Appendix A

Plasma Model with Electron Temperature Wave Perturbations

In this analysis, the plasma model described in Sec. 5.1 is modified as follows: a wave perturbation in the electron temperature is considered, an energy equation for the electron fluid introduced, electron-ion collisions neglected, a Gaussian ion density profile assumed \( q = 1 \), and attention restricted to flute modes. It is shown that solutions to the plasma model are either neutrally stable (for a cold plasma), or reduce to the centrifugal instability case described in Sec. 5.4.1. That is, the dispersion relation for the centrifugal instability is unaffected by the presence of perturbations in the electron temperature.

The appendix is structured as follows: Section A.1 modifies the assumptions, and presents the plasma model. Section A.2 introduces a perturbation to the electron temperature, and reduces the system of equations to a three row matrix equation. Finally, Sec. A.3 solves the model, and shows that the solutions reduce to either neutrally stable solutions (for a cold plasma) or the centrifugal instability described in Sec. 5.4.1.

A.1 Assumptions and Plasma Model

In this work, assumption (5) of Sec. 5.1.1, which asserted that the electron temperature was uniform, is weakened to apply only to the steady state. As with Sec. 5.4.1 electron-ion collisions are neglected by setting \( \delta \) to zero. Finally, a Gaussian ion density profile is assumed (i.e. \( q = 1 \)).
Across the field the electron heat conductivity is small, and thus the electron heat flux across the field is neglected in the energy equation for the electron fluid. Also, radiative energy loss from the plasma is neglected. With these simplifications the energy equation for the electron fluid [46] can be written,

\[
\frac{D}{Dt} \left( \frac{3}{2} n_e k T_e \right) = - \left( \frac{5}{2} n_e k T_e \right) \nabla \cdot \mathbf{v} + \mathbf{J}_e \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]  

(A.1)

where \( \frac{D}{Dt} \) is the convective derivative, and \( \mathbf{J}_e = -en_e \mathbf{v}_e \) is the electron current density.

As in Chapter Five, length is normalized to \( R \) and time to \( 1/\omega_{ic} \), where \( \omega_{ic} = ZeB_z/m_i \) is the ion cyclotron frequency. Expanding the convective derivative and dividing through by \( Z n_i \), followed by transformation into dimensionless form and multiplication by \( M R^2 \omega_{ic}^{-1} \) yields the dimensionless equation,

\[
\frac{3\psi Z}{2} \left( \frac{\partial \gamma}{\partial \tau} + \mathbf{u}_i \cdot \nabla \gamma \right) + \psi Z \gamma \nabla \cdot \mathbf{u}_i = -\mathbf{u}_e \cdot (\psi Z \nabla \chi + \mathbf{u}_i \times \mathbf{\hat{\gamma}})
\]  

(A.2)

in which the gradient operator \( \nabla \) is in dimensionless units, \( \gamma = T_e/T_0 \) and \( T_i = \lambda T_0 \), such that \( T_0 \) is the steady state electron temperature.

Equations of motion and continuity of the ion and electron fluids can thus be written:

\[
\frac{\partial \mathbf{u}_i}{\partial \tau} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i = -\psi (Z \nabla \chi + \lambda \nabla l_i) + \mathbf{u}_i \times \mathbf{\hat{\gamma}}
\]  

(A.3)

\[
\psi Z (\nabla \chi - \gamma \nabla l_i - \nabla \gamma) - \mathbf{u}_e \times \mathbf{\hat{\gamma}} = \mathbf{0}
\]  

(A.4)

\[
-\frac{\partial l_i}{\partial \tau} = \nabla \cdot \mathbf{u}_i + \mathbf{u}_i \cdot \nabla l_i
\]  

(A.5)

\[
-\frac{\partial l_i}{\partial \tau} = \nabla \cdot \mathbf{u}_e + \mathbf{u}_e \cdot \nabla l_i
\]  

(A.6)

In the absence of electron-ion collisions, only the equation of motion for the electron fluid is changed from Chapter Five [see Eq. (5.2)]. Finally, with \( \gamma_0 = 1 \), the steady state solution is unchanged from the steady state solution described in Sec. 5.1.3.
A.2 Plasma Waves

In this treatment, waves of the form \( \exp i (m\theta - \omega \tau) \) are considered, where \( m \) is the azimuthal mode number and \( \omega = \omega^r + i\omega^i \). These waves have no component in the \( z \) direction, and are thus flute modes. As with Chapter Five, the aim is to find dispersion relations \( \omega = \omega_M(m) \), where the subscript \( M \) denotes an arbitrary wave mode.

