Chapter 3

Analytical Description of the Collisional Plasma Column in a Vacuum Arc Centrifuge

In this chapter the effects of electron-ion collisions in the plasma column within the rotation region of a VAC [see Fig. 1-2] are modelled using a perturbation technique. The model used here is similar to that of an earlier numerical treatment by Yue and Simpson [36], but has several key simplifications: ion viscosity effects are ignored; terms greater than first order in the electron-ion collision frequency parameter $\delta$, are neglected; and to first order in $\delta$, the perturbed plasma parameters are only expanded to first order (linear dependence) in the axial position $z$. Nevertheless, comparisons of the analytic solutions derived here to those of the numerical treatment [36] suggest that the analytic model satisfactorily predicts important features, although the effects of ion viscosity could not be predicted. The advantage of the analytic approach is that the expressions found here provide a more efficient method of investigating the effects of electron-ion collisions.

Using the perturbed solutions, the change in steady state separative performance due to the effect of electron-ion collisions is discussed, and conditions under which separative power is maximized suggested. The non-uniformity in the axial magnetic field of a VAC caused by the azimuthal current is also investigated.
The chapter is organized as follows: Sec. 3.1 outlines the set of assumptions and presents the plasma fluid model. Section 3.2 solves the plasma model in the absence of electron-ion collisions, and investigates the non-uniformity in the magnetic field caused by the azimuthal current. Section 3.3 treats the influence of electron-ion collisions as a perturbation to the steady state, and solves the problem to first order. Approximate expressions are developed for general operating conditions, and validated by comparison to the exact solutions in particular cases. Section 3.4 presents a comparison to the numerical treatment of Yue and Simpson [36], and discusses plasma behavior. Finally, Section 3.5 proposes conditions that maximize the separative power, and contains concluding remarks.

3.1 Plasma Model

The plasma is described by a two fluid model in a cylindrical co-ordinate system \((r, \theta, z)\). Following earlier steady-state models of the VAC plasma [32, 51], the following assumptions are made about the plasma in the rotation region.

1. Ions of different charge can be treated as a single species with average charge \(Z\).

2. Quasi-neutrality, such that \(n_e = Zn_i\).

3. Steady-state and azimuthally symmetric plasma, which is axially uniform in the absence of electron-ion collisions (which are treated as a perturbation).

4. Zero-order fluid velocity \(\mathbf{v} = (0, \omega_0r, v_{z0})\), where \(\omega_0\) is the plasma rotation frequency and \(v_{z0}\), the axial streaming velocity. Ion rigid rotation is established from spectroscopic measurements [39], whilst uniform ion streaming velocity has been established from plasma deposition experiments [25, 33, 40]. To zero-order, classical diffusion of the plasma column is ignored.

5. Uniform ion and electron temperature, \(T_i\) and \(T_e\). Spectroscopic measurements of ion temperature from line broadening indicate that the ion temperature is uniform across the column [39].
(6) Zero-order Gaussian ion density distribution \( n_i = n_i(0) \exp \left(-\frac{r^2}{R^2}\right) \), where \( n_i(0) \) is the on-axis ion density and \( R \), the ‘characteristic’ radius, which is the radius at which the density is \( 1/e \) of its on-axis value. The Gaussian ion density distribution is established by Langmuir probe measurements of the ion saturation current [28].

(7) Neglect of finite Larmor radius and viscosity effects. Excepting for magnesium plasmas, estimates of the ion Larmor radius are significantly less than the characteristic radius over a wide range of VAC plasmas.

(8) Neglect of electron inertia.

With these assumptions, steady-state fluid model equations of continuity, charge continuity, momentum conservation and Ohm’s law may be written:

\[
\begin{align*}
n_i \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla n_i &= 0 \quad (3.1) \\
\nabla \cdot \mathbf{J} &= 0 \quad (3.2) \\
\rho (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla p + \mathbf{J} \times \mathbf{B} \quad (3.3) \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{\nabla p_e}{e n_e} &= \eta \tilde{\xi} \cdot \mathbf{J} + \frac{\mathbf{J} \times \mathbf{B}}{e n_e} \quad (3.4)
\end{align*}
\]

where \( \rho = M n_i \) is the ion mass density; \( \mathbf{J} \), the current density; \( \mathbf{B} \), the magnetic field; \( \eta \), the Spitzer [52] electrical parallel resistivity, and \( \tilde{\xi} = \text{diag}(\xi, \xi, 1) \), a diagonal tensor. The term \( \xi \) is the enhancement factor of electron-ion collisions perpendicular to the field, equivalent to the factor ‘\( f \)’ used in the numerical treatment of Yue and Simpson [36]. The terms \( p_i \) and \( p_e \) are the electron and ion pressures respectively, and \( p = p_i + p_e \) is the total pressure. Here, the Debye logarithm \( \ln \Lambda \) is corrected by 0.3 to account for shielding by positive ions [53].

