Chapter 6

Small Axial Wavelength Modes of the Resistive Plasma Column

In this chapter the small electron-ion collision parameter, \( \delta \), is retained in the linearized set of plasma equations [Eq. (5.25)], and treated as a small quantity. The first-order wave perturbed plasma parameters are each expanded again as a first-order series expansion for small \( \delta \). The term \( N \), appearing in the modified Kummer differential equation describing the radial variation of the wave [see Eq. (5.32)], is also expanded as a first-order Taylor series in \( \delta \) about \( \delta = 0 \).

In Chapter Five, a solution for the first-order wave perturbation \( l_{i1}(y) \) (the logarithm of the normalized ion density \( l_{i}(y) = \ln \frac{n_i(y)}{n_i(0)} \)) was found for the collisionless case, \( \delta = 0 \). In that chapter, a value of \( N \) was determined so that the solution for \( l_{i1}(y) \) satisfied radial boundary conditions.

In this chapter, the first-order wave perturbation \( l_{i1}(y) \) is expanded again to first order in \( \delta \), and solutions for the expansion derived. The expansion of \( N \) to first order about \( \delta = 0 \) is again selected so that the expansion of \( l_{i1}(y) \) satisfies radial boundary conditions. To simplify the algebra in this chapter, a Gaussian ion density profile is assumed (i.e. \( q = 1 \)).

It will be shown that the unstable modes found in this chapter increase in normalized growth rate \( \omega' \) with decreasing normalized axial wavenumber \( k_3 \), to where the first order expansion in \( \delta \) for \( N \) begins to break down. In this region, the solutions presented in Chapter Seven must be used. The solutions presented in this chapter are however valid for \( k_3 \gtrsim 0.2 \), and thus
complement the solutions found in Chapter Seven.

The chapter is organized as follows. In Sec. 6.1, the wave perturbed variables appearing in Eq. (5.25) are expressed to first order in delta, and subsequently, the linearized set of equations [Eq. (5.25)] also expanded to first order in delta. Section 6.2 outlines the method of solution, whilst Sec. 6.3 discusses radial boundary conditions and the eigen-modes treated in this Chapter. Sections 6.4 and 6.5 solve for flute modes and other modes respectively, whilst Sec. 6.6 presents dispersion curves and discusses the results. Finally, Sec. 6.7 contains concluding remarks.

6.1 Expansion of Wave Perturbation

In this treatment use is made of the property of the VAC plasma $\delta \ll 1$, to expand each wave perturbation as follows,

\[
\begin{pmatrix}
  l_{11}(y) \\
  X_1(y) \\
  \varphi_{11}(y) \\
  \varphi_{c1}(y) \\
  u_{e31}(y)
\end{pmatrix}
= 
\begin{pmatrix}
  l_{i10}(y) \\
  X_{i0}(y) \\
  \varphi_{i10}(y) \\
  \varphi_{c10}(y) \\
  u_{e310}(y)
\end{pmatrix}
+ \delta 
\begin{pmatrix}
  l_{i11}(y) \\
  X_{i1}(y) \\
  \varphi_{i11}(y) \\
  \varphi_{c11}(y) \\
  u_{e311}(y)
\end{pmatrix}
\tag{6.1}
\]

where $y = x^2$, and $x$ is the normalized radius. As discussed, the function $l_{11}(y)$ is the first-order wave perturbation in the logarithm of the normalized ion density. The function $X_1(y)$ is the difference between $l_{11}(y)$ and the perturbation in the normalized potential $\chi_1(y)$, all multiplied by $\frac{2}{\chi + Z}$. The functions $\varphi_{11}(y)$ and $\varphi_{c1}(y)$ are the perturbations in the radial components of the normalized ion and electron velocity divided by $x$, respectively. The function $u_{e31}(y)$ is the perturbation in the axial component of the normalized electron velocity. On the RHS of Eq. (6.1), the functions $l_{i10}(y), X_{i0}(y), \varphi_{i10}(y), \varphi_{c10}(y)$, and $u_{e310}(y)$, which in this chapter are referred to as the zero-order functions, are the solutions of $l_{i1}(y), X_1(y), \varphi_{i1}(y), \varphi_{c1}(y)$, and $u_{e31}(y)$ in the collisionless limit ($\delta = 0$). Finally, the functions $l_{i11}(y), X_{i1}(y), \varphi_{i11}(y), \varphi_{c11}(y)$, and $u_{e311}(y)$, which in this chapter are referred to as the first-order functions, are the first-order expansion terms in $l_{i1}(y), X_1(y), \varphi_{i1}(y), \varphi_{c1}(y)$, and $u_{e31}(y)$ about $\delta = 0$. 

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In Chapter Five, Eq. (5.25) was the reduced set of linearized equations describing the perturbed system. With the replacements for \( l_{11}, X_1, \varphi_{i1}, \varphi_{v1}, \) and \( u_{eq1} \) described above, the zero-order component of Eq. (5.25) may be written

\[
\begin{pmatrix}
\Psi (l'_{10} (y) - X'_{10} (y)) \\
y \varphi'_{i10} (y) \\
y \varphi'_{v10} (y) - i m \Psi X'_{10} (y) \\
0 \\
0
\end{pmatrix} = \tilde{A}_{10} \times 
\begin{pmatrix}
l_{10} (y) \\
X_{10} (y) \\
\varphi_{i10} (y) \\
\varphi_{v10} (y) \\
u_{ez10} (y)
\end{pmatrix}
\]

(6.2)

where \( \tilde{A}_{10} \) is the matrix

\[
\tilde{A}_{10} = \begin{pmatrix}
\frac{m \Psi C}{2 \omega y^2} & 0 & \frac{i \omega}{2} - \frac{i C^2}{2 \omega} & \frac{C^2}{2 \omega} & 0 \\
\frac{i}{2} \left( \omega - \frac{\Psi}{y} \left( \frac{m^2}{y^2} + k_i^2 \right) \right) & 0 & -1 + y - \frac{m \Psi C}{2 \omega y} & \frac{m}{2 \omega} & 0 \\
\frac{i}{2} (\omega - m \Omega_{10}^2 - 2m \Psi) & 0 & 0 & -1 + y - \frac{ik_i}{2} & 0 \\
0 & \frac{m \Psi}{2 y} & 0 & -\frac{i}{2} & 0 \\
0 & -ik_i \Psi & 0 & 0 & 0
\end{pmatrix}
\]