The system comprising Eqs. (A.2) to (A.6) can be linearized with perturbations of the form

\[
\xi(\tau, x, \theta) = \xi_0(x) + \varepsilon \xi_1(x) \exp i (m\theta - \omega \tau) \tag{A.7}
\]

where \( \xi(\tau, x, \theta) \) is any of \( u_i, u_e, l_i, \chi \) or \( \gamma \). The normalized electron temperature \( \gamma \) can therefore be written,

\[
\gamma = 1 + \varepsilon \gamma_1(x) \exp i (m\theta - \omega \tau) \tag{A.8}
\]

and the remaining plasma parameters \( u_{ix}, u_{i\theta}, u_{ix}, u_{i\theta}, u_{ix}, l_i \) and \( \chi \) written:

\[
\begin{align*}
    u_{ix} &= 0 + \varepsilon x \varphi_{11}(x) \exp i (m\theta - \omega \tau), \\
    u_{i\theta} &= x \Omega_0 + \varepsilon x \Omega_1(x) \exp i (m\theta - \omega \tau), \\
    u_{i3} &= u_{i0} + \varepsilon u_{i1}(x) \exp i (m\theta - \omega \tau), \\
    l_i &= -x^2 + \varepsilon l_{i1}(x) \exp i (m\theta - \omega \tau), \\
    \gamma &= \gamma_0(x) + \varepsilon \gamma_1(x) \exp i (m\theta - \omega \tau) \\
    \chi &= \chi_0(x) + \varepsilon \chi_1(x) \exp i (m\theta - \omega \tau)
\end{align*}
\]

with \( \gamma_0(x) \) and \( \Omega_{i\theta} \) described by Eqs. (5.5) and (5.6) respectively. In total, the plasma model has nine unknowns in the wave-perturbed variables \( u_i, u_e, l_i, \gamma \) and \( \chi \), and is now solved.

A.3 Solutions

Taking the difference between azimuthal components of Eqs. (A.3) and (A.4), and solving for \( \Omega_{i1}(x) \) yields,

\[
\Omega_{i1}(x) = \frac{1}{\omega} \left( m \phi \left( \frac{(\lambda + Z) l_{i1}(x) + Z \gamma_1(x)}{x^2} \right) + i (\varphi_{e1}(x) - C \varphi_{11}(x)) \right) \tag{A.9}
\]
Solving the radial and azimuth components of Eq. (A.4) for \( \Omega_{e1}(x) \) and \( \varphi_{e1}(x) \) respectively, yields

\[
\Omega_{e1}(x) = \psi Z \left( 2\gamma_1(x) - \frac{\gamma_1'(x) + l'_1(x) - \chi_1'(x)}{x} \right) \tag{A.10}
\]

\[
\varphi_{e1}(x) = i\psi Z \left( \frac{\gamma_1(x) + l_1(x) - \chi_1(x)}{x} \right) \tag{A.11}
\]

Changing the dependant variable to \( y = x^2 \), equations describing \( l_1(y), \chi_1(y) \) and \( \varphi_{11}(y) \) can be written in matrix form, as

\[
\begin{pmatrix}
\psi \left( \lambda_1(y) + Z \chi_1'(y) \right) \\
y \varphi_{11}'(y) \\
0
\end{pmatrix} = \tilde{A}_T \times
\begin{pmatrix}
l_1(y) \\
\chi_1(y) \\
\varphi_{11}(y)
\end{pmatrix} \tag{A.12}
\]

where \( \tilde{A}_T \) is the matrix

\[
\tilde{A}_T = \begin{pmatrix}
m\lambda_1 \psi C & mZ \psi C & \frac{i\psi}{Z} - \frac{ic^2}{2Z} \\
\frac{i}{\psi} \left( \bar{\omega} - \frac{m^2 \lambda_1 \psi}{2\psi} \right) & \frac{i m^2 \psi C}{2\psi} & -1 + y - \frac{mC}{\omega} \\
i \left( \bar{\omega} - m\Omega_{10}^2 - 2m\lambda_1 \psi \right) & -2i m Z \psi & 0
\end{pmatrix} \tag{A.13}
\]

and where Eqs. (A.9), (A.10) and (A.11) have been used. The first row equation is the difference between radial components of Eqs. (A.3) and (A.4). The second row equation is Eq. (A.5), and the third row equation, Eq. (A.6).