In this treatment it is convenient to separate the radial dependence of the electron density \( n_e \) and the radial dependence of the electron-ion collision frequency \( \nu_{ei} \). Writing \( \rho \simeq \frac{m_e \nu_{ei}}{n_e e^2} \), where \( m_e \) is the electron mass, the RHS of Ohm’s law, Eq. (3.4), can be rewritten

\[
\left( \frac{B}{e Z} \right) \left( \delta n_e \tilde{\xi} \cdot \frac{\mathbf{J}}{n_i} + \frac{\mathbf{J}}{n_i} \times \mathbf{b} \right)
\]
<table>
<thead>
<tr>
<th>cathode</th>
<th>Mg</th>
</tr>
</thead>
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<tr>
<td>axial magnetic field, $B_z$</td>
<td>0.1 T</td>
</tr>
<tr>
<td>electron temperature, $T_e$</td>
<td>5.5 eV</td>
</tr>
<tr>
<td>ion temperature, $T_i$</td>
<td>5.5 eV</td>
</tr>
<tr>
<td>average ion charge state, $Z$</td>
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<td>axial velocity, $v_{e0}$</td>
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</tr>
<tr>
<td>characteristic radius at $z = 0$, $R$</td>
<td>0.02 m</td>
</tr>
<tr>
<td>on-axis ion density at $z = 0$, $n_i(0)$</td>
<td>$4 \times 10^{19}$ m$^{-3}$</td>
</tr>
<tr>
<td>angular frequency at $z = 0$, $\omega_0$</td>
<td>$1.5 \times 10^5$ rad s$^{-1}$</td>
</tr>
<tr>
<td>final axial position, $z_f$</td>
<td>0.4 m</td>
</tr>
<tr>
<td>$\delta = \frac{\nu_{ei}(0)}{\omega_{ce}}$</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 3.1: Typical VAC plasma parameters.

where $\delta = \frac{\nu_{ei}(0)}{\omega_{ce}}$ is the ratio of the on-axis electron-ion collision frequency $\nu_{ei}(0)$ to the electron cyclotron frequency $\omega_{ce}$. $n_s = \nu_{ei}/\nu_{ei}(0)$, the normalized radial dependence of the electron-ion collision frequency, and $\mathbf{b}$, the unit magnetic field vector.

Table 3.1 shows typical VAC plasma conditions for a magnesium plasma. The values of Table 3.1 are taken from the theoretical treatment of Yue and Simpson [36], and based on the published experimental work of Del Bosco et al. [28] and Dallaqua et al. [54]. For such conditions, $\delta$ is of the order $10^{-2}$, and is treated as a small parameter here.

Constitutive equations describing the electron and ion pressure may be written $p_i = n_i k T_i$ and $p_e = n_e k T_e$. Replacing $\mathbf{E}$ with $-\nabla \phi$ from Faraday’s law, $\rho$ with $M n_i$ and using the constitutive equations, Ohm’s law, Eq. (3.4), becomes

$$-\nabla \phi + \mathbf{v} \times \mathbf{B} + \frac{k_B T_e \nabla n_i}{e n_i} = \frac{B}{eZ} \left( \delta n_s \mathbf{\xi} \cdot \frac{\mathbf{J}}{n_i} + \frac{\mathbf{J}}{n_i} \times \mathbf{b} \right)$$

(3.5)

### 3.2 Solutions without Electron-Ion Collisions

In this section the effects of electron-ion collisions are ignored (with $\delta = 0$), and solutions to the plasma model found. Following assumption (3), the plasma column has no azimuthal dependence nor any axial structure. Integrating the equation $\nabla \cdot \mathbf{B} = 0$ with respect to $r$ and solving for $B_r$ yields the solution $B_r \propto 1/r$, which diverges as $r \to 0$. Thus, it can be concluded
$B_r = 0$. In the VAC, no current flows into the collector plate, and so $J_z = 0$. Integrating the axial component of Ampere’s Law with respect to $r$, and solving for $B_\theta$ yields the solution $B_\theta \propto 1/r$, which again diverges as $r \to 0$. Thus, it follows that $B_\theta = 0$. Finally, using Ampere’s law with an entirely axial magnetic field $\mathbf{B} = (0,0,B_z(r))$ gives

$$\mathbf{J} = \left(0, -\frac{1}{\mu_0} \frac{\partial B_z(r)}{\partial r}, 0\right)$$

(3.6)

Replacing $\mathbf{J}$ with Eq. (3.6) substituting into Eq. (3.3), and integrating yields

$$B_z(r) = B_{z0} \sqrt{1 - \varepsilon \exp \left(-\frac{r^2}{R^2}\right)}$$

(3.7)

where $B_{z0}$ is the field strength at infinity (or the field strength without a plasma), and

$$\varepsilon = \frac{M n_0 e_0}{B_{z0}^2} \left[\frac{2 k_B (T_i + Z T_e)}{M} + \omega_c^2 R^2\right]$$

(3.8)