(6.3)

The first-order component of Eq. (5.25) can be written,

\[
\begin{pmatrix}
\Psi (l'_{i11} (y) - X'_{11} (y)) \\
y \varphi'_{i11} (y) \\
y \varphi'_{v11} (y) - i m \Psi X'_{11} (y) \\
0 \\
0
\end{pmatrix} = \tilde{A}_{10} \times 
\begin{pmatrix}
l_{i11} (y) \\
X_{11} (y) \\
\varphi_{i11} (y) \\
\varphi_{v11} (y) \\
u_{ez11} (y)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
-\Psi \xi_1 e^{-\gamma X'_{10} (y)} \\
0
\end{pmatrix} + \tilde{A}_{11} \times 
\begin{pmatrix}
l_{i10} (y) \\
X_{10} (y) \\
\varphi_{i10} (y) \\
\varphi_{v10} (y) \\
u_{ez10} (y)
\end{pmatrix}
\]

(6.4)
where $\tilde{A}_{11}$ is the matrix

$$
\tilde{A}_{11} = \begin{pmatrix}
0 & 0 & -\xi_\perp e^{-y} & \xi_\perp e^{-y} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{im\xi_\parallel e^{-y}}{2\omega} & \frac{im\xi_\parallel e^{-y}}{2\omega} & 0 \\
\xi_\perp e^{-y}\left(-\frac{m\Psi}{2\sigma_0} + \frac{Q}{2} + \Psi\right) & 0 & \frac{iC\xi_\perp e^{-y}}{2\omega} & -\frac{i\xi_\perp e^{-y}}{2\omega} & 0 \\
\frac{\xi_\perp e^{-y}k_0\Psi}{2\sigma_0} & 0 & 0 & 0 & -\xi_\parallel e^{-y}
\end{pmatrix}
$$

(6.5)

In matrix $\tilde{A}_{11}$ the term $\xi_\parallel$ arises because the diagonal tensor $\xi = \text{diag}(\xi_\perp, \xi_\perp, 1)$, introduced in Chapter Five to describe enhancement of resistivity transverse to the field, has been replaced with $\xi = \text{diag}(\xi_\perp, \xi_\perp, \xi_\parallel)$. This replacement allows terms associated with electron-ion collisions across and along the field to be more easily distinguished, and the effects of collisions across the field to be investigated in the absence of collisions parallel to the field ($\xi_\parallel = 0$).

### 6.2 Method of Solution

Solutions of Eq. (6.2) were investigated in Sec. 5.4 (for which $\delta = 0$), and so the zero-order functions are determined. Consequently, in Eq. (6.4), which describes the first-order functions, the RHS is known. Equation (6.4) can thus be treated as a set of inhomogeneous first order differential equations, where the LHS is the set of homogeneous differential equations and the RHS is the set of source or driving terms.

In Chapter Five it was shown that there existed three solution branches to Eq. (6.2): (a) $k_3 = 0$, flute modes; (b) $X_{10}(y) = 0$, other modes; and (c) $\Psi = 0$, a cold plasma. As with Chapter Five, solutions of Eq. (6.4) for case (c) are neutrally stable in this treatment, and are therefore not considered further.

It was shown in Chapter Five that for cases (a) and (b), Eq. (6.2) could be solved for $\varphi_{110}(y)$ and $\varphi_{e10}(y)$ in terms of $l_{110}(y)$ and $l'_{110}(y)$. Ignoring stable solutions, the second row in Eq. (6.2) could thus be reduced to the modified Kummer equation

$$
L(N) \left[ l_{110}(y) \right] = 0
$$

(6.6)
with the $L(N)$ operator given by Eq. (5.32), (here, with $q = 1$), and $N$ given by either $N_a$ [defined by Eq. (5.33) for case (a): flute modes], or $N_b$ [defined by Eq. (5.40) for case (b): other modes].

Similarly, Eq. (6.4) can be solved for $\varphi_{i1}(y)$ and $\varphi_{i11}(y)$ in terms of $l_{i1}(y)$, $l'_{i1}(y)$, and a combination of the zero-order functions, and their first derivatives. Substituting for the zero-order functions (determined in Sec. 5.4), the second row of Eq. (6.4) can be reduced to the second order inhomogeneous differential equation

$$L(N) [l_{i11}(y)] = -\mathcal{U}_{10}(y)$$  \hspace{1cm} (6.7)

where the term $\mathcal{U}_{10}(y)$ is a combination of the zero-order functions and their first and second derivatives, and thus represents a driving term. The homogenous differential equation for $l_{i11}(y)$ in Eq. (6.7) is the same as the differential equation for $l_{i10}(y)$ in Eq. (6.6), because the expansion of the plasma parameters in Eq. (6.1) was linear in $\delta$.

Finally, the complete equation for $l_{i1}(y) = l_{i10}(y) + \delta l_{i11}(y)$, can be written

$$L(N)[l_{i10}(y)] + \delta (L(N)[l_{i11}(y)] + \mathcal{U}_{10}(y)) = 0$$  \hspace{1cm} (6.8)

The term $N$ in Eqs. (6.6) to (6.8), given by either $N_a$ [Eq. (5.33)] or $N_b$ [Eq. (5.40)], is not an explicit function of $\delta$. However, $N$ is an explicit function of $\varphi$ (the complex frequency), which in turn can be written as an explicit function of $\delta$ (i.e. $N[\varphi(\delta)]$). Thus, $N[\varphi(\delta)]$ can be expanded as a Taylor series in $\delta$ about $\delta = 0$ as follows,

$$N[\varphi(\delta)] = N[\varphi(0)] + \delta \left[ \frac{\partial \varphi}{\partial \delta} \right] + O(\delta^2)$$

$$\equiv N_0 + \delta N_1(\varphi) + O(\delta^2)$$  \hspace{1cm} (6.9)

By applying radial boundary conditions to $l_{i10}(y)$ the eigenvalue equation $N_0 = 2n + |m|$ was obtained in the collisionless working of Chapter Five [$N(\varphi) = N_0$], for both case (a); flute modes [Eq. (5.36) with $q = 1$] and case (b); other modes [Eq. (5.41)]. Here, only the first-order term $N_1(\varphi)$, henceforth written $N_1$, remains unknown. In the following working of Sec. 6.2

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[case (a): flute modes] and Sec. 6.3 [case (b): other modes], Eq. (6.9) is written

\[ N_a = N_0 + \delta N_{a1}, \]
\[ N_b = N_0 + \delta N_{b1} \]

respectively.