Finally, taking the difference between Eq. (A.2) and Eq. (A.5) multiplied by \( 2\psi \), subtracting the third row of Eq. (A.12) multiplied by \( \psi \), and solving for \( \gamma_1(y) \) yields the expression

\[
\gamma_1(y) = \frac{2i}{\psi Z} \left( \frac{\Omega_{10} + m^2 \Omega_{10}^2 + 2m\lambda_1 \psi}{3\bar{\omega} + 2m\Omega_{10}^2 + 4m\lambda_1 \psi} \right) y \varphi_{11}(y) - \left( \frac{2m\Omega_{10}}{3\bar{\omega} + 2m\Omega_{10}^2 + 4m\lambda_1 \psi} \right) \chi_1(y)
\]

Thus, the electron temperature perturbation is determined from the solutions for \( \varphi_{11}(y) \) and \( \chi_1(y) \) in Eq. (A.12). It will now be shown that Eq. (A.12) exhibits the same solutions as Eq. (5.28).

If \( \psi = 0 \), the third row equation in Eq. (A.12) yields \( \bar{\omega} = m\Omega_{10}^2 \), which is stable and therefore discarded. If \( m = 0 \), the third row of Eq. (A.12) yields \( \bar{\omega} = 0 \), such that the perturbation
defined by (A.7) is no longer a wave. Restricting \( m \neq 0 \) and \( \psi \neq 0 \), the fourth row of Eq. (A.12) determines \( \gamma_1(y) \) in terms of \( l_{i1}(y) \), \( \chi_1(y) \) and \( \varphi_1(y) \). Manipulation of the other three row equations yields the modified Kummer equation,

\[
\frac{m}{\omega} \left( \frac{\omega^2 - C^2}{\omega - m\Omega_{a}} \right) L(N_a)[l_{i1}(y)] = 0
\]  
(A.14)

where \( L(N_a) \) is defined by Eq. (5.32) and \( N_a \) defined by Eq. (5.33). In Chapter Five \( l_{i1}(y) \) was replaced with \( g_1(y) \), where [Eq. (5.30) with \( q = 1 \)]

\[
l_{i1}(y) = \frac{-2m\Psi g_1(y)}{\omega - m\Omega_{a}}
\]

Using this relationship, \( l_{i1}(y) \) is replaced with \( g_1(y) \), and Eq. (A.14) rewritten to yield,

\[
\left( \frac{\omega^2 - C^2}{\omega} \right) L(N_a)[g_1(y)] = 0
\]  
(A.15)

This is the equation describing the radial variation of the centrifugal instability [Eq. (5.31)], in Sec. 5.4.1. Thus, it follows that the dispersion relation for the centrifugal instability is unaffected by the presence of electron temperature oscillations.
Appendix B

Numerical Solutions of Chapter Seven

In this appendix, further details are provided about the numerical code (described in Sec. 7.2.1), used to find solutions of Eq. (7.5). To assist in the explanation of aspects of the numerical code, the following terminology is introduced.

<table>
<thead>
<tr>
<th>term</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta F$</td>
<td>step size in $F$</td>
</tr>
<tr>
<td>$(\omega^r_0, \omega^t_0)$</td>
<td>trial frequency $\omega^r$, and growth $\omega^t$ at each new $F$</td>
</tr>
<tr>
<td>$(\omega^r, \omega^t)$</td>
<td>solution frequency $\omega^r$, and growth $\omega^t$ at each $F$</td>
</tr>
<tr>
<td>$\Delta \omega^r$</td>
<td>step size in $\omega^r_0$ for finding maximum growth envelope</td>
</tr>
<tr>
<td>$\Delta \omega^t$</td>
<td>step size in $\omega^t_0$ for finding maximum growth envelope</td>
</tr>
<tr>
<td>$y_c \ll 1$</td>
<td>lower limit of $y$ range</td>
</tr>
<tr>
<td>$Y &gt; 1$</td>
<td>upper limit of $Y$ range</td>
</tr>
<tr>
<td>$P = \begin{cases}</td>
<td>g_1(y_c)</td>
</tr>
<tr>
<td>$\epsilon \ll 1$</td>
<td>tolerance of penalty $P$</td>
</tr>
</tbody>
</table>
B.1 Tracking Eigen-Modes Across $F$

In this section the numerical code used to find solutions to Eq.(7.5) is explained in further detail.

In general the numerical code would be initiated at $F = 0$ or $F = \infty$, where exact solutions for $(\omega^r, \omega^t)$ could be found, and $F$ stepped up or down with either fixed or variable step size, $\Delta F$. In this, eigen-modes could be tracked.

For each $F$, a fifth order Runge-Kutta integration procedure with variable step size in $y$ [78] was used to integrate $g_1(y)$ inwards from the outer boundary at $y = Y$ [where $g_1(Y) = 0$]. To avoid numerical problems the integration was halted at a radial position $y = y_k$ arbitrarily close to the axis, and the penalty $P$ returned. In general, the value of $P$ would be far from zero, the on-axis boundary condition. The Davidon-Fletcher-Powell (variable metric) search algorithm [78] was then employed to vary $(\omega_0^r, \omega_0^t)$, and the integration procedure repeated until a local minima in $P$ could be found. Eigen-mode solutions were required to satisfy $P < \epsilon$.