Substituting $B_z(r)$ with Eq. (3.7) and solving for $J_\theta$ and $\phi(r)$, the steady state becomes:

$$\mathbf{B} = (0,0,B_{z0} f(r))$$

(3.9)

$$\mathbf{J} = \left(0, -J_{\theta 0} \frac{r \exp \left(-\frac{r^2}{R^2}\right)}{f(r)}, 0\right)$$

(3.10)

$$\phi = \phi_c + \frac{B_{z0}}{2} \left(\frac{\omega_i^2}{\omega_c^2} + \frac{2 k_B T_i}{M \omega_i R^2}\right) r^2 + \frac{\omega_0 B_{z0}}{2} r^2 (1 + g(r))$$

(3.11)

where $\phi_c$ is an arbitrary reference potential, $\omega_i$, the ion cyclotron frequency, and

$$f(r) = \sqrt{1 - \varepsilon \exp \left(-\frac{r^2}{R^2}\right)}$$

(3.12)

$$g(r) = -1 + \frac{R^2}{r^2} \ln \left(\frac{1 + f(r)}{1 + f(0)}\right) - 2 f(r) + 2 f(0)$$

(3.13)

with

$$J_{\theta 0} = e Z n_i(0) \left(\frac{\omega_i^2}{\omega_c^2} + \frac{2 k_B (T_i + Z T_e)}{M \omega_i R^2}\right)$$

(3.14)
Figure 3-1 plots the functions $f(r) = \frac{B_r}{B_{z0}}$ for a typical VAC plasma, as described by Table 3.1. The decrease in the on-axis magnetic field strength due to the azimuthal current is clearly visible, but the magnitude of the decrease is small, being 1.2% for the conditions considered here.

![Graph of f(r)](image)

Figure 3-1: Plots of the function $f(r)$ for a typical VAC plasma (see Table 3.1).

It is convenient to introduce the following change of variables:

$$l_i = \ln \frac{n_i}{n_i(0)}$$

(3.15)

$$w = \frac{J}{n_i}$$

(3.16)

Ignoring the small diamagnetic effect, the steady-state solution to the VAC plasma can now be written:

$$v = (0, \omega_0 r, v_z) = (0, \Omega_0 u_{ic} r, u_{z0} \omega_{ic} R)$$

$$w = \left(0, -\omega_0 r \exp \left( -\frac{r^2}{R^2} \right), 0 \right)$$

$$l_i = -\frac{r^2}{R^2}$$

35
\[
\phi = \phi_c + B_{z0} \left( \frac{kT_i}{M\omega_c R^2} + \frac{1}{2}\frac{\omega_0}{\omega_c} \left( 1 + \frac{\omega_0}{\omega_c} \right) \right) r^2
\]

where \( \omega_{00} = \frac{B_{z0}}{n_i(0)} \).

### 3.3 Solutions with Electron-Ion Collisions

In this section the small diamagnetic effect on the axial field is ignored, and the plasma model modified to include the effect of finite conductivity. With new variables \( l_i \) and \( \mathbf{w} \), the revised system equations [Eqs. (3.1) to (3.4)] can be written:

\[
\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla l_i = 0 \quad (3.17)
\]

\[
\nabla \cdot \mathbf{w} + \mathbf{w} \cdot \nabla l_i = 0 \quad (3.18)
\]

\[
(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{k_B(T_i + ZT_e)}{M} \nabla l_i + \frac{1}{M} \mathbf{w} \times \mathbf{B} \quad (3.19)
\]

\[
-\nabla \phi + \mathbf{v} \times \mathbf{B} + \frac{k_B T_e}{e} \nabla l_i = \frac{B}{eZ} \left( \delta n_s \xi \cdot \mathbf{w} + \mathbf{w} \times \mathbf{z} \right) \quad (3.20)
\]

#### 3.3.1 Perturbation Treatment

Assuming azimuthal symmetry, a perturbation to each of the plasma model parameters \( x(r, z) \) of the form

\[
x(r, z) = x_0(r) + \delta x_1(r) + \delta \frac{z}{R} X_1(r) + O(\delta^2) \quad (3.21)
\]

is considered. Here \( x_0(r) \) is the solution in the non-resistive limit. The perturbed component is expanded to first order (linear dependence) in the axial position \( z \), such that \( x_1(r) \) is the first order perturbation with \( z = 0 \), and \( X_1(r) \) is the coefficient of the axially linear component in the first order perturbation. The following working is to first order in \( \delta \).