With the substitution \( N = N_0 + \delta N_1 \), Eq. (6.8) can be written

\[
\left( L(N_0) [l_{i10}(y)] + \delta \frac{N_1}{2} l_{i10}(y) \right) + \delta \left( L(N_0 + \delta N_1) [l_{i11}(y)] + \mathcal{U}_{10}(y) \right) = 0
\]

which, to first order in \( \delta \) yields

\[
L(N_0) [l_{i10}(y)] + \delta \left( L(N_0) [l_{i11}(y)] + \frac{N_1}{2} l_{i10}(y) + \mathcal{U}_{10}(y) \right) = 0
\]

The first term on the LHS is the collisionless solution, and therefore zero. Thus, the equation

\[
L(N_0) [l_{i11}(y)] = - \left( \frac{N_1}{2} l_{i10}(y) + \mathcal{U}_{10}(y) \right)
\]

is obtained. This is a second order inhomogeneous differential equation for \( l_{i11}(y) \), with the driving term on the RHS. The radial dependence of the driving term is known, but constant \( N_1 \) remains unknown. Given the radial dependence of the driving term and the homogeneous solutions, the method of variation of parameters \[56\] can be used to find the particular solution.

Finally, application of the radial boundary conditions to the first-order solution \( l_{i11}(y) \) determines \( N_1 \). With an expression for \( N_1 \) found, solutions for \( \varpi \) can be obtained from Eq. (6.9).

### 6.3 Radial Boundary Conditions and Choice of Eigen-Modes

In Chapter Five it was noted that application of the weaker boundary condition of Rosenbluth et al. \[70\] on \( l_{i10}(y) \), which is the removal of any essential singularity at infinity, produces the same dispersion relation as application of the stronger boundary condition \( \lim_{Y \to \infty} l_{i10}(Y) = 0 \). In this chapter, it is more convenient to use the boundary condition of Rosenbluth et al. \[70\]
to constrain \( l_{11} \). Thus, the radial boundary condition as applied to \( l_{111}(y) \) becomes that \( l_{111}(y) \) possess no essential singularity at infinity.

For the wave perturbation treatment to remain valid, the wave perturbations must be continuous. Thus, expressions for \( l_{111}(y) \) must not possess any singularities for finite \( y \).

As discussed in the next chapter the inclusion of FLR effects acts to stabilize higher eigen-modes by limiting \( N \). In this chapter, only stability of the lowest eigen-modes, for which \( 2n + |m| \leq 2 \), is investigated. The notation \((m,n)\), where \( m \) is the azimuthal mode number and \( n \) the radial eigen-mode number, introduced by Rognlien [75], provides a convenient label with which to identify the various eigen-modes. In this notation, and with \( N_0 \leq 2 \), eigen-modes are restricted to the set \( \{(0, 0), (0, 1), (\pm 1, 0), (\pm 2, 0)\} \).

Finally, to reduce the algebra, the Fourier reality condition [79]

\[
\omega_M(-m, -k_j) = -\omega_M^*(m, k_j)
\]

is used, where \( \omega = \omega_M(m, k_j) \) is the dispersion relation, and the subscript \( M \) denotes an arbitrary wave mode. Thus, \( m < 0 \) solutions of \( \omega = \omega_M(m, k_j) \) with \( \omega^r \geq 0 \) can be found by solving \( \omega = -\omega_M^*(|m|, -k_j) \), or solving \( \omega = \omega_M(|m|, -k_j) \) for \( \omega^r \leq 0 \). Therefore, without any loss of generality, the dispersion relation Eq. (6.9) need only be determined for the eigen-mode set \( \{(0, 0), (0, 1), (1, 0), (2, 0)\} \).

### 6.4 Flute Mode Solutions

For \( m = 0 \), the method described in Sec. 6.2 need not be used, as an explicit calculation is possible. In the absence of electron-ion collisions, Sec. 5.3.1 concluded either \( \varpi = 0 \) or \( l_{i10}(y) = 0 \). If \( \varpi = 0 \), there can be no wave growth. If \( l_{i10}(y) = 0 \), then \( \varphi_{i10}(y) = \varphi_{e10}(y) = 0 \) and \( X_{10}(y) = c_{01} \). For these solutions, the fourth row equation of Eq. (6.4) yields \( \varphi_{e11}(y) = 0 \). Substitution into the third row of Eq. (6.4) yields \( \varpi l_{i11}(y) = 0 \), which gives either the stable solution \( \varpi = 0 \), or \( l_{i10}(y) = l_{i11}(y) = 0 \), for which the perturbation vanishes.

For \( m \neq 0 \) the method described in Sec. 6.2 is used. For flute modes, calculation of the driving term \( U_{10}(y) \) in Eq. (6.10) requires expressions for the zero-order functions\(^1\)

