{aligned} \quad (5.63)$$

(Note that we would have also recovered the diamagnetic drift if we had retained the term $(\mathbf{j} \times \mathbf{B})$ in the Ohm's law.) This is reminiscent of Fick's law

$$\Gamma_{\perp} = -D_{\perp} \nabla n \quad (5.64)$$

with the fully ionized magnetized plasma diffusion coefficient given by

$$D_{\perp} = \frac{\eta_{\perp} \sum n_s k_B T_s}{B^2} \quad (5.65)$$

We remark the following important properties:

1. The diffusion coefficient scales with $1/B^2$ as for the weakly ionized case, implying a random walk of step length r_L . To see this note that, for fixed resistivity, $D_{\perp} \sim T/B^2 \sim v^2/B^2 \sim v^2/\omega_c^2 \sim r_L^2$
2. η decreases with T , $\eta \sim T^{-3/2}$ so that diffusion *decreases* as the temperature *increases*! This is a favourable scaling.
3. Diffusion is automatically ambipolar. Both species diffuse at the same rate - this is a consequence of momentum conservation.

5.6.1 Coulomb collisions in magnetoplasma

We have seen that there is very little energy transfer between unlike species due to $m_e \ll m_i$. By contrast however, it is *only* collisions between unlike charges that can give rise to diffusion in a fully ionized magnetized plasma.

Because of conservation of momentum, the centre of mass in collisions between like-charged particles remains unchanged (see Fig. 5.4). For unlike particles, the biggest displacement is suffered for head-on collisions.

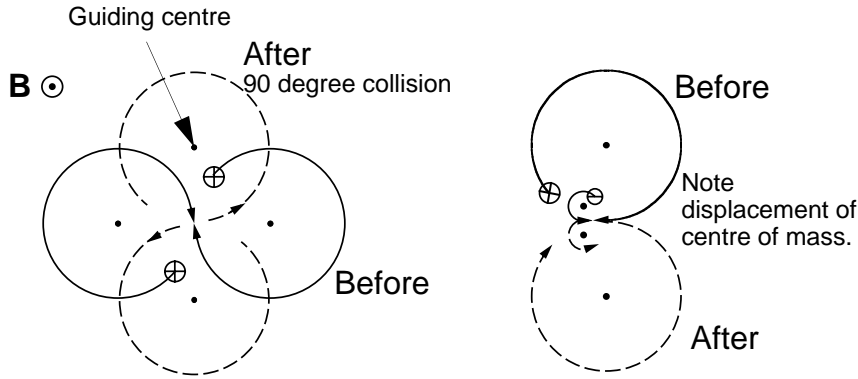


Figure 5.4: Left: Schematic showing particle displacements in direct Coulomb collisions between like species in a magnetized plasma. Right: Collisions between unlike particles effectively displace guiding centres.

Because of the mass disparity, electrons bounce off almost stationary ions and execute a random walk of step length r_L . In general, ions only move slightly, but very often. However, conservation of momentum implies that the diffusion rate for ions and electrons is the same (no charge separation, no electric fields).

5.6.2 Other types of diffusion

Experiments suggest that diffusion actually scales like $1/B$ rather than $1/B^2$. This is bad. The empirically determined *Bohm diffusion coefficient* is given by

$$D_B = \frac{1}{16} \frac{k_B T_e}{eB}. \quad (5.66)$$

Observe that this scales with temperature. How could this situation arise? It is now believed that excess, or *anomalous diffusion* is caused by instabilities and convective drifts that arise spontaneously in the plasma. If we let the escape velocity be proportional to the E/B drift velocity

$$\Gamma_{\perp} = nu_{\perp} \propto nE/B.$$

However,

$$e\phi \sim k_B T_e \quad \text{so} \quad E \sim \phi/L \sim k_B T_e / eL$$

and the driven flux is

$$\begin{aligned} \Gamma_{\perp} &\approx \gamma \frac{n}{L} \frac{k_B T_e}{eB} \\ &\sim -\gamma \frac{k_B T_e}{eB} \nabla n \\ &\approx -D_B \nabla n. \end{aligned} \tag{5.67}$$

We have also already encountered neoclassical diffusion. In this theory, particles make collisions (reflections) with bumps in the magnetic topology. This gives rise to banana orbits and a step length for diffusion that can be significantly greater than the Larmor radius. The variation of perpendicular diffusion constant with particle collision frequency for particle confinement in a toroidal magnetic confinement device is shown in Fig. 5.5.

