Tomography and reliable information

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Principles of tomography are developed and applied to the problem of two-view interferometry on a tokamak plasma. It is shown that M equispaced views, or projections, of a two-dimensional object yield precisely $M^2 + M$ numbers characterizing the object. This result is an extension of the previous work of Niland [J. Opt. Soc. Am. 72, 1677 (1982)], who proved that M^2 generalized Fourier coefficients, or moments of the object, could be retrieved by M-view tomography. Furthermore it is shown that only half of the alias-free numbers are useful in reconstructing a uniform image of the unknown object. Questions of sampling within a view are addressed and the aliasing contaminants explicitly identified. An algorithm using an orthogonal expansion in the frequency domain is used to examine the attributes of the image reconstructed using various subsets of the available Fourier coefficients.

1. INTRODUCTION

To resolve experimentally many of the issues relating to the physics of magnetically confined fusion plasmas, there is a great need for an accurate, high-resolution spatial and temporal measurement of the two-dimensional plasma electron density distribution. In general, the density distribution is inferred from interferometric measurements of the plasma-produced phase shift in one or more discrete beams of probing radiation. This phase shift is related to the integral of the electron density n_e along the line of sight by¹

$$\phi = r_e \lambda \int n_e \mathrm{d}l,\tag{1}$$

where r_e is the classic electron radius and λ is the wavelength of the probing radiation. In tokamak fusion devices, restricted diagnostic access and technical limitations have meant that measurements of the phase shift have usually been available for only a small set of parallel chords in a vertical cross section of the plasma. It is often a reasonable assumption that the plasma possesses circular symmetry so that $n_e(r)$ may then be recovered by Abel inversion of Eq. (1). In more recent tokamak devices, however, the plasma is more closely bean shaped or D shaped than circular. When the assumption of circular symmetry must be abandoned and not withstanding that several useful approaches have been developed,²⁻⁴ little can be determined with certainty about the plasma profile from a single view or projection (i.e., a complete set of line integrals at fixed angle) and in the absence of other a priori knowledge. To resolve the ambiguities one requires measurements of ϕ at other viewing angles so that tomographic reconstruction techniques can be used.

Advances in far-infrared detector technology,⁵ coupled with the application of imaging techniques, have recently culminated in the construction and operation of a two-dimensional phase-imaging interferometer capable of providing simultaneously forty or more channels of information in two orthogonal views of the UCLA Microtor tokamak plasma.⁶⁻⁸ This enhanced capability has permitted extraction for the first reported time of two-dimensional information about the plasma distribution that in no way depends on imposed *a priori* assumptions or constraints. Clearly, however, with just two complete projections or views of the plasma, the amount of reliably recovered information is small, and the reconstructed image will reveal only the gross features of the plasma.

This assertion is made precise by Niland,⁹ who shows that, given a system providing M complete views of an object that are equispaced in angle, one may extract from the projections M^2 real numbers that characterize the object. (A complete view is defined as the collection of all line integrals through the tomographic object along lines parallel to a given line.) These numbers are the generalized moments of the object and are free from angular aliasing contamination. More particularly, the moments can be shown to correspond to a well-defined subset of the expansion coefficients in an orthogonal polynomial representation of the source and its set of projections first developed by Cormack¹⁰ and often used in x-ray tomography on plasmas.

The question of reliable information and reconstruction from a finite set of complete projections has also been addressed by Klug and Crowther.¹¹ Adopting a Fourier-space approach, they pose the reconstruction process as an eigenvalue problem and identify three distinct boundaries or cutoffs in the eigenvalue spectrum. The first cutoff encompasses the collection of unaliased eigenfunctions, which yields a reconstruction that is uniformly resolved. A uniform reconstruction is one that reliably represents the object spectrum out to some maximum frequency ω_0 . The second cutoff includes the above functions as well as those unaliased eigenfunctions having the same condition for extraction of their associated eigenvalues from the projection data (and so subject to the same level of noise amplification) as any of those contributing to the uniform reconstruction. The radial terms in this set are characterized by a particular value of the parameter $\nu = l + 2s$, which, in addition to governing the noise susceptibility, determines the number of zeros and so to some extent the resolution within a projection obtained by the polynomial eigenfunction. This boundary is thus called the l + 2s cutoff. Finally, they identify the set consisting of all the functions that are uncontaminated by azimuthal aliasing.

The main thrust of this paper is to unify these two approaches and to consider the validity of the results in the usual case of discretely sampled projections. Since this investigation was motivated by a need to extract reliable information from the two-view Microtor interferometer (whose projections are band limited by diffraction and sampled at the Nyquist rate), some specific attention is given to the problems of such small M configurations.

The paper is structured as follows. In Section 2 we briefly examine the frequency-space properties of a bounded twodimensional object and develop an orthogonal expansion that, by way of the projection theorem, is shown to be related directly to the Cormack expansions for the source and its set of projections. By using this representation, it is shown that Niland's fundamental result can in fact be extended to include an extra M alias-free numbers that are associated with the Mth azimuthal term in the Fourier series expansion of the source. An intuitive explanation for Niland's result is provided by examination of the frequency-plane behavior of such a harmonic expansion.

In Section 3 we address the problem of reconstructing an image from the known set of reliable quantities. It is shown that only half of the $M^2 + M$ low-order reliable moments are required to produce an image that is uniform in the sense defined in Ref. 11. The *M*th-harmonic terms, in the absence of other knowledge about the object, are not useful for image reconstruction. Nevertheless the additional information, in a given application, may be extremely valuable, particularly when the number of views is small. It is found that the cutoff in the spectrum of moments required for a uniform reconstruction in this Cormack representation also includes all those unaliased functions having the same level of noise susceptibility as any member of the uniform set. In this picture, then, the uniform and the l + 2s cutoffs are no longer distinct.

When the M projections are not assumed complete, i.e., are sampled at only a finite number N of discrete chordal positions (the situation usually encountered in practice), the above results can no longer be strictly applied. What can be asserted in a given situation depends on the nature of the object (its bandwidth) as well as the number of projections and the sampling rate within the projections. These points are dealt with in some detail in Section 3. Finally, in Section 4 an algorithm for image reconstruction that employs the above-mentioned frequency-plane representation and that is appropriate when the numbers of samples M and N are large is proposed and demonstrated on some computer-generated phantoms. This approach permits manipulation of the Fourier image before inversion while retaining the favorable attribute of using only those numbers that are known to be relatively free of aliasing contamination.

2. TOMOGRAPHY AND RELIABLE INFORMATION

We now give a frequency-plane derivation of Niland's result. This approach yields the main result in a slightly more intuitive and insightful manner by consideration of the frequency-space properties of a bounded two-dimensional object. The projection theorem, which links the two-dimensional Fourier transform of the unknown function with the one-dimensional transform of its projections, provides a powerful tool for relating these properties to the projections. In addition, examination of the ω -space behavior of a reconstruction based on the quantities known to be free from aliasing contamination reveals how uniformly features of a given size are represented in the domain of the final image. This is important since it is desirable that a tomography system using a finite number of views act as an ideal low-pass filter and so transmit the spectrum of the source function uniformly, i.e., without distortion, out to some effective system band limit. Finally, a reconstruction algorithm that uses the Fourier-plane representation of the source function and can be effectively implemented on computer is shown to have some advantages.