\(^1\)For flute modes the zero-order term \( u_{e10}(y) \) does not contribute to \( U_{10}(y) \).
\( \varphi_{110}(y), \varphi_{e10}(y) \) and \( X_{10}(y) \). With \( k_1 = 0 \), rows one, three and four of Eq. (6.2) can be solved to give expressions for \( \varphi_{110}(y), \varphi_{e10}(y) \) and \( X_{10}(y) \) in terms of \( k_{10}(y) \) and \( l'_{110}(y) \). This yields,

\[
\varphi_{e10}(y) = im\Psi \frac{X_{10}(y)}{y} = \frac{i}{2} \left( -\omega + m \left( \Omega_{40}^2 + 2\Psi \right) \right) \frac{l'_{110}(y)}{y} \tag{6.12}
\]

\[
\varphi_{110}(y) = \frac{i}{m} \left( \frac{\omega - m \Omega_{40}^2}{\omega^2 - C^2} \right) \left( \frac{mC l'_{110}(y)}{2} - \omega l'_{110}(y) \right) \tag{6.13}
\]

where \( C = 1 + 2\Omega_{40} \) has been used. From Chapter Five, \( l_{110}(y) \propto y^{[m]/2} L_n^{[m]}(y) \), and generating functions are available for the associated Laguerre polynomials \( L_n^{[m]}(y) \) [55]. Thus, it is possible to derive general expressions to describe \( U_{10}(y) \) for arbitrary \((m, n)\) eigen-modes. As only a few eigen-modes are of interest however, it is arguably simpler to calculate \( U_{10}(y) \) explicitly for each case. This is done next for the \((m, n) = (1, 0)\) and \((m, n) = (2, 0)\) eigen-modes. For each eigen-mode, the method of variation of parameters will be used to find the general solution for \( l_{111}(y) \), and a solution for \( N_{a1} \) obtained by applying radial boundary conditions to \( l_{111}(y) \).

For \((m, n) = (1, 0)\), the solution \( l_{10}(y) = c_{10} \sqrt{y} \) (\( c_{10} \) is a constant), together with Eqs. (6.12) and (6.13) yields the complete driving term [RHS of Eq. (6.10)]

\[
\frac{N_{a1}}{2} l_{10}(y) + U_{10}(y) = c_{10} \left( \frac{N_{a1} \sqrt{y}}{2} + \xi_y e^{-\sqrt{y}} (d_{10} + y d_{11} + y d_{12}) \right)
\]

where

\[
d_{10} = \frac{4i}{\omega - \Omega_{40}^2} \left( \omega - 1 - 5 \left( \Omega_{40}^2 + 2\Psi \right) + \frac{C + 2\Omega_{40}^2}{\omega + C} \right),
\]

\[
d_{11} = \frac{-4i}{\omega - \Omega_{40}^2} \left( \omega - 1 - 8 \left( \Omega_{40}^2 + 2\Psi \right) + \frac{C + 2\Omega_{40}^2}{\omega + C} \right),
\]

\[
d_{12} = \frac{-8i \left( \Omega_{40}^2 + 2\Psi \right)}{\omega - \Omega_{40}^2}
\]

For this case, homogeneous solutions to Eq. (6.10) have the form

\[
l_{111}(y) = k_1 \sqrt{y} + k_2 \sqrt{y} \left( \text{Ei}(y) - \frac{e^y}{y} \right) \tag{6.14}
\]

where \( k_1 \) and \( k_2 \) are arbitrary constants and \( \text{Ei}(y) \) is the exponential integral function [55]. Applying the method of variation of parameters to Eq. (6.10) with homogeneous solutions
given by Eq. (6.14) yields the particular solution, and the general solution for \( l_{111}(y) \) can be written

\[
\frac{l_{111}(y)}{c_{10}} = k_1 \sqrt{y} + k_2 \sqrt{y} \left( \text{Ei}(y) - \frac{e^y}{y} \right) + \frac{N_{a1}}{2} \left( \frac{y \ln y - 1 - y}{\sqrt{y}} \right) + \frac{\xi y e^{-y}}{4} \left( \frac{(2d_1 + 5d_2) y + 2d_2 y^2}{\sqrt{y}} \right)
\]

The function \( \sqrt{y} \left( \text{Ei}(y) - \frac{e^y}{y} \right) \) has an essential singularity at infinity, which is removed by the selection \( k_2 = 0 \). The function \( (y \ln y - 1 - y) / \sqrt{y} \) has a regular singularity at the origin, which is removed by the selection \( N_{a1} = 0 \). Thus, for flute \((m, n) = (1, 0)\) eigen-modes, the dispersion relation is unaffected by electron-ion collisions, to first-order.

For the \((m, n) = (2, 0)\) eigen-mode, the solution \( l_{110}(y) = c_{20} y \) \((c_{20} \text{ is a constant})\) together with Eqs. (6.12) and (6.13) yields the complete driving term

\[
\frac{N_{a1}}{2} l_{110}(y) + \mathcal{U}_{10}(y) = c_{20} y \left( \frac{N_{a1}}{2} + \xi_y e^{-y} (d_{20} + d_{21} y + d_{22} y^2) \right)
\]

where

\[
\begin{align*}
    d_{20} &= \frac{6 i}{\omega - 2 \Omega_{i0}^2} \left( \frac{\omega - 1 - 7 (\Omega_{i0}^2 + 2 \Psi)}{\omega + C} + \frac{C + 2 \Omega_{i0}^2}{\omega + C} \right), \\
    d_{21} &= \frac{-4 i}{\omega - 2 \Omega_{i0}^2} \left( \frac{\omega - 1 - 11 (\Omega_{i0}^2 + 2 \Psi)}{\omega + C} + \frac{C + 2 \Omega_{i0}^2}{\omega + C} \right), \\
    d_{22} &= \frac{-8 i (\Omega_{i0}^2 + 2 \Psi)}{\omega - 2 \Omega_{i0}^2}
\end{align*}
\]

For this case, homogeneous solutions to Eq. (6.10) can be written

\[
l_{111}(y) = k_1 y + k_2 \left( y \text{Ei}(y) - e^y \left( 1 + \frac{1}{y} \right) \right) \quad (6.15)
\]

Variation of parameters yields the particular solution, and the general solution can be written

\[
\frac{l_{111}(y)}{c_{20}} = k_1 y + k_2 \left( y \text{Ei}(y) - e^y \left( 1 + \frac{1}{y} \right) \right) + \frac{N_{a1}}{4} \left( \frac{2 y^2 \ln y - 3 y^2 - 4 y - 2}{y} \right) - \frac{\xi_y e^{-y}}{2} (d_{21} + 3 d_{22} + d_{22} y)
\]

The function \( k_2 \left( y \text{Ei}(y) - e^y \left( 1 + \frac{1}{y} \right) \right) \) has an essential singularity at infinity, which is removed
by the selection \( k_2 = 0 \). The function \( N_{d1}\left(\frac{2y^2 \ln y - 3y^2 - 4y - 2}{y}\right) \) has a simple pole at the origin, which is removed by the selection \( N_{a1} = 0 \). Hence, the dispersion relation for flute \((2, 0)\) eigen-modes are also unaffected by electron-ion collisions, to first-order.