Solving the axial components of Eq. (3.19) and Eq. (3.20) for \( V_{z1}(r) \) and \( \Phi_1(r) \) respectively yields

\[
\frac{V_{z1}(r)}{\omega_{ic} R} = -\frac{\Psi}{u_{z0}} L_{i1}(r) \quad (3.22)
\]

\[
\Phi_1(r) = \frac{k_B T_e}{e} L_{i1}(r) \quad (3.23)
\]
where
\[
\Psi = \frac{k_B (T_i + Z T_e)}{M \omega_i^2 R^2}
\] (3.24)

Equation (3.17) can then be written
\[
\left( \frac{1}{r} - \frac{2r}{R^2} + \frac{\partial}{\partial r} \right) \left( \frac{v_{r1}(r)}{\omega_i R} + \frac{z V_{r1}(r)}{R \omega_i R} \right) + \left( \frac{u_{z0}^2 - \Psi}{u_{z0}} \right) \frac{L_{i1}(r)}{R} = 0
\] (3.25)

The \( z \) coefficient in Eq. (3.25) can be solved for \( V_{r1}(r) \) yielding
\[
V_{r1}(r) \propto \exp \left( \frac{\sigma}{r} \right)
\]
which diverges as \( r \to 0 \) and \( r \to \infty \). Both properties are unphysical, and so \( V_{r1}(r) = 0 \). Expanding the azimuthal component of Eq. (3.19), equating \( z \) coefficients, substituting \( V_{r1}(r) = 0 \) and solving for \( W_{r1}(r) \) yields \( W_{r1}(r) = 0 \). This result is consistent with the absence of a radial boundary, as there is no axial return current path.

Solving the azimuthal components of Eqs. (3.19) and (3.20) yields an expression for \( v_{r1}(r) \) in terms of \( n_s(r) \) and \( L'_{i1}(r) \) [see Eq. (3.27)]. Substituting for \( v_{r1}(r) \), the \( z \) independent part of Eq. (3.25) yields a second order differential equation for \( L_{i1}(r) \)
\[
R^2 L''_{i1}(r) + \left( \frac{R}{r} - \frac{2r}{R} \right) R L'_{i1}(r) + C^2 \left( \frac{1}{u_{z0}^2} - \frac{1}{\Psi} \right) L_{i1}(r) =
\]
\[
\frac{\xi_{1C}}{\Psi u_{z0}} \left( \frac{w_0}{eZ \omega_i} \right) \left( 2 n_s(r) \left( 1 - \frac{r^2}{R^2} \right) + n'_s(r)r \right)
\] (3.26)
where \( C = 1 + 2 \Omega_0 \).

The remaining radial and azimuthal perturbed components of \( v \) and \( w \) are determined by solving the radial and azimuthal components of Eqs. (3.19) and (3.20). This yields expressions for \( w_{r1}(r) \), \( V_{\theta1}(r) \) and \( W_{\theta1}(r) \) in terms of \( n_s(r) \) and \( L'_{i1}(r) \),
\[
\begin{pmatrix}
  v_{r1}(r) \\
  w_{r1}(r) \\
  V_{\theta1}(r) \\
  W_{\theta1}(r)
\end{pmatrix} = \frac{\Psi}{C} \begin{pmatrix}
  \frac{w_0 r}{eZ} & - \frac{e Z n_s k \omega_i R}{C} \\
  -2 \Omega_0 w_0 r & - \frac{e Z n_s k \omega_i R}{C} \\
  0 & \omega_i R \\
  0 & e Z \omega_i R
\end{pmatrix} \begin{pmatrix}
  \xi_{1} n_s(r) \\
  R L'_{i1}(r)
\end{pmatrix}
\] (3.27)

37
and \( v_{\theta 1}(r) \) and \( w_{\theta 1}(r) \) in terms of \( l'_{i1}(r) \) and \( \phi'_{i1}(r) \),

\[
\begin{pmatrix}
  v_{\theta 1}(r) \\
  w_{\theta 1}(r)
\end{pmatrix} = \frac{\Psi \omega_i R}{C (\lambda + Z)} \begin{pmatrix}
  eZ (\lambda + CZ) & -2eZ^2 \Omega_0 \\
  \lambda & Z
\end{pmatrix} \begin{pmatrix}
  R l'_{i1}(r) \\
  e \rho \phi'_{i1}(r)
\end{pmatrix}
\]

(3.28)

where \( \lambda = T_i/T_e \), the ratio of ion to electron temperatures.

Finally, Eq. (3.18) is solved for \( W_{z1}(r) \) in terms of \( w_{r1}(r) \) and \( w'_{r1}(r) \). Substitution for \( w_{r1}(r) \) from Eq. (3.27) yields an expression for \( W_{z1}(r) \) in terms of \( L''_{i1}(r) \), \( L'_{i1}(r) \), \( n'_{s}(r) \) and \( n_{s}(r) \),

\[
W_{z1}(r) = \frac{eZ \omega_i R \Psi w_{0}}{C^2} \left( R^2 L''_{i1}(r) + \frac{R}{r} - \frac{2r}{R} \right) R L'_{i1}(r) + \frac{2 \xi_{w0} R \Omega_0}{C} \left( 2 \left( 1 - \frac{r^2}{R^2} \right) n_{s}(r) + r n'_{s}(r) \right)
\]

(3.29)

