1. Lecture 9:

- **Phase-space Lagrangian:**

  Define \( L_{\text{ph}}(q, \dot{q}, p, t) \equiv p \cdot \dot{q} - H(q, p, t) \),

  \[
  = p \cdot [\dot{q} - u(q, p, t)] + L(q, u(q, p, t), t). \tag{1}
  \]

  with \( L_{\text{ph}}(q, \dot{q}, p, t) = L(q, \dot{q}, t) \) iff \( \dot{q} = u(q, p, t) \).

- **Modified Hamiltons principle:** \( S_{\text{ph}} \) stationary iff

  \[
  \frac{\delta L_{\text{ph}}}{\delta q} = \frac{\partial L_{\text{ph}}}{\partial q} - \frac{d}{dt} \frac{\partial L_{\text{ph}}}{\partial \dot{q}} = -\frac{\partial H}{\partial q} - \dot{p} = 0, \tag{2}
  \]

  \[
  \frac{\delta L_{\text{ph}}}{\delta p} = \frac{\partial L_{\text{ph}}}{\partial p} = \dot{q} - \frac{\partial H}{\partial p} = 0.
  \]

- **Gauge Trans.:** Introduce \( L^2_{\text{ph}} \equiv L^1_{\text{ph}} + M \) \& \( M = -q \cdot p \). Then

  \[
  L^1_{\text{ph}} \equiv p \cdot \dot{q} - H(q, p, t). \tag{3}
  \]

  \[
  L^2_{\text{ph}} = -q \cdot \dot{p} - H(q, p, t). \tag{4}
  \]

  produce same equations of motion.
• **Canonical Transformations**: A transformation \((q, p) \mapsto (Q, P)\) that obeys Hamilton’s equations of motion, with new Hamiltonian \(K(Q, P)\).

\[
L'_{\text{ph}}^{(1)}(Q, \dot{Q}, P, t) \equiv P \cdot \dot{Q} - K(Q, P, t) .
\]  \hspace{2cm} (5)

• **Generating Functions**: \(S_{\text{ph}}[q, p]\) and \(S'_{\text{ph}}[Q, P]\) must be the same, as they describe the same dynamics. Requires existence of \(F\) such that

\[
L_{\text{ph}}(q, \dot{q}, p, \dot{p}, t) = L'_{\text{ph}}(Q, \dot{Q}, P, \dot{P}, t) + \frac{dF}{dt} ,
\]  \hspace{2cm} (6)

\(F\) can be chosen to be a combination of *old* \((q, p)\) and *new* \((Q, P)\) coordinates

\[
F_1(q, Q, t), \quad F_2(q, P, t), \quad F_3(p, Q, t), \quad F_4(p, P, t) .
\]  \hspace{2cm} (7)
• Generating Functions: (cont.)

Type 1: \( F_1(q, Q, t), L_{ph} = L^{(1)}_{ph}, L'_{ph} = L''^{(1)}_{ph} \)

\[
p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}, \quad K = H + \frac{\partial F_1}{\partial t}
\] (8)

Type 2: \( F_1(q, P, t), L_{ph} = L^{(1)}_{ph}, L'_{ph} = L''^{(2)}_{ph} \)

\[
p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}, \quad K = H + \frac{\partial F_2}{\partial t}
\] (9)

Type 3: \( F_1(q, P, t), L_{ph} = L''^{(2)}_{ph}, L'_{ph} = L''^{(1)}_{ph} \)

\[
q = -\frac{\partial F_3}{\partial q}, \quad P = -\frac{\partial F_3}{\partial Q}, \quad K = H + \frac{\partial F_3}{\partial t}
\] (10)

Type 4: \( F_1(q, P, t), L_{ph} = L''^{(2)}_{ph}, L'_{ph} = L''^{(2)}_{ph} \)

\[
q = -\frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}, \quad K = H + \frac{\partial F_4}{\partial t}
\] (11)
2. Identity transformation

The identity transformation is defined by

\[ Q = q, \quad P = p. \quad (12) \]

Type 1 transformations, with generating functions \( F_1(q, Q, t) \), do not include the identity because they assume \( \dot{Q} \) is functionally independent of \( \dot{q} \), which is manifestly false.