### 6.5 Other Mode Solutions

This class of modes is characterized by \( X_{10}(y) = 0 \), and is solved by the method described in Sec. 6.2. Calculation of the driving term \( U_{10}(y) \) in Eq. (6.10) requires description of the zero-order functions \( \varphi_{e10}(y), \varphi_{e10}(y) \) and \( u_{e10}(y) \). Rows one, three and four of Eq. (6.2) can be solved to give expressions for \( \varphi_{e10}(y), \varphi_{e10}(y) \) and \( u_{e10}(y) \) in terms of \( l_{110}(y) \) and \( l'_{110}(y) \). This yields,

\[
\begin{align*}
\varphi_{e10}(y) &= 0, \quad (6.16) \\
\varphi_{110}(y) &= \left(\frac{2i \Psi}{\omega^2 - C^2}\right) \left(\frac{mC}{2} l_{110}(y) - \omega l'_{110}(y)\right), \quad (6.17) \\
u_{e110}(y) &= \left(\omega - m \left(\Omega_{10}^2 + 2\Psi\right)\right) \left(\frac{l_{110}(y)}{k_3}\right) \quad (6.18)
\end{align*}
\]

For the \((m, n) = (0, 0)\) eigen-mode, the solution \( l_{110}(y) = c_{00} \) \((c_{00} \text{ is a constant})\), together with Eqs. (6.16) to (6.18), gives the complete driving term

\[
\frac{N_{b1} l_{110}(y)}{a_{01}} + U_{10}(y) = \frac{N_{b1}}{2} - i e^{-y} \left(\frac{1 - 2y}{y}\right) \left(\xi_\perp C \left(\frac{\Omega_{10}^2 + 2\Psi}{2\omega \Psi}\right) + \xi_\parallel \left(1 - \frac{\omega^2}{k_3^2 \Psi}\right)\right)
\]

Applying the method of variation of parameters to Eq. (6.10) with \((m, n) = (0, 0)\) homogeneous solutions

\[
l_{111}(y) = k_1 + k_2 \text{Ei}(y)
\]

yields the particular solution, and the general solution to \( l_{111}(y) \) can be written

\[
\frac{l_{111}(y)}{c_{00}} = k_1 + k_2 \text{Ei}(y) + \frac{N_{b1}}{2} \ln y + i e^{-y} \left(\xi_\perp C \left(\frac{\Omega_{10}^2 + 2\Psi}{2\omega \Psi}\right) + \xi_\parallel \left(1 - \frac{\omega^2}{k_3^2 \Psi}\right)\right)
\]

The function \( k_2 \text{Ei}(y) \) has an essential singularity at infinity, which is removed by the selection \( k_2 = 0 \). The function \( \frac{N_{b1}}{2} \ln y \) has a regular singularity at \( y = 0 \), which is removed by the
selection $N_{b1} = 0$. Thus, the dispersion relation for $(m, n) = (0, 0)$ modes with $X_{10}(y) = 0$ (other modes) is unaffected by the affects of electron-ion collisions, to first-order.

For the $(m, n) = (0, 1)$ eigen-mode, the solution $l_{i10}(y) = c_{01}(y - 1) (c_{01}$ is a constant), together with Eqs. (6.16) to (6.18) gives the complete driving term

$$
N_{b1} \frac{l_{i10}(y)}{c_{01}} + \Omega_{10}(y) = \frac{N_{b1}(1 - y)}{2} - \frac{2i\xi_{\|} e^{-y}}{y\omega} \left( 1 - \frac{\omega^2}{k_y^2 \Psi} \right) (1 - 3y + y^2) + \\
\xi_{\perp} e^{-y} \frac{i}{\omega} \left( \left( \frac{\omega^2 + C^2}{\omega^2 - C^2} \right) (1 - 2y) - C \left( \frac{\Omega_{10}^2 + 2 \Psi}{2 \Psi} \right) (1 - 4y + 2y^2) \right)
$$

Applying the method of variation of parameters to Eq. (6.10) with $(m, n) = (0, 1)$ homogeneous solutions

$$
l_{i11}(y) = k_1 (1 - y) + k_2 ((1 - y) \text{Ei}(y) + e^y)
$$
yields the particular solution, and the general solution for $l_{i11}(y)$ can be written

$$
\frac{l_{i11}(y)}{c_{01}} = k_1 (1 - y) + k_2 (e^y + (1 - y) \text{Ei}(y)) + \frac{N_{b1}}{2} ((y - 1) \ln y - y - 1) - \\
\frac{i\xi_{\perp}}{4\omega} \left( \frac{\omega^2 + C^2}{\omega^2 - C^2} \right) (y - 1) \text{Ei}(-y) + \left( 3 \left( \frac{\omega^2 + C^2}{\omega^2 - C^2} \right) + C \left( \frac{\Omega_{10}^2 + 2 \Psi}{\Psi} \right) (1 - 2y) \right) e^{-y} + \\
\frac{i\xi_{\|}}{4\omega} \left( 1 - \frac{\omega^2}{k_y^2 \Psi} \right) (y - 1) \text{Ei}(-y) + (-5 + 4y) e^{-y}
$$

The selection $k_2 = 0$ removes the essential singularity at infinity in $k_2 (e^y + (1 - y) \text{Ei}(y))$. The functions $\ln y$ and $\text{Ei}(-y)$ both have regular on-axis singularities, which combine in $l_{i11}(y)/c_{01}$ as follows

$$
\lim_{y \to 0} \frac{l_{i11}(y)}{c_{01}} = \left( - \frac{N_{b1}}{2} + \frac{i\xi_{\perp}}{4\omega} \left( \frac{\omega^2 + C^2}{\omega^2 - C^2} \right) - \frac{i\xi_{\|}}{4\omega} \left( 1 - \frac{\omega^2}{k_y^2 \Psi} \right) \right) \lim_{y \to 0} \ln y
$$

Removal of the on-axis singularity thus requires the selection

$$
N_{b1} = \frac{i\xi_{\perp}}{2\omega} \left( \frac{\omega^2 + C^2}{\omega^2 - C^2} \right) - \frac{i\xi_{\|}}{2\omega} \left( 1 - \frac{\omega^2}{k_y^2 \Psi} \right)
$$

(6.19)

Hence, the dispersion relation for $(m, n) = (0, 1)$ is affected by electron-ion collisions, to first-order. Dispersion curves for this case are presented in Sec. 6.6.