Let the object $\psi(x, y)$ be contained in the unit circle, and let $G(t, \phi)$ be the line integral of $\psi(x, y)$ along the straight line $y \sin \phi + x \cos \phi = t$, where $0 \le \phi < 2\pi$ and $-1 \le t \le 1$. The function *G* is called the shadow of the object ψ , and the set of line integrals or rays at fixed ϕ for $-1 \le t \le 1$ is called a view or projection of ψ . The projection, or central-slice, theorem then gives the following fundamental result:

$$\int_{-\infty}^{\infty} dt G(t, \phi) \exp(-i\omega t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \psi(x, y)$$
$$\times \exp[-i\omega(x \cos \phi + y \sin \phi)]$$
$$= \Psi(u, v), \tag{2}$$

where $\Psi(u, v)$ is the two-dimensional Fourier transform of $\psi(x, y)$ and $(u, v) = (\omega \cos \phi, \omega \sin \phi)$ are the spatial frequency coordinates in ω space. A set of M equispaced views thus generates a pattern of 2M lines or spokes radiating from the origin and separated in angle by π/M in the Fourier-transform plane of ψ . This is depicted for M = 2 views in Fig. 1. Without loss of generality, we take these lines to be at angles

$$\{\phi_m: \phi_m = m\pi/M, \quad m = 0, 1, 2, \dots, 2M-1\},\$$

corresponding to a set $\{G_M(t)\}$ of equispaced views at the first M angles ϕ_m :

$$\{G_m(t):G_m(t) = G(t,\phi_m), m = 0, 1, 2, \dots, M-1\}.$$

Note that the projections are assumed complete.

Since ψ is contained within the unit circle, it may also be expressed in terms of polar coordinates. We choose now to denote this object $\psi(r, \theta)$, and the natural periodicity in azimuth facilitates expansion as a Fourier series:

$$\psi(r,\theta) = \sum_{l=-\infty}^{\infty} \exp(il\theta) f_l(r).$$
(3)

An alternative expression for Ψ is obtained by expressing the right-hand side of Eq. (2) in terms of polar coordinates and substituting for $\psi(r, \theta)$ from Eq. (3). Writing $\Psi = \Psi(\omega, \phi)$, we find that

$$\Psi(\omega,\phi) = \sum_{l} \exp(il\phi) F_{l}(\omega), \qquad (4)$$

where the Fourier components are

$$F_{l}(\omega) = 2\pi (-i)^{l} \int_{0}^{\infty} r \mathrm{d}r f_{l}(r) J_{l}(\omega r)$$
(5)



Fig. 1. Geometry for the two-view tomography problem. (a) Orthogonal views of the object $\psi(r, \theta)$ at angles ϕ_1 and $\phi_2 = \phi_1 + \pi/2$. (b) The two views determine the value of the transform of the unknown object along radial lines through the origin at angles ϕ_1 and ϕ_2 .

and are proportional to the Hankel transform of $f_l(r)$ of order l.

Since the object ψ is spatially bounded, its transform Ψ is analytic over the whole ω plane¹² so that the Fourier component F_l can be expanded as a Taylor series:

$$F_l(\omega) = \sum_{n=0}^{\infty} c_{ln} \omega^n$$

It is desirable to orthogonalize this basis, and the expansion in terms of Bessel functions 13

$$\omega^n = 2^n \sum_{s=0}^{\infty} \frac{(n+2s)\Gamma(n+s)}{s!} J_{n+2s}(\omega)$$
(6)

proves useful for this purpose, enabling us to write

$$F_{l}(\omega) = \sum_{\nu} Q_{l\nu} W_{\nu}(\omega), \qquad (7)$$

where

$$W_{\nu}(\omega) = A_{\nu}\omega^{-1}J_{\nu+1}(\omega) \tag{8}$$

and

$$A_{\nu} = [2(\nu + 1)]^{1/2}.$$

The complex coefficients $Q_{l\nu}$ are nonzero only for values of the index ν satisfying

$$\nu = |l| + 2s, \qquad s = 0, 1, 2, \dots$$

The functions $W_{\nu}(\omega)$ exhibit no explicit dependence on l and are orthonormal on $[0, \infty]$ with weight ω . The functions $w_{l\nu}$ = $W_{\nu}(\omega)\exp(il\phi)$ therefore form a complete orthonormal basis for the Hilbert space defined on the Fourier-transform plane $[0, \infty] \times [0, 2\pi)$ having the inner product

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^\infty \omega \mathrm{d}\omega \Psi_1 \Psi_2^*. \tag{9}$$

We may now more concisely write

$$\Psi = \sum_{l,\nu} Q_{l\nu} w_{l\nu}, \qquad (10)$$

where the indices range over the allowable values of l and ν and where $Q_{l\nu} = \langle \Psi, w_{l\nu} \rangle$ are the Fourier coefficients, or moments, of Ψ with respect to the chosen basis $\{w_{l\nu}\}$.

The minimum significant spatial frequency attained by an object bounded by the unit disk and exhibiting an *l*thorder azimuthal variation $(l \neq 0)$ occurs for structures at the periphery of the disk and so is of the order of $\omega \sim l$ rad per unit length. The energy in the azimuthal component $F_l(\omega)$ up to frequency ω for frequencies less than this minimum frequency becomes vanishingly small as $\omega \rightarrow 0$. As a consequence, the Fourier components F_l are largely decoupled for sufficiently low spatial frequencies ($\omega \leq l$). If the number of views (i.e., slices in the frequency plane) is sufficient to satisfy the Nyquist sampling criterion for, say, the *l*th azimuthal harmonic, it would seem that some information about Ψ may be obtained without significant aliasing contamination from higher-order (l' > l) azimuthal features $F_{l'}$ when $\omega < l$.

It is remarkable that $M^2 + M$ low-order moments $Q_{l\nu}$ of the source distribution may be retrieved *exactly* from M projections of ψ , entirely free of angular aliasing contamination. This information, as would be expected from the plausibility argument above, relates to the low-frequency content of the transform. It is not, however, enough to characterize fully the transform in this region since by the analyticity of Ψ exact knowledge of the transform in any region is sufficient to determine the behavior of Ψ on the complete (ω, ϕ) plane. Nevertheless the relative contribution from higher-order components l' > l becomes exceedingly small whenever $\omega < l$, and this is reflected in the behavior of the basis functions $w_{l\nu}$. Gray-scale contour diagrams for some members of this set are shown in Fig. 2. Note that the higher ν functions are localized farther from the origin. Before determining the



Fig. 2. Contours of the real part of the three lowest-order basis functions $w_{l\nu}(\omega, \phi)$ for l = 3 and $\nu = 3, 5, 7$.

privileged subset of basis functions that can be recovered from a finite number of views, however, we relate the moments $Q_{l\nu}$ of Ψ to corresponding quantities characterizing the object ψ and its shadow G.

We first require a result by Born and Wolf¹⁴ that links the functions $W_{\nu}(\omega)$ to the generalized Hankel transform of polynomials orthogonal on the unit disk:

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$$\int_{0}^{1} r dr R_{\nu}^{l}(r) J_{l}(\omega r) = (-1)^{\frac{\nu-1}{2}} W_{\nu}(\omega).$$
(11)

The functions R_{ν}^{l} are Zernike polynomials

$$R_{\nu}^{l}(r) = A_{\nu} \sum_{n=0}^{\frac{\nu-|l|}{2}} \frac{(-1)^{n}(\nu-n)!r^{\nu-2n}}{n!\left(\frac{\nu+l}{2}-n\right)!\left(\frac{\nu-l}{2}-n\right)!}$$

and for given l are orthonormal with weight r on [0, 1]. In a similar fashion to Niland and Schmidt-Harms,¹⁵ we define a Hilbert space on the unit disk $[0, 1] \times [0, 2\pi)$ having the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \int_0^1 r \mathrm{d}r \psi_1 \psi_2^*.$$
 (12)

The functions $u_{l\nu} = R_{\nu}^{l}(r)\exp(il\theta)$ are a complete orthonormal basis spanning this space, so that ψ can be expanded as

$$\psi = \sum_{l=-\infty}^{\infty} \exp(il\theta) \sum_{\nu} q_{l\nu} R_{\nu}^{\ l}(r)$$
$$= \sum_{l,\nu} q_{l\nu} u_{l\nu}, \qquad (13)$$

where $q_{l\nu} = \langle \psi, u_{l\nu} \rangle$ are the moments of ψ with respect to the basis $\{u_{l\nu}\}$. Taking the *l*th-order Hankel transform of $f_l(r)$ and using Eq. (11), we compare with Eq. (7) to obtain

$$Q_{l\nu} = \zeta_{l\nu} q_{l\nu}, \tag{14}$$

where

 $\zeta_{l\nu} = 2\pi (-i)^{\nu}$

and

 $\zeta_{-l\nu} = (-1)^l \zeta_{l\nu}^*,$

so that the moments of ψ are related in a simple way to the moments $Q_{l\nu}$ of its transform. The proportionality constant $\zeta_{l\nu}$ ensures the appropriate symmetry relations between the object and its transform. For example, take Ψ Hermitian. Then $\Psi^*(\omega, \phi) = \Psi(\omega, \phi + \pi)$, requiring that $Q_{-l\nu} = (-1)^l Q_{l\nu}^*$ so that $q_{-l\nu} = q_{l\nu}^*$ as is necessary, since ψ must be real. Since $\zeta_{l\nu}$ shows no explicit l dependence, however, we henceforth simply write ζ_{ν} .