3.3.2 Boundary Conditions

For finite \( r \), the steady state solutions \( x_0(r) \) are all finite. The series perturbation must be well defined at all finite \( r \), and therefore both \( x_1(r) \) and \( X_1(r) \) must each remain finite for finite \( r \). If \( x_0(r) \) has a singularity at infinity (e.g., \( l_i \) and \( \phi \)), the ratio \( \left| \frac{x_1(r)}{x_0(r)} \right| \) and \( \left| \frac{X_1(r)}{X_0(r)} \right| \) must remain finite. Applied to \( l_i(r) \), it is necessary for \( L_{i1}(r) \) to have no singularities for finite \( r \), and the ratios \( \left| \frac{L_{i1}(r)}{r^2} \right| \) and \( \left| \frac{L'_{i1}(r)}{r^2} \right| \) must not diverge as \( r \to \infty \).

As with Yue and Simpson [36], the boundary constraint on axial current density \( J_z = W_z n_i \) is simply that \( J_z = 0 \) at the collector plate, \( z = z_f \). That is,

\[
W_z(r, z_f) = \delta w_{z1}(r) + \delta \frac{z_f}{R} W_{z1}(r) = 0
\]

(3.30)

This equation fixes \( J_z \) at the anode mesh.

3.3.3 Initial Conditions

In this work, initial conditions at \( z = 0 \) (the anode mesh) for \( l_i \), \( v_\theta \), \( w_\theta \) and \( v_z \) are selected to match those in the numerical treatment of Yue and Simpson [36], in which the ion density took a Gaussian profile, the plasma displayed rigid rotation, and the axial streaming velocity was
independent of radius. Thus,
\[ l_{\vartheta 1}(r) = v_{\vartheta 1}(r) = v_{z 1}(r) = 0 \] (3.31)
from which it follows that \( w_{\vartheta 1}(r) = \phi'_1(r) = 0 \) [see Eq. (3.28)].

In this treatment the initial radial velocity and radial current are specified by their steady state solutions at the anode mesh. This differs from the work by Yue and Simpson [36], in which initial \( v_r/r \) and \( J_r \) were set arbitrarily at the anode mesh. In their treatment [36] two possible initial conditions were investigated: (a) no radial outflow \( (v_r/r = 0) \) and an initial radial current, and (b) zero radial current \( (J_r = 0) \) and an initial outward diffusion. For case (b), the radial electric field at the anode mesh takes the same profile assumed in this work.

3.3.4 Solutions

Equation (3.26) is non-dimensionalized through the variable replacement \( y = r^2/R^2 \), yielding
\[ y L''_{\vartheta 1}(y) + (1 - y) L'_{\vartheta 1}(y) - a L_{\vartheta 1}(y) = b \left( (1 - y) n_s(y) + y n'_s(y) \right) \] (3.32)
where
\[ a = \frac{C^2}{4} \left( \frac{1}{\Psi} - \frac{1}{u_{z0}} \right), \quad b = \frac{\xi_{\vartheta 1} C}{u_{z0}} \left( \frac{\Omega_0^2}{2\Psi} + 1 \right) \] (3.33)

Exact Solutions

Replacing \( n_s(y) \) with the exact radial dependence of the electron-ion collision frequency, \( n_s(y) = e^{-y} \), Eq. (3.32) becomes
\[ y L''_{\vartheta 1}(y) + (1 - y) L'_{\vartheta 1}(y) - a L_{\vartheta 1}(y) = b (1 - 2y) e^{-y} \] (3.34)

The homogeneous equation is Kummer’s equation with two independent solutions [55],
\[ L_{\vartheta 1}(y) = c_1 M(a, 1, y) + c_2 U(a, 1, y) \] (3.35)
where \( c_1, c_2 \) are arbitrary constants, and \( M \) and \( U \) are the Kummer functions [55].

Using the method of variation of parameters [56], the particular solution can be written in
integral form as

\[
\frac{L_{i1}(y)}{c_1 c_2 b \Gamma(a)} = M(a, 1, y) \int \left( \frac{1 - 2y}{y} \right) e^{-2y} U(a, 1, y) dy - U(a, 1, y) \int \left( \frac{1 - 2y}{y} \right) e^{-2y} M(a, 1, y) dy
\]  

(3.36)

In general the two indefinite integrals that appear in Eq. (3.36) are not easily expressed in closed form, except when \( a \) is integral. For convenience, only integral values of \( a \) are considered in this treatment. For the typical experimental conditions of Table 3.1, \( a \approx 0.7 \), and so the cases \( a = 1 \) and \( a = 2 \) are solved.

Performing the integrations for the case \( a = 1 \) the complete solution for \( L_{i1}(y) \) can be written

\[
L_{i1}(y) = b \left( e^y \text{Ei}(-2y) - e^{-y} \right) + c_1 e^y - c_2 e^y \text{Ei}(-y)
\]

where \( \text{Ei}(y) \) is the exponential integral function [55]. Applying the boundary conditions described in Sec. 3.3.2, the function \( L_{i1}(y) \) has a regular singularity at \( y = 0 \), which is eliminated by the selection \( c_2 = b \), and an essential singularity at infinity, which is removed by the selection \( c_1 = 0 \).