For the $(m, n) = (1, 0)$ eigen-mode, the solution $l_{i20}(y) = c_{20} \sqrt{y}$, together with Eqs. (6.16)
to (6.18) gives the complete driving term

\[
\frac{N_{b1}}{2} l_{110}(y) + \mathcal{O}_{10}(y) = c_{10} \sqrt{y} \left( \frac{N_{b1}}{2} + \frac{e^{-y}}{y} \left( \xi_{\perp} (e_{10} + e_{11}y) + \xi_{\parallel} (f_{10} + f_{11}y) \right) \right)
\]

where,

\[
e_{10} = \frac{i}{\omega} \left( \frac{\omega - C}{\omega + C} - (\omega + 3C) \left( \frac{\Omega_{00}^2 + 2\Psi}{4\Psi} \right) \right),
\]

\[
e_{11} = \frac{iC}{\omega} \left( \frac{\Omega_{00}^2 + 2\Psi}{\Psi} \right),
\]

\[
f_{11} = \left( \frac{4\omega}{C - 5\omega} \right) f_{10} = 2i \left( \frac{1}{\omega} + \frac{\Omega_{00}^2 + 2\Psi}{k^2\omega} \right)
\]

Applying the method of variation of parameters to Eq. (6.10) with homogeneous solutions given by Eq. (6.14) yields the particular solution, and the general solution can be written

\[
\frac{l_{111}(y)}{c_{10}} = k_1 \sqrt{y} + k_2 \sqrt{y} \left( \text{Ei}(y) - \frac{e^y}{y} \right) + \frac{N_{b1}}{2} \left( \frac{y \ln y - 1 - y}{\sqrt{y}} \right) + \frac{\xi_{\perp}}{4} \left( (e_{10} + e_{11})y \text{Ei}(-y) - (e_{10} + e_{11} + 2e_{11}y) e^{-y} \right) + \frac{\xi_{\parallel}}{4} \left( (f_{10} + f_{11})y \text{Ei}(-y) - (f_{10} + f_{11} + 2f_{11}y) e^{-y} \right)
\]

As for the flute mode case, the essential singularity at infinity in \(k_2 \sqrt{y} \left( \text{Ei}(y) - \frac{e^y}{y} \right)\) is removed by the selection \(k_2 = 0\). The general solution \(l_{111}(y)/c_{10}\) has a regular on-axis singularity, appearing as

\[
\lim_{y \to 0} \frac{l_{11}(y)}{c_{10}} = \frac{\delta}{4} \left( -2N_{b1} - \xi_{\perp} (e_{10} + e_{11}) - \xi_{\parallel} (f_{10} + f_{11}) \right) \lim_{y \to 0} \frac{1}{\sqrt{y}}
\]

and so removal requires the selection
\[ N_{b1} = -\xi_\perp \left( \frac{e_{20} + e_{11}}{2} \right) - \xi_\parallel \left( \frac{f_{20} + f_{11}}{2} \right) \]

\[ = i\xi_\perp \left( \frac{\omega - C}{2\omega} \right) \left( \Omega_{20}^2 + 2\Psi \frac{\omega^2}{4\Psi} - \frac{1}{\omega + C} \right) + \]

\[ \xi_\parallel \left( \frac{\omega - C}{4\omega} \right) \left( \frac{1}{\omega} + \frac{\Omega_{20}^2 + 2\Psi - \omega}{k_3^2\Psi} \right) \]  

(6.20)

Thus, for the \((m, n) = (1, 0)\) eigen-mode, the dispersion relation is also affected by electron-ion collisions, to first-order. Again, dispersion curves for this mode are presented in Sec. 6.6.

Finally, for the \((m, n) = (2, 0)\) eigen-mode, the solution \(l_{110}(y) = c_{20} y\), together with Eqs. (6.16) to (6.18) gives the complete driving term

\[ \frac{N_{b1}}{2} l_{110}(y) + \mathcal{U}_{10}(y) = c_{20} y \left( \frac{N_{b1}}{2} + \frac{e^{-y}}{y^2} (\xi_\perp (e_{21} y + e_{22} y^2) + \xi_\parallel (f_{21} y + f_{22} y^2)) \right) \]

where

\[ e_{21} = \frac{2i}{\omega} \left( \frac{\omega - C}{\omega + C} - (\omega + 2C) \left( \frac{\Omega_{20}^2 + 2\Psi}{4\Psi} \right) \right) \]

\[ e_{22} = e_{12}, \]

\[ f_{22} = \left( \frac{2\omega}{C - 4\omega} \right) f_{21} = 2i \left( \frac{1}{\omega} + \frac{2\Omega_{20}^2 + 4\Psi - \omega}{k_3^2\Psi} \right) \]

Applying the method of variation of parameters to Eq. (6.10) with homogeneous solutions given by Eq. (6.15) yields the particular solution, and the general solution can be written

\[ \frac{l_{111}(y)}{c_{20}} = k_1 y + k_2 \left( y \text{Ei}(y) - e^{y} \left( 1 + \frac{1}{y} \right) \right) + \frac{N_{b1}}{4} \left( 2y^2 \ln y - 3y^2 - 4y - 2 \right) + \]

\[ \frac{\xi_\perp}{16} \left( (2e_{21} + 3e_{22}) y^2 \text{Ei}(-y) - ((2e_{21} + 3e_{22}) (1 + 3y) + 8e_{22} y^2) e^{-y} \right) + \]

\[ \frac{\xi_\parallel}{16} \left( (2f_{21} + 3f_{22}) y^2 \text{Ei}(-y) - ((2f_{21} + 3f_{22}) (1 + 3y) + 8f_{22} y^2) e^{-y} \right) \]