By taking the inverse Fourier transform of Eq. (2), it can be verified that with Ψ expanded as in Eq. (10), the shadow $G(t, \phi)$ can also be expressed as a Fourier series:

$$G(t,\phi) = \sum_{l=-\infty}^{\infty} \exp(il\phi)g_l(t),$$
(15)

where

$$g_{l}(t) = \sum_{\nu} p_{l\nu} V_{\nu}(t)$$
 (16)

and

$$V_{\nu}(t) = \sqrt{2/\pi} (1 - t^2)^{1/2} U_{\nu}(t), \qquad (17)$$

where $U_{\nu}(t)$ is a Chebyshev polynomial of the second kind of degree ν . The $V_{\nu}(t)$ are orthonormal on [-1, 1] with weight

 $(1 - t^2)^{-1/2}$ so that the functions $v_{l\nu} = V_{\nu}(t)\exp(il\phi)$ are a complete orthonormal basis for the projection Hilbert space $[-1, 1] \times [0, 2\pi)$ with the inner product

$$\langle G_1, G_2 \rangle = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_{-1}^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}} G_1 G_2^*.$$
 (18)

With these definitions we have

$$G = \sum_{l,\nu} p_{l\nu} v_{l\nu}, \tag{19}$$

where

$$p_{l\nu} = \langle G, v_{l\nu} \rangle$$

= $\lambda_{\nu} q_{l\nu}$, $\lambda_{\nu} = 2\sqrt{2\pi}/A_{\nu}$. (20)

Therefore the moments of $\psi(r, \theta)$ and $\Psi(\omega, \phi)$ are related directly to the moments of the shadow G of ψ through Eqs. (14) and (20). It will be shown that a well-defined set of moments is still exactly retrievable when only a finite set of projections $\{G_M(t)\}$ is available.

The reason for the chosen representation of Ψ in terms of the functions $W_{\nu}(\omega)$ is now evident. The corresponding expansions for ψ and G in terms of Zernike and weighted Chebyshev polynomials are those first given by Cormack.¹⁰ These polynomials provide a natural pair of bases for the projection and disk Hilbert spaces in the sense that they yield the singular value decomposition of the projection operator that maps the object ψ onto its shadow $G:G = P\psi$.¹⁶ Thus

$$G = P\psi = \sum_{l,\nu} \frac{\langle G, v_{l\nu} \rangle}{\lambda_{\nu}} P u_{l\nu}, \qquad (21)$$

so that

 $Pu_{l\nu} = \lambda_{\nu}v_{l\nu},$

and the scalars $\{\lambda_{\nu}\}$ are the singular values for recovery of ψ from its projections. This representation is often used in tomography on plasmas where the important features such as magnetohydrodynamic modes tend to be localized in ω space and so require only relatively few terms in such a harmonic expansion of the source function.¹⁷

In passing, note that if $\psi(r, \theta)$ is real, it can be expanded as

$$\psi(r,\theta) = \sum_{l=0}^{\infty} \left\{ \left[\sum_{\nu} a_{l\nu} R_{\nu}^{l}(r) \right] \cos l\theta + \left[\sum_{\nu} b_{l\nu} R_{\nu}^{l}(r) \right] \sin l\theta \right\},\$$

where the coefficients are real and are given by

$$a_{l\nu} = (q_{l\nu} + q_{-l\nu}),$$

$$b_{l\nu} = i(q_{l\nu} - q_{-l\nu}),$$
(22)

and the prime indicates that the l = 0 term is to be halved. We shall have cause to refer to the quantities $a_{l\nu}$ and $b_{l\nu}$ explicitly below.

We now show that a special subset of these Fourier coefficients, or moments, is immune to angular aliasing contamination and so in principle is exactly retrievable from M-view tomography. Since the azimuthal part of the Fourier series expansion of Ψ is known only at the 2M points $\{\phi_m\}$, harmonics no higher than the (M-1)th order can be determined fully. The slices in the Fourier plane provided by the projections will in general be aliased by contributions from harmonics of order higher than M-1 so that the functions $\hat{F}_l(\omega)$ obtained from the projections may not reliably represent the true components $F_l(\omega)$. The $\hat{F}_l(\omega)$ can be recovered from Eq. (4) by using the discrete Fourier transform on the set of 2M sampled values of Ψ :

$$\hat{F}_{l}(\omega) = \frac{1}{2M} \sum_{m=0}^{2M-1} \exp(-il\phi_{m})\Psi(\omega,\phi_{m}).$$

On substitution for Ψ from Eq. (4) it follows that

$$\hat{F}_{l} = F_{l} + \sum_{j=1}^{-} (F_{l+2Mj} + F_{l-2Mj}), \qquad (23)$$

where the sum over *j* represents the infinite set of aliases of *l*. The superscript $\hat{}$ is hereafter used to designate quantities that are recovered from a finite number of views and are so subject to aliasing contamination. By substituting for the F_l from Eq. (7) and using the orthogonality of the functions $W_{\nu}(\omega)$, we produce

$$\hat{Q}_{l\nu} = Q_{l\nu} + \sum_{j} (Q_{l+2Mj,\nu} + Q_{l-2Mj,\nu}), \qquad (24)$$

where j is such that $\nu = |l| + 2s = |l \pm 2Mj| + 2s'$ for some integer $s' \ge 0$. Note that the contaminating quantities must have the same index ν as the recovered term.

The important result is that

$$\hat{Q}_{l\nu} = Q_{l\nu}, \qquad |l| = 0, 1, \dots, M-1,$$

$$\nu = |l|, |l| + 2, \dots, 2M - |l| - 2, \qquad (25)$$

so that M equispaced views of an object yield M^2 real quantities $a_{l\nu}$ and $b_{l\nu}$ that are free from azimuthal aliasing contamination. The result follows from Eq. (24) and the fact that the coefficients $Q_{l\nu}$ are zero when $\nu \neq |l| + 2s$ for any positive integer s. This is the result first obtained by Niland. For notational ease, we denote this special set of coefficients by $\{C_M\}$.

Note, however, that Eq. (24) also admits another set of recoverable quantities, namely,

$$\hat{Q}_{M\nu} = \hat{Q}_{-M\nu} = Q_{M\nu} + Q_{-M\nu}$$

= $\zeta_{\nu} a_{M\nu}, \qquad \nu = M, M+2, \dots, 3M-2.$ (26)

By contrast, the $b_{M\nu}$ coefficients remain invisible for the chosen *M*-view geometry. In the Fourier-space representation (Fig. 2), the slices in the frequency plane coincide with the zero lines for the $b_{M\nu}$ basis functions and so provide no information. Hereafter we denote this extra set of real numbers by $\{c_M\}$ and the complete set of $M^2 + M$ alias-free real numbers by $\{S_M\} = \{C_M\} \cup \{c_M\}$.