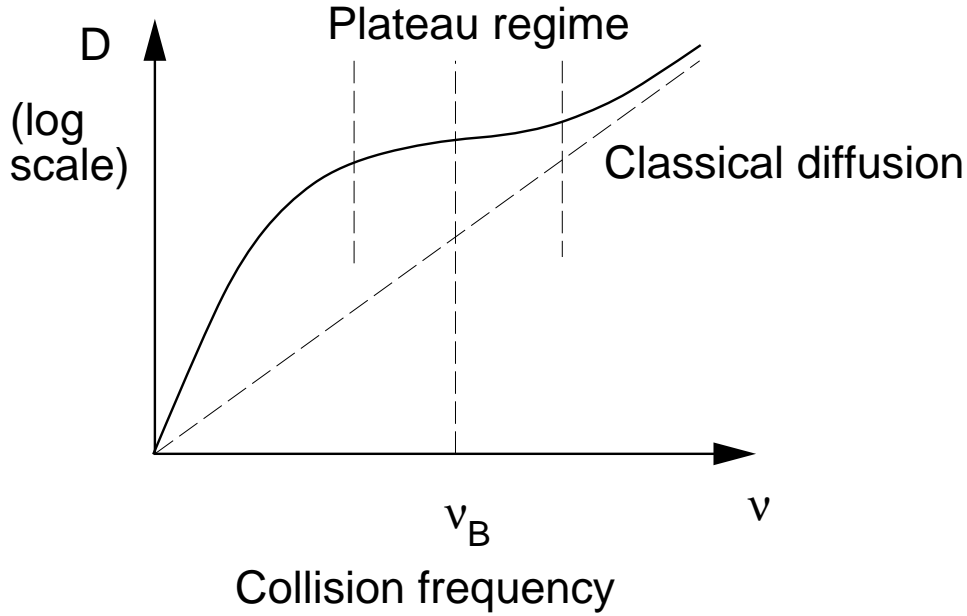


Figure 5.5: The theoretical perpendicular diffusion coefficient versus collision frequency for a tokamak. The region of enhanced diffusion occurs in the so called "plateau" regime centered about the particle bounce frequency in the magnetic mirrors.

Problems

Problem 5.1 A cylindrical plasma column of radius a contains a coaxial magnetic field $\mathbf{B} = B_0 \hat{\mathbf{z}}$ and has pressure profile

$$p = p_0 \cos^2 (\pi r / 2a)$$

- (a) Calculate the maximum value of p_0
- (b) Using this value of p_0 calculate the diamagnetic current $\mathbf{j}(r)$ and the total field $\mathbf{B}(r)$.
- (c) Show $j(r)$, $B(r)$ and $p(r)$ on a graph.

Problem 5.2 A cylindrically symmetric plasma column in a uniform B-field has

$$n(r) = n_0 \exp(-r^2/r_0^2) \quad \text{and} \quad n_i = n_e = n_0 \exp(e\phi/kT_e)$$

- (a) Show that the E/B drift and the electron diamagnetic drift are equal and opposite.
- (b) Show that the plasma rotates as a solid body
- (c) Find the diamagnetic current density as a function of radius.
- (d) Evaluate j_D in A/m² for $B = 0.4\text{T}$, $n_0 = 10^{16}\text{m}^{-3}$, $kT_e = kT_i = 0.25\text{eV}$ and $r = r_0 = 1\text{ cm}$.