As for the flute mode case, the essential singularity at infinity in \( k_2 \left( y \text{Ei}(y) - e^{y} \left( 1 + \frac{1}{y} \right) \right) \) is removed by the selection \( k_2 = 0 \). The general solution \( l_{111}(y)/c_{20} \) has a regular on-axis
singularity, appearing as
\[
\lim_{y \to 0} \frac{b_{111}(y)}{c_{20}} = \frac{\delta}{16} \left( -8 N_b - \xi_\perp (2 e_{21} + 3 e_{22}) - \xi_\parallel (2 f_{21} + 3 f_{22}) \right) \lim_{y \to 0} \frac{1}{y}
\]

and so removal requires the selection
\[
N_{b1} = -\xi_\perp \left( \frac{2 e_{21} + 3 e_{22}}{8} \right) - \xi_\parallel \left( \frac{2 f_{21} + 3 f_{22}}{8} \right)
= i \xi_\perp \left( \frac{\omega - C}{2 \omega} \right) \left( \frac{\Omega_{\omega}^2 + 2 \Psi}{4 \Psi} - \frac{1}{\omega + C} \right) + i \xi_\parallel \left( \frac{\omega - C}{4 \omega} \right) \left( \frac{1}{\omega} + \frac{2 \Omega_{\omega}^2 + 4 \Psi - \omega}{k_s^2 \Psi} \right)
\]

Thus, as with the \((m,n) = (0,1)\) and \((0,1)\) cases, the \((m,n) = (2,0)\) eigen-mode dispersion relation is affected by electron-ion collisions, to first order. Dispersion curves for these eigen-modes are presented in the next section.

### 6.6 Dispersion Curves

For the \(0 \leq N_0 \leq 2\) eigen-modes considered here, only the dispersion relation for the \(X_{10}(y) = 0\) (other mode) branch with \(N_0 = 1, 2\) is modified to \(O(\delta)\). To completely specify solutions for the complex frequency \(\omega\), the normalized plasma rotation frequency \(\Omega_{\omega}\), normalized thermal velocity \(\Psi\), and electron-ion collision parameter \(\delta\) are required [see Eq. (5.40) with \(q = 1\), and Eqs. (6.19),(6.20) and (6.21)]. As for Sec. 5.4.1, the value \(\Omega_{\omega} = 0.59\) is used. In the experiments described in Chapter Eight, radial profiles of the electron temperature and ion density profile are reported for a magnesium plasma. An estimate of the electron temperature, \(T_e = 2.9\) eV comes from an average of the profile across the bulk of the plasma column, whilst the characteristic radius, \(R = 14.3\) mm is inferred from a Gaussian fit to the ion density profile. Assuming equal ion and electron temperatures (i.e. \(\lambda = 1\)), a value of the normalized thermal velocity, \(\Psi = 1.6\) is estimated for the magnesium plasma. An estimate of the on-axis ion density, \(n_i(0) = 5.2 \times 10^{19} \text{ m}^{-3}\) is also inferred from the Gaussian fit to the ion density profile. Together with estimates for the electron temperature and average charge state \((Z = 1.5)\) for a magnesium plasma [25], the normalized resistivity parallel to the field is estimated to be
\[ \delta = \frac{\epsilon Z n_0}{B_\alpha} \frac{\eta_L}{\gamma_E} \approx 0.03, \] where \( \eta_L \) is the electrical resistivity of an ideal gas, and \( \gamma_E \) the ratio of the conductivity for a charge state \( Z \) to that in a Lorentz gas [52]. As discussed in Chapter Five, the Debye logarithm \( \ln \Lambda \) is here corrected by 0.3 to account for shielding by positive ions [53]. These plasma parameters are later summarized in Table 8.3 of Chapter Eight. Finally, with the plasma parameters determined, dispersion curves can plotted for the most unstable ‘other mode’ solutions.

Figure 6-1(a) is a plot of the \((m, n) = (1, 0)\) eigen-mode dispersion curve as a function of the normalized axial wavenumber \( k_3 \). On the left axis is plotted the normalized frequency \( \omega^r \) and normalized growth rate \( \omega^\epsilon \) in the frame of the ion fluid. On the right axis the slip \( s \) is plotted. As introduced in Chapter Five, the slip is defined as the ratio of the difference between the instability and rotation frequencies, with respect to the rotation frequency. That is,

\[ s = \frac{\omega^r - \Omega_0}{\Omega_0} \]

In the frame of the ion fluid, the dispersion relation is an even function of \( k_3 \), resulting in the symmetry of \( \omega \) with \( k_3 \). In the laboratory frame, the frequency is Doppler shifted by \( \omega^r = \omega^r + m \Omega_0 + k_3 u_0 \), resulting in the linear offset of \( \omega^r \) (and therefore the slip, \( s \)) with \( k_3 \). In this work the ion axial velocity \( v_2 \) was not measured, and so a value of \( 10^4 \) ms\(^{-1}\) has been taken for magnesium, based on measurements by Dallaqua \textit{et al.} [25]. Using estimates of \( \omega_{ic} \) and \( R \) (295 krad s\(^{-1}\) and 14.3 mm, respectively), the normalized ion axial velocity is estimated to be \( u_0 = \frac{\nu_{ic}}{\omega_{ic} R} = 2.37 \).