For instances of the code where a constant step size $\Delta F$ was used, the trial solution $(\omega_0^r, \omega_0^t)$ at each new $F$ was extrapolated from the $(\omega^r, \omega^t)$ solutions at the previous three $F$ values by Neville's algorithm for polynomial extrapolation [78]. In instances where a variable step size $\Delta F$ was used, the trial solution $(\omega_0^r, \omega_0^t)$ at each new $F$ was taken from the $(\omega^r, \omega^t)$ solution at the previous $F$ and a binary chop technique employed to adjust $\Delta F$ until $P < 1$.

In this configuration, the procedure tracked eigen-modes. To find the most unstable eigenmode, a second instance of the procedure was performed, which found the most unstable eigenmode over the range $|\omega_0^r| < 1$. This procedure is described in the next section.

B.2 Finding the Maximum Growth Envelope

In this section the procedure used to find the most unstable mode for a given $F$ is described. By sweeping the procedure across $F$ the maximum growth envelope could be found. The procedure employed relies on the assumption that maximum increase in $\omega^t$ with increasing $F$ is less than 10 (i.e. $\frac{d\omega^t}{dF} < 10$). Providing that this is true, the procedure is functionally equivalent to gridding the initial frequency $\omega_0^r$ and growth $\omega_0^t$ at a resolution of $\Delta \omega^r = 0.05$ and $\Delta \omega^t = 0.01$, and scanning for peak growth over the range $|\omega_0^r| < 1$. 

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The procedure was started at $F = 0$, where the most unstable eigen-mode could be found analytically, and a fixed step size of $\Delta F = 0.05$ used. At each new $F$ a trial growth rate of $\omega'_0 = \omega^i + 10\Delta F$ was selected, where $\omega^i$ was the largest growth rate solution at the previous $F$. The initial frequency $\omega'_0$ was then stepped down from $\omega'_0 = 1.0$ to $\omega'_0 = -1.0$ in $\Delta \omega^i$ increments, and the local minima in $P$ found. At each of the local minima in $P$, the variable metric search algorithm was used with trial conditions $(\omega^r_0, \omega^l_0)$ to find a solution $(\omega^r, \omega^l)$. If the final penalty $P$ was less than a prescribed tolerance $\epsilon$, and the solution growth rate $\omega^i$ exceeded the trial growth rate $\omega'_0$, the code returned with the solution $(\omega^r, \omega^l)$ as the most unstable mode for that $F$. Initially, in every case either $P > \epsilon$ or $\omega^i < \omega^i_0$ across $|\omega^i_0| \leq 1$, and so $\omega^i_0$ would be decremented by $\Delta \omega^i$ and the scan through $\omega^i_0$ repeated until $P < \epsilon$ and $\omega^i > \omega^i_0$. 
Appendix C

Image Processing and Cross-Correlation Techniques used in Chapter Eight.

In this appendix, the simple image processing techniques used in Sec. 8.2 are described, and further details of the cross correlation procedure outlined in Sec. 8.3.3, provided.

C.1 Mylar Film Analysis

The image processing techniques described in this section were used to locate the plasma column centre (with respect to the geometric grid) in mylar film images of the deposition. For each image, a Labview data capture utility was used to locate the co-ordinates of important features. On command, the data capture utility returned the pixel co-ordinates \((x, y)\) of the target pointer, and saved them to file. Here, the \((x, y)\) origin is the top left hand corner of the image, with the \(x\) axis pointing to the right, and \(y\) axis pointing down. The procedure used is described below and shown graphically in Fig. C-1.

(a) Select four successive grid points \((x_i, y_i)\); whose three intervals are averaged to established the scale in pixels per meter, \(p_r = \sqrt{ \left( \frac{1}{3} \sum_{i=1}^{3} |x_i - x_{i+1}| / 3 \right)^2 + \left( \frac{1}{3} \sum_{i=1}^{3} |y_i - y_{i+1}| / 3 \right)^2} . \)
(b) Select top \((x_1,y_1)\) and bottom \((x_2,y_2)\) vertical extremities of the grid; to reference the clockwise rotation of the image with respect to the film, \(\varphi_r = - \arctan \left( \frac{x_2-x_1}{y_2-y_1} \right)\).