For the case \( a = 2 \),

\[
L_{i1}(y) = b \left( e^{-y} + 4(1 + y) e^y \text{Ei}(-2y) \right) + c_1 (1 + y) e^y - c_2 (1 + (1 + y) e^y \text{Ei}(-y))
\]

In this case, the function \( L_{i1}(y) \) has a regular singularity at \( y = 0 \), which is eliminated by the selection \( c_2 = 4b \), and an essential singularity at infinity, which is removed by the selection \( c_1 = 0 \).

The solutions for \( a = 1 \) and \( a = 2 \) can thus be written

\[
L_{i1}(y) = b \left( e^y \left( \text{Ei}(-2y) - \text{Ei}(-y) \right) - e^{-y} \right) \quad |a = 1 \quad (3.37)
\]

\[
L_{i1}(y) = b \left( 4e^y (1 + y) \left( \text{Ei}(-2y) - \text{Ei}(-y) \right) + e^{-y} - 4 \right) \quad |a = 2 \quad (3.38)
\]
Approximate Solutions

For practical calculations, a convenient approximate solution would be useful. This can be achieved by slightly modifying the physical description. In this treatment, the analytical form of the electron-ion collision frequency is modified by multiplying it by a finite polynomial in y. Thus, solutions of the form $L_{ii}(y) = e^{-y}f(y)$, $n_s(y) = e^{-y}g(y)$, are considered, where $n_s(y)$ is the normalized radial dependence of the electron-ion collision frequency. Under these replacements, Eq. (3.32) transforms to

$$yf''(y) + (1 - 3y)f'(y) + (2y - a - 1)f'(y) = b(1 - 2y)g(y) + bg'(y) \quad (3.39)$$

The simplest trial solution is $g(y) = 1 + c_1y$, for which solutions to Eq. (3.39) are:

$$L_{ii}(y) = \frac{b}{3 + a} \left(1 + 2y\right)e^{-y} \quad (3.40)$$

$$n_s(y) = e^{-y} \left(1 + \frac{2y}{3 + a}\right) \quad (3.41)$$

For $y > \frac{3 + a}{2}$ the trial form for $n_s(y)$ is negative, and the approximation to the physical situation is clearly poor. However, close to the column axis where $y$ is small, the approximation is reasonable. For $a = 1$, $n_s(y)$ and the electron-ion collision frequency cross zero at 1.4 times the characteristic radius.

A comparison to the exact solution is possible when $a = 1$. On the left axis, Fig. 3-2 shows a plot of the exact and approximate solutions for $L_{ii}(y)/b$; and on the right axis, the exact and approximate solutions for $n_s(y)$ are shown. For $y < 2$, agreement between the exact and approximate solutions is reasonable. Beyond this radius, the approximate solution to the radial dependence $n_s(y)$ of the electron-ion collision parameter is small but has the wrong sign, and so the approximate solution is not expected to be valid.

Approximate solutions for $n_i, \phi, v$ and $w$ can now be found. Using Eqs. (3.27) to (3.29), boundary conditions (3.30), initial conditions (3.31), and approximate solutions for $L_{ii}$ and $n_{s1}$ given by Eqs. (3.40) and (3.41) the solutions to $n_i, \phi, v$ and $w$ can be written as follows:

41
\[ n_i(r, z) = n_i(0) \exp \left( -\frac{r^2}{R^2} \right) \left( 1 - \frac{\delta b}{3 + a} \left( 1 - 2 \frac{r^2}{R^2} \right) \exp \left( -\frac{r^2}{R^2} \right) \frac{z}{R} \right) \]

\[ \frac{e\phi(r, z)}{k_B T_e} = \frac{e\phi_e}{k_B T_e} + \left( \frac{\lambda}{Z} + \Omega_0 (1 + \Omega_0) \left( \frac{\lambda + Z}{2 \Psi Z} \right) \right) \frac{r^2}{R^2} - \frac{\delta b}{3 + a} \left( 1 - 2 \frac{r^2}{R^2} \right) \exp \left( -\frac{r^2}{R^2} \right) \frac{z}{R} \]

\[ \frac{\mathbf{v}}{Re_{w_c}} = \begin{pmatrix} 0 \\ \Omega_0 \frac{r}{R} \\ \frac{\delta b \Psi}{3 + a} \exp \left( -\frac{r^2}{R^2} \right) \left( \frac{2 \omega_{w_c}}{C^2} \left( \frac{3 - 2 \frac{r^2}{R^2}}{R^2} \right) \frac{z}{R} \right) \end{pmatrix} \]

\[ \frac{\mathbf{w}}{Re_{w_c}} = \begin{pmatrix} 0 \\ -\frac{\omega_{w_c}}{\omega_{w_c}} \exp \left( -\frac{r^2}{R^2} \right) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\delta b \Psi}{3 + a} \frac{2eZ}{C} \exp \left( -\frac{r^2}{R^2} \right) \\ \frac{u_w}{C} \left( -3C - 2a\Omega_0 + 2C \frac{r^2}{R^2} \right) \\ \left( 3 - 2 \frac{r^2}{R^2} \right) \frac{2\omega_{w_c}}{C} \left( 2a\Omega_0 + 3C - (10C + 4a\Omega_0) \frac{r^2}{R^2} + 4C \frac{r^4}{R^4} \right) \end{pmatrix} \]