Since the $b_{M\nu}$ are unknown, a reconstruction that includes the extra set of M retrievable moments $a_{M\nu}$ will be necessarily nonuniform. Nevertheless the availability of such untainted information may be of considerable significance, depending on the shape and orientation of the important features of the tomographic object. This is especially true for the tokamak plasma application of interest here, where, for example, rotating (l = 2)-type features would be revealed by the two-view system as periodic variations in the amplitude



Fig. 3. Q table and the sets $\{S_M\}$, $\{U_M\}$, and $\{A_M\}$ for (M = 4)-view tomography.

of $a_{M\nu}$ at twice the rotation frequency of the l = 2 mode. Some measure of radial information is also obtained, since both c_{22} and a_{24} are retrievable when M = 2. When the plasma cross section is noncircular, two views are sufficient to yield the plasma elongation in the direction parallel to one of the views.

As an aid to visualizing the set of coefficients $\{S_M\}$, they are displayed in tabular form in Fig. 3. For convenience, the case M = 4 has been chosen to demonstrate some of the properties of the Q table,⁹ and attention is restricted to the moments for which l > 0. Only the nonzero coefficients lying upon and above the diagonal boundary $\nu = l$ are shown. The members of the moment set $\{S_M\}$ are linked by solid lines. Elements upon and inside the first and second dashed lines in the table constitute the special sets $\{U_M\}$ and $\{A_M\}$ described in Section 3. It becomes clear, when presented in this format, why the set $\{S_M\}$ is privileged. The possible aliases for the moments having l < M are those quantities (having the same value of ν) that when folded about the position l = M coincide with, and so corrupt, the desired term. There remains a triangular portion of the table, which, because of the nullity of terms below the diagonal, is entirely free of this effect. As already noted, the terms obtained for l = M yield only the real part of the true coefficients and, when ν is sufficiently large, are themselves finally tainted by nonzero negative l terms.

It is appropriate at this point to note that the low-order moments obtained by two-view tomography on a plasma, for example, can be related in a simple fashion to physically more intuitive quantities. We first define the generalized moments by

$$\mu_{lk} = \int \frac{\mathrm{d}\theta}{2\pi} \int r \mathrm{d}r r^k \exp(-il\theta)\psi$$
$$= \frac{1}{2} (\alpha_{lk} + i\beta_{lk}). \tag{27}$$

The average electron density \bar{n}_e in the chosen cross section is therefore given directly from the lowest-order generalized moment of the electron density distribution. Similarly we

may write for the center of mass $(x_c, y_c) = (\alpha_{11}/2\mu_{00},$ $-\beta_{11}/2\mu_{00}$) and for the mean square deviation from the center of mass $\sigma^2 = \mu_{02}/\mu_{00} - (x_c^2 + y_c^2)$. The extra privileged term α_{22} taken together with μ_{02} permits separate determination of the mean square deviations σ_x^2 and σ_y^2 , giving a measure of the elongation of the plasma distribution in a direction parallel to one of the views. These quantities, which can also be written in terms of the unaliased $a_{l\nu}$ and $b_{l\nu}$ in a simple way, have been explicitly calculated for the twoview interferometer data and provide considerable insight into the physical processes determining the plasma behavior.⁸ In passing, we note that a system using only a single view unambiguously yields both a_{00} , the area under the projection or equivalently the total mass of the object ψ on the unit disk, and a_{11} , which is proportional to the displacement of the center of mass of ψ transverse to the direction of view.

3. SAMPLING CONSIDERATIONS

The results obtained in Section 2 assume that the views $G_m(t)$ are complete. In this ideal circumstance, the collection of $M^2 + M$ real numbers $\{S_M\}$ is the most that can be asserted from the data in the absence of other knowledge. In practice, however, the F_l provided by M views are usually known at only a finite set of points [as would be yielded, e.g., by a fast Fourier transform (FFT) of the projection data] so that the $Q_{l\nu}$ are subject to radial aliasing contamination and can no longer be exactly determined. Without some sort of a priori knowledge about the object, such as the maximum significant spatial frequencies that are likely to occur, little can be extracted with confidence from a finite data set.

Fortunately many physical objects are effectively band limited in the sense that the power contained in the spectrum above some cutoff frequency ω_0 is negligibly small. Equivalently, it may be that it is the measuring system itself that sets the effective system bandwidth ω_s . This is the case for the two-view plasma interferometer, in which the optical system forms an image that is band limited by diffraction and is sampled at the Nyquist frequency by an image plane detector array. It is of crucial importance therefore to ascertain the number of views M and samples per view Nrequired to characterize an object that is effectively band limited. The informal Fourier-space argument presented below (see also Refs. 11 and 18) yields the answer in a simple way.

Assume that $\Psi(\omega, \phi)$ is small for $\omega > \omega_0$. Let ψ be such an object of effective bandwidth ω_0 . By the projection theorem, the G_m are also effectively band limited and so can be approximately recovered from a finite number N of samples of G_m that are equispaced by π/ω_0 on the interval [-1, 1]. Such a sampling would be free from aliasing contamination to the extent that neglect of higher-frequency components than ω_0 is valid. A discrete Fourier transform performed on the sampled projection will yield values of the transform Ψ at N equispaced points separated by $\Delta \omega = 2\omega_0/N = \pi$ rad per unit length along a cut through the origin. Since the object is confined to the unit circle, the transform Ψ contains no component corrugations having more than a single cycle per unit length in the (ω, ϕ) plane. Consequently the number of equispaced views (or cuts) needed to specify uniquely the two-dimensional transform to the same bandwidth ω_0 is fixed by the Nyquist condition that the distance between

points in the plane not exceed $\Delta\omega$. This yields the approximate relation $M \sim (\pi/2)N$, so that the total number of measurements required to recover ψ is $MN \sim (2/\pi)\omega_0^2$. Evidently, since $N = (2/\pi)\omega_0$, the number of equispaced complete views sufficient to recover the spectrum of an object effectively band limited to ω_0 is $M = \omega_0$. This is consistent with the simple-minded argument following Eq. (10), which suggests that circular frequencies at least up to $\omega \sim M$ rad per unit length should be resolvable without aliasing ambiguity from M complete views.

Below we briefly discuss retrieval of reliable information from discretely sampled projections. First, however, we examine the question of image uniformity and show that a well-defined subset $\{U_M\}$ of the reliable coefficients $\{S_M\}$ is useful for producing an image that is resolved more uniformly than a reconstruction based on the full set $\{S_M\}$.

A. Image Uniformity and Reconstruction Bandwidth

Given M complete views of an arbitrary object, we ask to what extent the numbers $\{S_M\}$ permit reconstruction of an undistorted image of the object. In other words, what is the correspondence between the object spectrum and the privileged moment set discussed in Section 2? To resolve this question we examine the finite dimensional subspace S of the unit disk Hilbert space that is spanned by the basis functions corresponding to the collection of unspoiled moments provided by M views of the object. The projection onto S of the delta function $\delta(\theta - \theta_0)\delta(r - r_0)/r$,

$$h_{S}(r,\theta;r_{0},\theta_{0}) = \sum_{|S_{M}|} u_{l\nu}^{*}(r_{0},\theta_{0})u_{l\nu}(r,\theta)/2\pi,$$
(28)

where the $u_{l\nu}$ are members of the set spanning S, is a measure of the response of this subspace to an impulsive source. It is easy to show¹⁵ that h_s is the unique function in S having the same moments as the delta-function source and hence is the most compact function in the disk Hilbert space that matches the reliable moments. Alternatively h_S is Green's function for the linear operator that maps an object onto its reconstruction in this minimum norm sense, so that the reconstruction ψ_S of an arbitrary function ψ from its reliable moment set $\{S_M\}$ is given by

$$\psi_{S}(\xi,\eta) = \iint r dr d\theta h_{S}(\xi,\eta;r,\theta)\psi(r,\theta).$$
(29)

The Green function may be interpreted as the point-spread function for reconstruction in the $(M^2 + M)$ -dimensional subspace S. Explicit calculations show that h_S is not spatially invariant on the unit disk but has contours that are approximately elliptical with eccentricity a function of r_0 (for the first M^2 moments only¹⁵). When the additional M moments in the set $\{c_M\}$ are included, h_S also depends on θ_0 .