Consider type 2 generating functions \( F_2(q, P, t) \):

\[ Q = \frac{\partial F_2}{\partial P}, \quad p = \frac{\partial F_2}{\partial q}, \quad K = H + \frac{\partial F_2}{\partial t}. \quad (13) \]

Try

\[ F_2 = q \cdot P, \quad (14) \]

in eq. (13). This gives \( Q = \partial F_2/\partial P = q \) and \( p = \partial F_2/\partial q = P \), which is the same as eq. (12). That is, eq. (14) generates the identity transformation.
3. Example : Harmonic Oscillator

Problem : Consider the harmonic oscillator with

\[ L = \frac{1}{2} m (\dot{q}^2 - \omega^2 q^2), \quad H = \frac{p^2}{2m} + m\omega^2 q^2 / 2 \]  \hspace{1cm} (15)

with \( \omega \) fixed.

Idea : \( H \) is the sum of two squares \( \rightarrow \) suggests canonical transformation

\[ p = f(P) \cos Q, \quad q = \frac{f(P)}{m\omega} \sin Q \] \hspace{1cm} (16)

for which \( K = H = f^2(P)/(2m) \).

Solution : Trial

\[ F_1 = \frac{1}{2} m\omega q^2 \cot Q \] \hspace{1cm} (17)

Substitute into Eqs. (8), providing

\[ p = m\omega q \cot Q, \quad P = \frac{1}{2} m\omega q^2 \cosec^2 Q \] \hspace{1cm} (18)
which can be solved, producing

\[ p = (2m\omega P)^{1/2} \cos Q, \quad q = \left( \frac{2P}{m\omega} \right)^{1/2} \sin Q \]  

(19)

Compare to Eqs. (16), yields \( f(P) = \sqrt{2m\omega P} \), and new Hamiltonian

\[ K = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} = \omega P \]  

(20)

Equations of motion are thus

\[ \dot{Q} = \frac{\partial K}{\partial P} = \omega \]  

(21)

\[ \dot{P} = -\frac{\partial K}{\partial Q} = 0 \].  

(22)

If \( \epsilon = 0 \), then \( \dot{Q} = \omega \), with solution \( Q = \omega t + \alpha, \ P = K/\omega \).
Interpretation: Solutions for new and old coordinates:

\[ q = \sqrt{\frac{2H}{m\omega^2}} \sin(\omega t + \alpha), \quad p = \sqrt{2mH} \cos(\omega t + \alpha) \]  
\[ Q = \omega t + \alpha, \quad P = K/\omega \]  

\( q \) and \( p \) are the new coordinates, \( Q \) and \( P \) are the old coordinates.

Relationship to Quantum mechanics: Replace \( K \leftrightarrow E \) and \( P \leftrightarrow N\hbar \), with \( N \) the number of photons. Hence (Einstein, 1911),

- \( E = N\hbar\omega \), the energy of \( N \) photons at frequency \( \omega \).
- \( P = N\hbar \) is a constant of the motion. That is, the number of quanta is conserved in the system.
4. Identity-connected transformations

Suppose we move continuously away from the identity among a family of canonical transformations parameterized by $\epsilon$, say, such that $\epsilon = 0$ is the identity. The generating function can be written in the form

$$F_2 = q \cdot P + \sigma(q, P, t) \epsilon + O(\epsilon^2) ,$$

where the notation $O(\epsilon^2)$ means a term scaling like $\epsilon^2$ in the limit $\epsilon \to 0$, and thus negligibly small compared with the $O(\epsilon)$ term. The term $\sigma$ is called the infinitesimal generator of the family of transformations.