Figure 3-2: Plots of $L_{i1}(y)/b$ (left axis) and $n_s(y)$ (right axis) for both exact and approximate solutions when $a = 1$. 

42
3.4 Results and Discussion

The approximate solution can now be compared to results of the numerical treatment of Yue and Simpson [36], which used the conditions of Table 3.1, together with $\xi = 2$. For various plasma parameters, Figs. 3-3 and 3-4 compare axial and radial profiles of the approximate and numerical solutions respectively.

![Graph showing ion density and rotation frequency as functions of axial position.](image)

Figure 3-3: Plots of on-axis ion density $n_i(0)$ (left axis) and rotation frequency at the characteristic radius $\omega(R)$ (right axis) as a function of axial position, $z$. Case (i) is taken from Yue and Simpson [36], and case (ii) is this work.

The left axis of Fig. 3-3 shows the on-axis ion density $n_i(0)$, and the right axis shows the rotation frequency at the characteristic radius $\omega(R)$. In Fig. 3-3, oscillations can be seen in $n_i(0)$ and $\omega(R)$ of the numerical treatment by Yue and Simpson [36]; the amplitude of these oscillations is related to the initial ion radial velocity and the radial current density, which were set arbitrarily at the anode mesh in that work. In contrast, for the solutions here, perturbations around the steady state are considered, the starting point at $z = 0$ m is the steady state solution and there is no initial transient approach to quasi-steady state. By about $z = 0.1$ m the oscillations have diminished, and the ion density and rotation frequency agree

43
reasonably with the approximate solutions. At larger $z$, nonlinear behavior can be seen in $n_i(0)$ (Fig. 3-3) of the numerical treatment by Yue and Simpson [36]. Such behavior is not present in this analysis, which expands the perturbed component to linear dependence only.

Figure 3-4: Figure (a) plots ion density $n_i$ (left axis) and rotation frequency $\omega$ (right axis) as a function of radial position, $r$. Figure (b) plots radial current $J_r$ (left axis) and axial current $J_z$ (right axis) as a function of radial position, $r$. Case (i) is taken from Yue and Simpson [36], and case (ii) is this work.

Figure 3-4(a) plots the ion density radial profile (left axis) and the rotation frequency (right axis) at $z = 0.1$ m; whilst Fig. 3-4(b) plots the radial current density (left axis) and the axial current density (right axis), also at $z = 0.1$ m. Turning first to the ion density in Fig. 3-4(a),
the approximate solution, like the numerical solution, is close to Gaussian. Turning next to the angular velocity profiles in Fig. 3-4(a), the profile of Yue and Simpson [36] is smoother than the profile of the approximate solution, due to the effect of viscosity which has been ignored in the present treatment.

The current density profiles in Fig. 3-4(b) have the correct magnitude and profile, but the magnitudes of \( J_r/r \) and \( J_z \) are slightly larger than the numerical predictions at the centre, and lower at large radius. This occurs due to the smoothing of the rotation profile by viscosity, visible in the angular frequency plot [Fig. 3-4(a)].

In a VAC, the separation factor \( \alpha \) is defined to be the ratio of matter flux of the heavy to light isotope at a given radius, divided by the on-axis ratio of matter flux. That is,

\[
\alpha(r) = \frac{n_{i2}v_{z2}}{n_{i1}v_{z1}}_r / \left[ \frac{n_{i2}v_{z2}}{n_{i1}v_{z1}} \right]_0
\]

where the subscripts \( i_1 \) and \( i_2 \) denote different isotope species. In Chapter Two, it was shown that the separation reduces in the steady state to

\[
\alpha(r) = \exp \left( A \times \frac{r^2}{R^2} \right)
\]

where \( A = \frac{\Delta m \omega^2 R^2}{2kT} \), which is sometimes referred to as a separative figure of merit [35]. The quantity \( A \) is a useful gauge of performance, as it is the logarithm of the enrichment ratio where a reasonable amount of material is present. The term \( \frac{\Delta m \omega^2 R^2}{2kT} \) is maximized for a rapidly rotating cold plasma. The results presented in this work describe the change in \( A \), due to electron-ion collisions. A Gaussian fit of the predicted ion density profiles here at \( z = 0.4 \) m indicate negligible radial diffusion, and so it is reasonable to treat \( R \) as invariant along \( z \). The change in the separative figure of merit due to electron-ion collisions is

\[
\frac{\Delta A}{A} = \frac{\omega^2(R)}{\omega_0^2} - 1 \quad (3.42)
\]

\[
= 2\delta \frac{z}{R} \frac{\omega_1(R)}{\omega_0} + O(\delta^2) \quad (3.43)
\]