This is an undesirable situation, since features of a given dimension will be reconstructed with varying resolution at different points on the disk. We thus seek a subspace U of Sthat will capture ψ in a more uniform fashion. Uniform in this sense means that the tomography system act as an ideal low-pass filter, with cutoff determined by the number of views. In such circumstances the Green function $h_U(r, \theta; 0,$ 0) would take the familiar Airy form and be position invariant. However, unless $M \to \infty$, the point-spread function h_U



Fig. 4. Energy E_{ν} contained in the ν th basis function $W_{\nu}(\omega, \phi)$ below frequency ω for $\nu = 0-30$.

because of distortion at the boundary of the unit disk. We therefore relax the criterion for uniformity by seeking a subspace U that is *sufficient* to transmit the object spectrum accurately up to some maximum frequency determined by the number of views. In other words, we discard those functions that are not necessary to ensure that Ψ is recovered up to this cutoff, since it is the higher-frequency information that distorts the reconstructed image. Since the Mth-harmonic terms clearly do not belong to this space U, we hereafter consider only the subspace C of S spanned by the M^2 functions corresponding to the coefficients constituting the set $\{C_M\}$.

Now consider Eq. (10) for the object spectrum Ψ . The energy $E_{\nu}(\omega)$ contained in the basis functions W_{ν} below circular frequency ω can be easily calculated as

$$E_{\nu}(\omega) = J_0^{2}(\omega) + J_{\nu+1}^{2}(\omega) + 2\sum_{k=1}^{\nu} J_k^{2}(\omega)$$
(30)

and is shown plotted in Fig. 4 for even values of the index between $\nu = 0$ and $\nu = 30$ and for ω up to a maximum of 60 rad per unit length. The maximum radial order v that is obtained by all angular components in the expansion (10) is $v_{\text{max}} = l_{\text{max}} = M - 1$. From Fig. 4, inclusion of all functions up to and including $\nu = \nu_{max}$ in the reconstruction ensures that the spectrum is reliably recovered up to a limiting frequency $\omega \sim \nu_{\text{max}} = M - 1$. The elements having $\nu > \nu_{\text{max}}$ are not available for all the angular harmonics and so distort the image and are discarded. It is therefore the $(M^2 + M)/2$ functions in *C* having $\nu \leq \nu_{\max}$ that span the space *U*, and we denote by $\{U_M\}$ the subset of corresponding coefficients in $\{C_M\}$. For example, the elements on and to the left of the first dashed line in Fig. 3 constitute the set $\{U_4\}$. It follows that the maximum number of views necessary to resolve fully an object of effective bandwidth ω_2 without aliasing contamination is $M = l_{max} + 1 = v_{max} + 1$, where v_{max} is such that

$$\sum_{\nu > \nu_{\max}} |q_{l\nu}|^2 < \epsilon, \tag{31}$$



Fig. 5. The Green function, or point response, for (M = 16)-view tomography. (a), (c) The Green function h_S calculated for radial positions $r_0 = 0.0$ and $r_0 = 0.5$, respectively. In (b) and (d) the $(M^2 + M)/2$ functions in U have been used to calculate \bar{h}_U for $r_0 = 0.0$ and $r_0 = 0.5$. Note that \bar{h}_U is more nearly uniform on the unit disk than is the point response h_C calculated on the complete subspace C.



Fig. 6. Modulus of the transmitted spectrum of a point source located at three different radii on the unit disk for (M = 16)-view tomography. (a), (b) Response obtained from functions spanning the space U along the u and v axes, respectively; (c) and (d) show the frequency response obtained in the space C.

where the quantity ϵ is a measure defining the degree to which ψ is effectively band limited.

For general objects, reconstruction in the space U is governed by the properties of the point-spread function on the unit disk. Figure 5 compares the point-spread functions h_U and h_S for $r_0 = 0$ and $r_0 = 0.5$ when M = 16. The greater uniformity displayed by h_U is obviously achieved at the expense of spatial resolution. Even so, h_U also necessarily distorts near the edge of the disk where the high ν moments dominate. A measure of the increased distortion of h_S compared with h_U as r_0 increases is the percentage of total power represented by the $\nu > \nu_{\text{max}}$ moments. This is given by the ratio

$$\left(\sum_{|S|-|U|} |q_{l\nu}|^2\right) \left\| \left(\sum_{|S|} |q_{l\nu}|^2\right) \right\|$$

and, for moderate M (>10), is found to increase in an almost linear fashion from a minimum of 0.25 at $r_0 = 0$ to ~0.6 at the edge of the disk.

The transmission properties of the tomographic system are perhaps better illustrated by observing how reliably the recovered spectrum $H_C(\omega, \phi; r_0, \theta_0)$ of the object $\delta(\theta - \theta_0)\delta(r - r_0)/r$ represents the flat spectrum of the delta-function source. The frequency-space behavior of h_C is described simply, using Eqs. (10), (14), and (28), as

$$H_{C}(\omega, \phi; r_{0}, \theta_{0}) = \sum_{|C_{M}|} (-i)^{\nu} u_{l\nu}^{*}(r_{0}, \theta_{0}) w_{l\nu}(\omega, \phi).$$
(32)

In Fig. 6 the magnitudes of the transmitted spectra, $|H_U|$ and $|H_C|$, are plotted along lines $\phi = 0^\circ$ and $\phi = 90^\circ$ for an (M = 16)-view system. The disparity of the response $|H_C|$ in the orthogonal directions u and v becomes increasingly significant with increasing r_0 . On the other hand, the uniformity of the response $|H_U|$ suffers only marginally as the impulsive source moves to the edge of the disk.

The trade-off between image uniformity and spatial resolution must be determined by the requirements of the application at hand. However, two further points should also be considered. First, note that the cutoff condition for a uniform reconstruction is equivalent to a cutoff in the spectrum of moments such that $v = |l| + 2s \leq v_{max}$. Consequently inclusion of functions for which $v > v_{max}$, in addition to giving rise to extra distortion in the image ψ_S , will also result in greater noise amplification since the singular values λ_v for recovery of the functions u_{lv} from the projection data decrease as v increases [cf Eq. (21)]. Furthermore, as shown below, when only a discrete set of samples of the projections is available, the coefficients for the functions in U are less susceptible to aliasing contamination and so more likely to yield a reconstruction free of serious artifacts.

B. Aliasing and Discretely Sampled Projections

In practice, the M projections are never complete, so that in general the approximations to the privileged moments derived from discretely sampled projections are no longer strictly immune to aliasing contamination. Below we examine what can be asserted for objects that are effectively band limited and explicitly identify the aliasing contaminants for given object bandwidths and viewing geometries.