(c) Select points \((x_k,y_k)\) equidistant to the geometric centre along each of the four grid lines that intersect the geometric centre; to establish the co-ordinates of the geometric centre 
\[ (x_g, y_g) = \frac{1}{8} \sum_{i=1}^{8} (x_i, y_i). \]

(d) Select points \((x_l,y_l)\) of intersection of the plasma and each of the four grid lines that intersect the geometric centre; to establish the position of the plasma column centre 
\[ (x_p, y_p) = \frac{1}{8} \sum_{i=1}^{8} (x_i, y_i). \]

Figure C-1: Sequence of image processing steps to estimate position of column centre from vessel axis.
The position of the plasma centre relative to the geometric centre is then given by 
\( (x_{pg}, y_{pg}) = (x_p - x_r, y_p - y_r) \). This is converted to polar co-ordinates, \((\Delta r, \varphi_{pg})\), where \(\Delta r\) is the radial distance from the vessel axis to the plasma column centre, and \(\varphi_{pg}\) is the angle of clockwise rotation from the \(-y\) axis. Transformation of co-ordinates yields:

\[
\Delta r = \frac{\sqrt{x_{pg}^2 + y_{pg}^2}}{p_r} \\
\varphi_{pg} = \begin{cases} 
\frac{\pi}{2} + \arctan \frac{y_{pg}}{x_{pg}}, & x_{pg} \geq 0 \\
\frac{3\pi}{2} + \arctan \frac{y_{pg}}{x_{pg}}, & x_{pg} < 0
\end{cases}
\]

The final position of the plasma column with respect to the geometric centre is 
\((\Delta r, \varphi) = (\Delta r, \varphi_{pg} + \varphi_r)\), where \(\varphi = 0\) is the top of the machine.

### C.2 Cross-Correlation

In this section further details of the cross-correlation procedure outlined in Sec. 8.3.3 are provided. The un-normalized cross-correlation \(r_{ab}(k\Delta t)\) of probe signals \(a\) and \(b\) is given by

\[
r_{ab}(k\Delta t) = \lim_{T_0 \to \infty} \int_{-T_0/2}^{T_0/2} a(t)b(t + k\Delta t)dt = \sum_{i=1}^{N} x_i y_{i+k}
\]

where \(\Delta t\) is the time spacing between adjacent data, and \(k\) is a positive integer. In this work \(\Delta t = 1\mu s\), and a data window of 4.5ms was selected (\(N = 4500\)). The extended cross correlation was achieved by \(r_{ab}(-k\Delta t) = r_{ba}(k\Delta t)\). For any given pulse, the first advanced and retarded peaks in the cross correlation were found \([r_{ab}(t_{adv})\) and \(r_{ba}(t_{rel})\) respectively], and the spacing between peaks \((T = t_{adv} + t_{rel})\) taken as the period for that pulse. The mean period \(\bar{T}\), was calculated by averaging \(T\) over eight pulses. For each pulse the phase \(\Delta \theta_i\) using the mean period \(\bar{T}\) and the offset of the closest (and usually strongest) peak from zero.

Typically three observations confirmed that the signal phase difference did not vary during the discharge. Firstly, spot inspection of the signal over the 5ms duration revealed the phase difference to be qualitatively invariant with time. Secondly, the correlation with \(j = 0\) and \(N = 4500\) exhibits well defined structure, which would not be present if the phase were varying.

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throughout the discharge. Finally, the extended cross-correlation was performed with different 1 ms windowing, 0 to 1 ms, 1 to 2 ms, 2 to 3 ms and 3 to 4 ms; such that

\[
\tau_{ab}(k\Delta t) \approx \sum_{i=1+j}^{N+j} x_i y_{i+k}
\]

where \( j = 0, 1000, 2000 \) and 3000, \( N = 600 \), and \( 0 \leq k \leq 200 \). The phase difference measured from each of these different windowing was in agreement.

The final phase difference for each position was taken to be the mean of eight pulses respectively. The mean \( \overline{\Delta \theta} \) was calculated by performing the summation

\[
\overline{\Delta \theta} = \frac{1}{P} \sum_{i=1}^{P} \left( \Delta \theta_i + 360\alpha_i \right)
\]

where \( \alpha_i = 1, 0, -1 \), selected such that the standard deviation of \( \overline{\Delta \theta} \) is minimized. Physically, this procedure amounts to selecting an azimuth for the degenerate co-ordinate so as to minimize the standard deviation. For most phase measurements, the scatter of phase data was small, and so the operation was reduced to selecting each \( \alpha_i \) so as to place the degenerate 180° in azimuth from the mean. For exceptional cases in which phase data was distributed over more than 180°, Eq. C.1 was calculated so as to minimize the standard deviation.