Figure 3-5 plots the change in the separative figure of merit, \( \Delta A/A \) for the conditions of Table 3.1 as a function of \( z \), for both the numerical [36] and approximate solutions. For the
Figure 3-5: The fractional improvement in the separative figure of merit $\Delta A/A$, as a function of axial position, $z$. Case (i) is taken from Yue and Simpson [36], and case (ii) is this work.

approximate solution, the effect of electron-ion collisions is to improve the separative figure of merit by 16% at the collector plate. The discrepancy in the slope between the profiles is due to the neglect of viscosity in the present treatment. Nonetheless, agreement is sufficient to predict trends of $\Delta A/A$ with varying normalized rotation frequency $\Omega_0$, square of the normalized thermal speed $\Psi$, and normalized axial streaming velocity, $u_z$. For the conditions of Table 3.1, these normalized parameters are $\Psi = 0.40$, $\Omega_0 = 0.26$ and $u_z = 0.87$.

Figure 3-6 is a mesh plot of $\Delta A/A$ with normalized rotation frequency $\Omega_0$ and square of the normalized thermal speed $\Psi$, for $u_z = 0.87$. The $1/\omega_0$ dependence of $\Delta A/A$ means that the incremental improvements in separation at large rotation are small. For low rotation frequency however, such that $\Omega_0^2 \ll 2\Psi$ the improvement in separation is significant: an improvement in $A$ of over 40% is predicted for $\Psi = 0.2$, $\Omega_0 = 0.05$.

Finally, Fig. 3-7 is a plot of the $\Delta A/A$ with normalized streaming velocity $u_z$ for $\Psi = 0.40$ and $\Omega_0 = 0.26$. Thermal shock wave formation is likely in the limit $u_z \rightarrow \sqrt{\Psi}$, where this first order perturbation treatment becomes invalid. Nevertheless, trends can be identified providing the curve is restricted to supersonic flow, for which $u_z > \sqrt{\Psi}$. In general, as $u_z$ is reduced the radial diffusion velocity is reduced, and the plasma experiences greater acceleration in its
Figure 3-6: Mesh plot of $\Delta A/A$ as a function of the square of a normalized thermal speed $\Psi$ and normalized rotation frequency $\omega_0$, for $u_{z0} = 0.87$.

Figure 3-7: Plot of $\Delta A/A$ as a function of the square of a normalized thermal speed $\Psi$ and normalized axial flow velocity $u_{z0}$. The plot is restricted to supersonic flow, for which $u_{z0} > \sqrt{\Psi}$.
azimuthal and axial components, leading to an increase in separation. The improvement in separation with decreasing axial flow velocity is consistent with an earlier prediction for the improvement in separation due to a 10% increase in the $B_2$ field [32].

3.5 Conclusions

In this Chapter, the steady-state plasma has been investigated both with and without electron-ion collisions. In a calculation which ignored the effects of electron-ion collisions, the diamagnetic effect caused by the azimuthal current was determined, and found to negligible. The effects of electron-ion collisions were included as a perturbation to the steady-state plasma. When the radial dependence of the electron-ion collision frequency was Gaussian, closed form perturbation solutions could be found only for certain plasma conditions. For general plasma conditions an approximate technique was used. The approximate treatment satisfactorily predicted important features, although the effect by viscosity could not be predicted.

Comparison of the fractional improvement in the separative figure of merit $\Delta A/A$, due to electron-ion collisions, in numerical and approximate treatments is reasonable. In this chapter the effect of varying temperature, rotation frequency and axial streaming velocity on the fractional change in the separative figure of merit $\Delta A/A$, was investigated. For plasmas that rotate slowly compared to the thermal speed, a substantial improvement in separation is predicted, and the fractional improvement in separation increases with decreasing temperature. For such slowly rotating plasmas however, the separative figure of merit $A$, is small in any case. For typical VAC plasma conditions a reduced (but still supersonic) streaming velocity is predicted to improve separative performance, and the improvement increases with decreasing temperature.

In the remainder of this work, attention is turned to the oscillations observed in electric probe signals. In the next chapter, the oscillations in electric probe signals are reviewed, and compared to oscillations observed in other rotating plasmas. Chapter Five introduces the plasma model and a wave perturbation, and solves for the wave perturbation in the absence of electron-ion collisions (i.e. $\delta = 0$). In Chapters Six and Seven the effects of electron-ion collisions are retained, and solutions for large and small axial wavelengths found, respectively. Finally, in Chapter Eight a detailed set of experiments on the PCEN device at the Brazilian
National Space Research Institute [25] is described, and the results compared to the theory of Chapters Five to Seven.