For now, take $M \sim \omega_0$, so that M complete projections are sufficient to capture the highest-frequency structures featured in the effectively band-limited object $\overline{\psi}$. Noting that the orthogonal expansion for the harmonic coefficients $g_l(t)$ can, with the change of variable $t = -\cos \tau$, be alternatively expressed as a sine series:

$$g_{l}(\tau) = (-1)^{l} \sqrt{2/\pi} \sum_{\nu=|l|}^{\infty} p_{l\nu} \sin[(\nu+1)\tau], \qquad (33)$$

we take the projection as sampled at the set of N positions specified by

$$\{\tau_n: \tau_n = n\pi/N, \quad n = 1, 2, ..., N\}.$$

This particular disposition of rays offers a convenient and natural framework for extraction of the possibly aliased moments by using Fourier techniques. With this arrangement the available data are sufficient to determine a maximum of N-1 possibly aliased quantities through the inverse discrete sine transform:

$$\hat{p}_{l\nu} = (-1)^l \frac{\sqrt{2\pi}}{N} \sum_{n=1}^{N-1} g_l(\tau_n) \sin\left[(\nu+1)\frac{n\pi}{N}\right].$$
 (34)

The extracted $\hat{p}_{l\nu}$ are simply all those for which $\nu \leq N-2$ so that recovery of the set $\{\hat{C}_M\}$ requires $N \geq 2M$ (here again the superscript denotes possibly alias-tainted quantities). In fact $\{\hat{C}_M\}$ forms a subset of the complete set $\{\hat{A}_M\}$ of coefficients recoverable in this case. The set $\{A_M\}$, as indicated in Fig. 3 for M = 4, consists of those terms for which $l \leq M-1$ and $\nu \leq N-2 = 2M-2$. Below we examine the point-spread function for reconstruction in the space A spanned by the functions corresponding to the possibly contaminated set $\{A_M\}$. Substituting Eq. (33) for $g_l(\tau)$ identifies the aliases as

$$\hat{p}_{l\nu} = p_{l\nu} + \sum_{\lambda} p_{l\lambda} - \sum_{\mu} p_{l\mu}, \qquad (35)$$

where

$$\lambda + 1 = 2Nj + (\nu + 1), \qquad j = 1, 2, \dots$$

and

$$\mu + 1 = 2Nk - (\nu + 1), \qquad k = 1, 2, \dots$$

The lowest alias for the ν th moment is from the term $\mu = 2N$ - $\nu - 2$. For an object of effective bandwidth ω_0 , we may neglect aliases from functions for which $\nu > \nu_{\max}$ so that the number of noise-free samples sufficient to recover the $p_{l\nu}$ is given by the condition $\mu = \nu = \nu_{\max} + 2$ so that $N = \nu_{\max} + 2 =$ M + 1. The total number of samples in the projection set is then $MN = M^2 + M$, which is simply twice the number of extractable functions for an object that is band limited to ω_0 .

When $M < \omega_0$ the collection of M^2 moments $\{C_M\}$ can still be recovered without contamination from discretely sampled projections, provided that $2M - 2 < (2N - \nu_{\max} - 4)$. Here ν_{\max} is given by condition (31) and 2M - 2 represents the maximum ν value attained by any member of the set $\{C_M\}$.

In the general case, the extracted coefficient can be explicitly related to the true quantity and its set of contaminants by using Eqs. (14), (20), (24), and (34):

$$\hat{q}_{l\nu} = q_{l\nu} + \sum_{j=1}^{\infty} \frac{\lambda_{2Nj+\nu}}{\lambda_{\nu}} q_{l,2Nj+\nu} - \sum_{k=1}^{\infty} \frac{\lambda_{2Nk-\nu-2}}{\lambda_{\nu}} q_{l,2Nk-\nu-2} + \sum_{i=1}^{i_{max}} \left\{ \sum_{j=0}^{\infty} \frac{\lambda_{2Nj+\nu}}{\lambda_{\nu}} q_{l\pm 2Mi,2Nj+\nu} - \sum_{k=1}^{\infty} \frac{\lambda_{2Nk-\nu-2}}{\lambda_{\nu}} q_{l\pm 2Mi,2Nk-\nu-2} \right\}.$$
(36)

The upper-index i_{\max} is fixed by the condition $|l \pm 2Mi_{\max}| \le \nu$. When the $q_{l\nu}$ are restricted to lie within the set $\{S_M\}$, only the radial aliases survive. It is clear that the extra bandwidth obtained by retaining all the moments extractable from, say, N = 2M samples per view (i.e., the set $\{A_M\}$) is acquired only at the risk of increased azimuthal aliasing contamination. However, it is significant that the higher the order ν of the contaminant, the less strongly it is coupled to the desired quantity.

The effects of aliasing on image reconstruction can be assessed by examining the properties of the point-spread function \hat{h}_A calculated from the contaminated members $\hat{q}_{l\nu}$ of the set $\{A_M\}$ at various positions on the unit disk and under different sampling conditions M and N. Without loss of generality, we may take $\theta_0 = 0$ so that the expression equivalent to Eq. (28) for the contaminated point spread is

$$\hat{h}_A(r,\theta) = \sum_{|A_M|} \hat{q}_{l\nu} u_{l\nu}(r,\theta)/2\pi,$$

where $\hat{q}_{l\nu}$ is given by Eq. (36) and the contaminating elements are simply $q_{l\nu} = u_{l\nu}^*(r_0, 0) = R_{\nu}^l(r_0)$.

Figures 7(a)–7(c) show the calculated point response \hat{h}_A for $r_0 = 0.8$ with M = 32 and N = 64. The azimuthal and radial aliases can be examined separately by, in the first case, allowing only the sum over j = 0 to contribute to the calculated terms $\hat{q}_{l\nu}$, and in the second case, by setting $i_{max} =$ 0. This is equivalent to letting N and M, respectively, become infinitely large. The results are depicted in Figs. 7(a) and 7(b). For the radial aliases the sums over j and k were truncated above j = k = 10, corresponding to a maximum order $\nu = 1342$ contributing to \hat{h}_A . Inclusion of more terms produced little discernible change in the resulting image. In Fig. 7(c) all allowable radial and azimuthal aliases are combined. The gray-scale images have been clipped at $\pm 10\%$ of the peak height to highlight the aliasing artifacts. In Figs. 7(d)-7(f) similar calculations have been performed for the point source at $r_0 = 0.3$. Observe that finite *M* artifacts are



Fig. 7. Point response \hat{h}_A calculated using the aliased coefficients $\hat{q}_{l\nu}$ (M = 32, N = 64) for functions in the space A. The point source is located at $r_0 = 0.8$ for (a)–(c) and at $r_0 = 0.3$ for (d)–(f). For the leftmost image in each case only azimuthal contaminants are included. The middle images show the effects of radial aliasing contamination alone, while the rightmost images include all azimuthal and radial aliases (up to order j = k = 10).

more pronounced for objects near the edge of the disk but that the effects of undersampling within a view appear to be the more damaging. The streaking and oscillatory structures apparent in these images are clearly reproduced in reconstructions of synthetic test objects described in Section 4.

C. Band-Limited Projections

We now consider the case in which the effective band limit for the projections is fixed by the measuring apparatus. For example, because of diffraction, the optical system used for the phase-imaging interferometer acts as an ideal low-pass filter with cutoff frequency $\omega_s \approx 30$ rad per unit length. In this case the required sampling rate within a view is fixed by ω_s , the system bandwidth, rather than by ω_0 . To capture exactly the image of the band-limited projection \bar{G}_m , we need to sample at the Nyquist rate across the full spatial extent $(-\infty, \infty)$ of the image plane. Given such a sampling we wish to ascertain that information that can be recovered reliably from the projections. As described before, a discrete *l*-fold harmonic analysis over the angles ϕ_m yields the true band-limited projection component \bar{g}_l and its set of aliases:

$$\hat{g}_{l}(t) = \bar{g}_{l}(t) + \sum_{j} \bar{g}_{l\pm 2Mj}(t).$$
 (37)

Equation (16) is not appropriate for the g_l so that the coefficients $p_{l\nu}$ cannot be related so simply to the measured quantities $\hat{p}_{l\nu}$. In principle it is nonetheless possible to restore g_l from its band-limited counterpart, given the knowledge that the original object and thus its shadow are spatially bounded.¹² This is facilitated by introducing functions $\sigma_k(t)$, the prolate spheroidal wave functions, which satisfy the integral equation

$$\int_{-1}^{1} \mathrm{d}\tau \sigma_k(\tau) \, \frac{\sin[\omega_s(t-\tau)/2\pi]}{\pi(t-\tau)} = d_k \sigma_k(t),$$

where d_k are the corresponding eigenvalues and possess the useful property of orthogonality on both the intervals [-1, 1] and $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} dt \sigma_k(t) \sigma_j(t) = \delta_{kj},$$
$$\int_{-1}^{1} dt \sigma_k(t) \sigma_j(t) = d_k \delta_{kj}.$$