Figure 6-1(b) plots \( \Delta N_\perp = \frac{\delta N_3 (\xi_1 = 0)}{N_0} \) and \( \Delta N_\parallel = \frac{\delta N_3 (\xi_1 = 0)}{N_0} \), the magnitudes of the fractional change in the perpendicular and parallel components of \( N \) respectively due to electron-ion collisions (to first order), as a function of \( k_3 \). Over the range of the plot \((|k_3| \leq 0.7)\), \( \Delta N_\parallel \gg \Delta N_\perp \) suggesting that the effects of electron-ion collisions perpendicular to the field do not significantly modify \( N \). Thus, solutions of the dispersion relation \( \omega = \omega_M (m, k_3) \) will also be insensitive to the effects of electron-ion collisions perpendicular to the field.
Figure 6-1: Dispersion, slip and $\Delta N$ curves for the most unstable $(m,n) = (1,0)$ eigen-mode, as a function of normalized axial wave-number, $k_s$. Figure (a) is a plot of the dispersion curve in the frame of the ion fluid (left axis) and laboratory frame slip (right axis). Figure (b) is a plot of $|\Delta N|$, the magnitude of the change in the function $N$ due to electron-ion collisions. Plasma parameters were taken from Table 8-3, with $\Omega_{e0} = 2.18$, $\psi = 0.64$, $\delta = 0.03$ and $u_{q0} = 2.37$. 
For the Taylor series expansion of $N$ [Eq. (6.9)] to remain valid, the total fractional change $\Delta N = \left| \frac{\xi_{N_{\|}}}{N_0} \right|$ due to first order electron-ion collisions effects, must remain small. As $\Delta N_{\|} \gg \Delta N_{\perp}$, it follows that $\Delta N \simeq \Delta N_{\|}$. In Fig. 6-1(b) it can be seen that $\Delta N_{\|} \gtrsim 1$ for $|k_{\parallel}| \lesssim 0.2$, and so for this region the Taylor series expansion fails. Returning to Fig. 6-1(a), this region is where peak growth is located. Thus, the model is only valid in the region where the eigen-mode is marginally unstable.

Figure 6-2 plots (a) the dispersion curves (left axis) and slip (right axis), and (b) the fractional change in $N$, as a function of $k_{\parallel}$ for the most unstable $(m,n) = (2,0)$ eigen-mode. The general features are very similar to Fig. 6-1. Whilst the $k_{\parallel}$ interval over which $|\Delta N|$ is small fails is slightly reduced, the value of $k_{\parallel}$ at peak growth has correspondingly dropped, and the model still fails in the region of peak instability.

Finally, Fig. 6-3 plots (a) the dispersion curves, and (b) the fractional change in $N$, as a function of $k_{\parallel}$ for the most unstable $(m,n) = (0,1)$ eigen-mode. Unlike Figs 6-1 and 6-2, a mode cross-over in Fig. 6-3 at $k_{\parallel} = 0.06$ occurs, where the frequency and growth rate of the most unstable mode cross. This accounts for the discontinuity in the slope of the dispersion curve at $k_{\parallel} = 0.06$.

In Fig. 6-3(b), the trend of $\Delta N_{\parallel} \to \infty$ as $k_{\parallel} \to 0$ can be understood by reference to the expression for $N_{b1}$ [Eq. (6.19)] repeated below,

$$N_{b1} = \frac{i \xi_{\perp}}{2 \omega} \left( \frac{\omega^2 + C^2}{\omega^2 - C^2} \right) - i \frac{\xi_{\parallel}}{2 \omega} \left( 1 - \frac{\omega^2}{k_{\parallel}^2 \Psi} \right)$$

In the $\xi_{\parallel}$ component, no selection of $\omega$ will remove the pole in Eq. (6.19) at $k_{\parallel} = 0$. This contrasts to the case for the $(m,n) = (1,0)$ and $(2,0)$ eigen-modes, where $N_{b1}$ could be held finite as $k_{\parallel} \to 0$ by the solutions $\omega = \Omega_{10}^2 + 2 \Psi$ and $\omega = 2 \Omega_{10}^2 + 4 \Psi$ respectively.
Figure 6-2: Dispersion, slip and $\Delta N$ curves for the most unstable $(m, n) = (2, 0)$ eigen-mode, as a function of normalized axial wave-number, $k_3$. Figure (a) is a plot of the dispersion curve in the frame of the ion fluid (left axis) and laboratory frame slip (right axis). Figure (b) is a plot of $|\Delta N|$, the magnitude of the change in the function $N$ due to electron-ion collisions. Plasma parameters were taken from Table 8-3, with $\Omega_0 = 2.18$, $\psi = 0.64$, $\delta = 0.03$ and $u_{q0} = 2.37$. 

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Figure 6-3: Dispersion and $\Delta N$ curves for the most unstable $(m, n) = (0, 1)$ eigen-mode, as a function of normalized axial wave-number, $k_i$. Figure (a) is a plot of the dispersion curve in the frame of the ion fluid. Figure (b) is a plot of $|\Delta N|$, the magnitude of the change in the function $N$ due to electron-ion collisions. Plasma parameters were taken from Table 8-3, with $\Omega_{i0} = 2.18$, $\psi = 0.64$, $\delta = 0.03$ and $u_{ij0} = 2.37$. 
6.7 Concluding Remarks

In this chapter the effects of electron-ion collisions were retained in the plasma model as follows: the plasma variables were expressed to first order in the electron-ion collision parameter $\delta$, and the set of linearized plasma equations [Eq. (5.25)] expanded to first order in $\delta$. Solutions were found for the $(m,n)$ eigen-modes with $2n + |m| \leq 2$.

It was found that, compared to the collisionless working of Chapter Five, only the dispersion relation for the ‘other modes’ case with $n \neq 0$ and $m \neq 0$ was modified. For the ‘other modes’ case, dispersion curves of the $(m,n) = \{(0,1),(1,0),(2,0)\}$ eigen-modes were plotted for the most unstable mode. The dispersion curves show that the growth rate of these modes increases with decreasing normalized axial wavenumber, to where the first order expansion in $\delta$ for $N$ begins to break down. Thus, the treatment in this chapter is valid only for small axial wavelengths such that $k_3 > 0.2$, where the most unstable modes are only marginally unstable.

In the next chapter, the normalized axial wavenumber will be scaled by the electron-ion collision parameter, $\delta$, so as to provide a consistent ordering in the limit of large axial wavelength ($k_3 \ll 1$). It will be shown the dispersion curves generated in Chapter Seven for the most unstable $m = 1$ and $m = 2$ modes match the dispersion curves of the most unstable $(m,n) = (2,0)$ and $(m,n) = (1,0)$ eigen-modes at $k_3 \approx 0.3$. Thus, the combined treatments provide a complete description of the dispersion curve. Finally, in Chapter Eight, a detailed set of experiments on the PCEN device at the Brazilian National Space Research Institute is described, and the results compared to the predicted properties of the $m = 1$ modes identified in the next chapter.