Exploiting these properties enables us to write

$$g_l(t) = \sum_{k=0}^{\infty} \frac{c_{lk}}{d_k} \sigma_k(t), \qquad (38)$$

$$\bar{g}_l = \sum_{k=0}^{\infty} c_{lk} \sigma_k(t), \qquad (39)$$

with the coefficients obtained by forming the appropriate inner product with the expansion eigenfunctions. Substituting Eq. (39) for \bar{g}_l into Eq. (37) yields

$$\hat{c}_{lk} = c_{lk} + \sum_{j=1}^{\infty} c_{l \pm 2Mj,k}.$$
(40)

The moments $p_{l\nu}$ may therefore be recovered in terms of the c_{lk} by using Eqs. (16) and (38) for $g_l(t)$:

$$\dot{p}_{l\nu} = \sum_{k=0}^{\infty} c_{lk} \rho_{k\nu}, \qquad (41)$$

where we have taken

$$\rho_{k\nu} = \frac{1}{d_k} \sqrt{\frac{2}{\pi}} \int_{-1}^1 \mathrm{d}t \sigma_k(t) U_\nu(t).$$

The quantities $\rho_{k\nu}$ and d_k can be explicitly computed. Substituting from Eq. (40) into Eq. (41) confirms that the untainted set $\{S_M\}$ can, at least in principle, be recovered from M band-limited views of the object. This is of course simply a restatement of the fact that g_l can be recovered from \bar{g}_l given the knowledge that ψ is spatially bounded.

Any physically realizable apparatus will in practice yield only a finite set of samples of the band-limited projections $\{\bar{G}_m\}$. In addition, these measurements will inevitably be corrupted by noise. For example, the two-dimensional interferometer provides 20 detector channels per view, which sample at approximately the Nyquist frequency within the most intense portion of the image of the plasma cross section. In this case estimates of the coefficients are reliable only to the extent that energy in the unsampled part of the image can be ignored. Equivalently, the error in the evaluation of the moments incurred by fitting the band-limited data to the basis functions $V_{\nu}(t)$ on the finite interval [-1, 1]will be small, provided that ν is sufficiently small compared with the system bandwidth ω_c . The fraction of energy contained in the interval [-1, 1] for $\bar{g}_l(t)$ can be shown to be

$$\left(\sum d_k c_{lk}^2\right) / \left(\sum c_{lk}^2\right) \cdot$$

It is simpler, however, to note that the energy contained in the two-dimensional spectrum $w_{l\nu}(\omega, \phi)$ of the basis functions $u_{l\nu}(r, \theta)$ above the cutoff ω_c is simply the quantity $1 - E_{\nu}(\omega_c)$, with $E_{\nu}(\omega)$ given by Eq. (30). As a first estimate, $1 - E_{\nu}$ will be representative of the energy residing outside the finite interval [-1, 1] for the band-limited projection basis function $V_{\nu}(t)$. From Fig. 4, observe that for the omitted energy at $\omega_s = 30$ rad per unit length to not exceed 10%, the recovered moments should be restricted so that $\nu \leq 4$.

4. IMAGE RECONSTRUCTION

A. Extraction of the Moments

With each of the M projections composed of N samples equispaced in angle τ , a discrete harmonic analysis over the first M angles $\{\phi_m\}$ yields the coefficients $\hat{g}_l(\tau_n)$ for n = 1, $2, \ldots, N$. In turn, the moments $\hat{p}_{l\nu}$ can be extracted according to Eq. (34), with the complete process being efficiently implemented on a computer using two-dimensional FFT techniques. When not all the available moments are required, some level of noise reduction may be achieved by performing FFT's on selected subsets of the N data points and averaging to obtain a better estimate of the spectrum $\{p_{l\nu}\}$. Of course, the reduction in noise will necessarily occur at the expense of bandwidth and a possible increase in aliasing contamination. For small to moderate values of N, the spectrum is more often extracted by fitting to the basis functions using general linear least-squares techniques.

It is instructive to evaluate explicitly the Fourier coefficients $\hat{g}_l(t)$ for the M = 2 interferometer geometry. From the band-limited views \bar{G}_0 and \bar{G}_1 we generate the four sets of Fourier coefficients $\bar{G}_0 + \bar{G}_1$ (l = 0), $\bar{G}_0 \pm i\bar{G}_1$ $(l = \pm 1)$, and $\bar{G}_0 - \bar{G}_1$ $(l = \pm 2)$. For the moderate bandwidth of the imaging interferometer, it is a good approximation to take $\bar{V}_{\nu} = V_{\nu}$ (at least for the functions in S) and so fit to the simpler weighted Chebyshev polynomials. With N = 20 points per view, it is clear that the moments $\{S_M\}$ will be strongly determined by a least-squares fit of the functions V_{ν} to the measured data.

B. Image Reconstruction and Two-View Plasma Interferometry

Given this finite number of what we hope are relatively pure quantities extracted from the projections, we desire to create an image of the source function. However, for a two-view system, though the moments are intrinsically of value, they provide only $M^2 + M = 6$ constraints on the reconstructed image of the plasma density distribution. Clearly one could simply use these numbers as coefficients in a minimum norm or Cormack reconstruction of ψ on the unit disk. However, such an approach does not utilize the extra information that is available. First, the reconstructed density distribution must be everywhere positive, and second, there can be no plasma outside the square vacuum vessel walls. It is therefore desirable to reconstruct a function that is confined within a square (circumscribed by the unit disk), is positive, and matches the known moments. An attractive approach is to construct a function that maximizes the entropy^{19,20}

$$S = -\sum \psi_j \ln \psi_j$$

of ψ (suitably discretized on a square pixel grid) while satisfying the set of constraints derived from the projection data. Nevertheless an image constrained by at most six numbers will convey limited information.

The constraint set can be significantly augmented if one makes the assumption that the object (in this case the plasma) possesses no harmonic content higher than l = 2. Such an assumption may be supported, for example, by signals obtained from other diagnostics, such as x-ray detector arrays, or theoretical estimates of the plasma shape. The full set of l = 0 and l = 1 moments and half of the l = 2 moments are then in principle retrievable, though a practical upper limit is set by the instrument bandwidth and by the presence of noise on the measured line integrals. It is also required that the center of the plasma and the center of the unit disk coincide so that spurious multipole moments of the assumed low-order l = 0 and l = 1 features are not generated. A discussion of the range and validity of such assumptions, the effects of noise, and maximum-entropy reconstruction of the interferometer data is deferred to a later publication.

C. Fourier Techniques for Image Reconstruction

The implementation of the maximum-entropy reconstruction technique can become prohibitively expensive when a large number of coefficients are available. In such cases it is more reasonable to evaluate explicitly the basis functions $u_{l\nu}$ on a polar grid by using the appropriate recurrence relation for the Zernike polynomials¹⁰ followed by a FFT generation of the angular factors. Although such an approach can be made highly efficient (especially if the radial terms are pretabulated), the resulting reconstruction will exhibit the ringing artifacts normally associated with polynomial expansions [see Fig. 11(b)] while also suffering from the (possible) inconvenience of having to be displayed in polar fashion. Direct evaluation on a Cartesian grid is undesirable because of the necessity of having to calculate basis functions for each value of *l* and *v* at each grid point.

An alternative approach is first to calculate the transform $\overline{\Psi}$ of the image on a Cartesian mesh (u_i, v_i) using the orthogonal expansions developed in Section 2 and invert this using standard two-dimensional FFT techniques to obtain $\overline{\psi}$. This approach has the attraction of permitting frequency-space manipulation of the image before the final reconstruction is produced. One is also free to choose the coefficient subset for image reconstruction, thereby either maximizing uniformity or resolution or reducing the risk of aliasing contamination and/or noise corruption. It is instructive to rearrange Eqs. (4) and (7) into the form

$$\Psi(\omega,\phi) = 2\pi \sum_{\nu} W_{\nu}(\omega)(-i)^{\nu} \sum_{0 \le l \le \nu} (a_{l\nu} \cos l\phi + b_{l\nu} \sin l\phi).$$
(42)

In a given application it is necessary to calculate the requisite functions only once for the chosen grid configuration and store these values in an appropriate array or look-up table. In particular, the Bessel functions $W_{\nu}(\omega_i)$ can be computed accurately for all orders by means of the recurrence

$$J_{\nu}(\omega) + J_{\nu+2}(\omega) = A_{\nu}W_{\nu}(\omega),$$

and, being explicitly independent of l, need only be calculated to the highest required value of v. In addition, for a $2K \times 2K$ uniform mesh in the (u, v) plane, only $(K^2 + K)/2$ of the grid points have distinct ω_i values, while the Hermitian properties of the transform necessitate explicit calculation for only one half of the plane. Nevertheless, calculation of the transform is computer intensive and limits the size of the reconstruction grid to $2K \sim 128$ for moderate computational cost (CPU times ~ 10 min on a VAX 780 computer for M = 64 and N = 128).

A reconstruction algorithm using these ideas has been implemented as described below for reconstruction of various synthetic source functions. Crawford and Kak²¹ have chosen an ellipse phantom for the demonstration of aliasing artifacts due to incomplete radial and azimuthal sampling of the shadow G. Figure 8 shows gray-scale contour plots of the same test ellipse phantom and its reconstruction from 64 views (equispaced in ϕ) comprising 128 samples per view (equispaced in τ) for various subsets of the extracted Fourier coefficients. The chosen spacing of grid points in the transform plane $(2K = 128, \Delta \omega = \pi)$ ensures adequate sampling of the transform out to the circular cutoff frequency $\omega_0 \sim M$ rad per unit length. The images have been clipped (following Ref. 21) at $\pm 10\%$ of the ellipse height to emphasize the aliasing streaks and other artifacts, while the display grid size matches the number of reconstructed pixels. Figure



Fig. 8. Reconstructions of a test ellipse phantom for various subsets of the Fourier coefficients extracted from the projections. See text for details.

8(a) shows the reconstruction of the phantom using all the coefficients extracted from the projections (the set $\{A_{64}\}$). The strong streaking is primarily the result of an insufficient number of samples per view. By retaining only those functions immune to angular aliasing (the set $\{C_{64}\}$), the severity of the streaking is marginally reduced [Fig. 8(b)]. However, when only those terms for which $\nu \leq \nu_{max} = M - 1$ are used (the set $\{U_{64}\}$), the resulting image is largely free of such artifacts [Fig. 8(c)] and closely resembles the ideal band-limited reconstruction for a cutoff at $\omega_0 = M = 64$ rad per unit length [Fig. 8(d)].

Location of an ellipse phantom close to the edge of the disk results in marked distortion of the reconstruction when all the available moments extracted for M = 32 and N = 64 are used to produce the image [Fig. 9(a)]. Finite M artifacts are much more prominent in this case, so that restriction to the set $\{C_{32}\}$ results in a substantial improvement in image quality [Fig. 9(b)]. As exhibited in Fig. 9(c), however, the degree of uniformity is greatest when only those functions spanning the space U are used for the reconstruction, as can be seen by comparison with the band-limited ($\omega_0 = 32$ rad per unit length) ellipse phantom shown in Fig. 9(d).

The level of image degradation when the line integrals are corrupted by noise is also significantly reduced when only the functions in U are used to produce the image. Figures 10(a) and 10(b) compare the reconstructions obtained from corrupted projections of the centered ellipse phantom for the sets $\{C_{64}\}$ and $\{U_{64}\}$, respectively. The standard deviation of the normally distributed noise is chosen as 1% of the maximum line integral. The quality of the two reconstructions can be compared by using the measure

$$\sigma = \left(\frac{\sum_{i,j} \left[\psi_R(i,j) - \psi(i,j)\right]^2}{\sum_{i,j} \left[\psi(i,j) - \psi_{av}\right]^2}\right)^{1/2}$$

where ψ_R is the reconstruction and ψ_{av} is the mean of the original phantom ψ . The values of σ for these two images are 0.321 and 0.315, respectively. The greater susceptibility to noise and aliasing artifacts in this case apparently outweighs the advantages of increased bandwidth afforded by reconstruction using all the available coefficients.

Finally, to confirm that the object structure is indeed reliably transmitted by the tomographic system up to bandwidth $\omega_0 \sim M$, we show reconstructions of the Shepp-Logan²² "head phantom" with and without the use of tapered



Fig. 9. Reconstructions of a test ellipse phantom near the edge of the unit disk for M = 32 and N = 64. Note the distortion of the image in (a) when all the available Fourier coefficients $\{A_{32}\}$ are used to produce the image. In (b) and (c) the sets $\{C_{32}\}$ and $\{U_{32}\}$, respectively, are used for the reconstructed image. For comparison the band-limited phantom ($\omega_0 = 32$ rad per unit length) is shown in (d).



Fig. 10. Reconstructions of the ellipse phantom of Fig. 8 when the projections are corrupted with noise. (a), (b) Reconstructions using the functions in the spaces C and U, respectively.



Fig. 11. Reconstructions of the Shepp-Logan head phantom. (a) Shows the input function, while (b) shows its unfiltered reconstruction using the basis functions in U. (c) Reconstruction after application of a tapered window before inversion. For comparison (d) shows a similarly filtered version of the test function.

windows for smoothing of the final image. For these reconstructions a grid of dimension 2K = 256 was used. The resulting images were clipped at the maximum and minimum levels of the features interior to the phantom. Figure 11(a) depicts the input function, while Fig. 11(b) shows its reconstruction from 100 views and 128 samples per view using the Zernike polynomials corresponding to the space U. The image of Fig. 11(b) is essentially identical to that obtained using the Fourier representation on the two-dimensional grid. In Fig. 11(c) the reconstructed spectrum was passed through a Hanning window with a cutoff at $\omega_0 = 100$ rad per unit length before inversion; the result can be compared with the input function similarly filtered [Fig. 11(d)].

5. CONCLUSION

This paper was motivated by the need to recover reliable information from the two-view interferometer installed for measurement of the electron density distribution in the UCLA Microtor tokamak plasma. Some of the results of Cormack,¹⁰ Klug and Crowther,¹¹ and Niland⁹ have been extended and tied together in this work. Specifically it has been shown that M complete equispaced projections of an unknown object yield $M^2 + M$ real numbers, free of angular aliasing contamination, that characterize the object. These numbers are the generalized moments of the source function and also correspond to a well-defined subset of the expansion coefficients for an orthogonal polynomial representation of the source and its projections. For small M configurations, such as the interferometer, the availability of the extra M terms associated with the Mth azimuthal harmonic in the expansion is especially significant.

The frequency-plane properties of the space spanned by the functions immune to aliasing contamination have been examined. It is demonstrated that a special subset $\{U_M\}$ of the privileged moment set $\{S_M\}$, comprising those members for which $\nu \leq \nu_{max}$, can be used to create an image of the source function that is more nearly uniform (free of distortion) and less susceptible to noise artifacts than an image constructed from the full set of $M^2 + M$ unaliased functions. These functions are also less susceptible to radial aliasing contamination when the projections are discretely sampled. The contaminants in the case of discretely sampled projections are explicitly identified.

Finally, a reconstruction algorithm based on the frequency-plane representation of the source function has been implemented and tested for various artificial objects. The reconstructions obtained illustrate the trade-off among the conflicting requirements of image resolution, uniformity, noise immunity, and freedom from aliasing artifacts.

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