Inserting eq. (26) in eq. (13) we have

$$Q = \frac{\partial F_2}{\partial P} = q + \frac{\partial \sigma(q, P, t)}{\partial P} \epsilon + O(\epsilon^2) ,$$

$$p = \frac{\partial F_2}{\partial q} = P + \frac{\partial \sigma(q, P, t)}{\partial q} \epsilon + O(\epsilon^2) .$$
5. Infinitesimal canonical transformations

Solve eq. (27) by iteration to give, up to first order in $\epsilon$,

$$\begin{align*}
Q(q, p, t, \epsilon) &= q + \frac{\partial \sigma(q, p, t)}{\partial p} \epsilon + O(\epsilon^2), \\
P(q, p, t, \epsilon) &= p - \frac{\partial \sigma(q, p, t)}{\partial q} \epsilon + O(\epsilon^2).
\end{align*}$$

(28)

Also, from eq. (13),

$$K(Q, P, t, \epsilon) = H(q, p, t) + \frac{\partial \sigma(q, p, t)}{\partial t} \epsilon + O(\epsilon^2).$$

(29)

This is again a mixed expression in the perturbed and unperturbed variables so we use eq. (28) to get an explicit expression

$$K(q, p, t, \epsilon) = H(q, p, t) + \left( \frac{\partial \sigma(q, p, t)}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial \sigma(q, p, t)}{\partial p} \frac{\partial H}{\partial q} \right) \epsilon + O(\epsilon^2).$$

(30)
6. Time evolution

Take $\epsilon = \delta t$. Now consider the choice $\sigma = H$ in eq. (28). To $O(\delta t)$,

$$Q(q, p, t, \delta t) = q + \frac{\partial H}{\partial p} \delta t = q(t) + \dot{q}(t) \delta t,$$

$$P(q, p, t, \delta t) = p - \frac{\partial H}{\partial q} \delta t = p(t) + \dot{p}(t) \delta t. \quad (31)$$

i.e. $Q(q, p, t, \delta t) = q(t + \delta t)$,

$$P(q, p, t, \delta t) = p(t + \delta t). \quad (32)$$

Hence, the Hamiltonian is the infinitesimal generator for dynamical evolution in time.
7. Hamilton-Jacobi Equations

Canonical transformations can be used to provide a general procedure to solve problems. Two possibilities are:

- If \( H \) is conserved, then transform to a new canonical coordinates that are all cyclic \( \rightarrow \) equations of motion with trivial solutions.

- Seek a canonical transformation from coordinates and momenta \((q, p)\) at time \( t \) to a set of new quantities, which can be initial values \((q_0, p_0)\) at \( t = 0 \). That is, we want

\[
q = q(q_0, p_0, t), \quad p = p(q_0, p_0, t) \tag{33}
\]

**Procedure**: Insist that the transformed Hamiltonian, \( K = 0 \). Then

\[
\frac{\partial K}{\partial P} = \dot{Q} = 0, \quad -\frac{\partial K}{\partial Q} = \dot{P} = 0 \tag{34}
\]

\( K \) is related to \( H \) and generating function via

\[
K = H(q, p, t) + \frac{\partial F}{\partial t} = 0 \tag{35}
\]
Hamilton-Jacobi: (cont.)

Using a type 2 generating function \( F(q, P, t) \), then \( p = \partial F_2 / \partial q \), and so

\[
H \left( q, \frac{\partial F_2}{\partial q}, t \right) + \frac{\partial F_2}{\partial t} = 0
\]  

(36)

which is known as the *Hamilton-Jacobi equation*. The solution of \( F_2 \) is often referred to as *Hamilton’s principal function*, and labeled \( S \).

Equation (36) is a 1st order PDE in \( n + 1 \) variables. Suppose there exists a solution of form

\[
S = S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)
\]

(37)

where the \( \alpha_1, \ldots, \alpha_n \) are independent constants of integration. We are free to choose these to be the new constant momenta, \( P_i = \alpha_i \). The \( F_2 \) transformation equations (9) give

\[
p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i}
\]

(38)

\[
Q_i = \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}
\]

(39)
Hamilton-Jacobi : (cont.)

The constant $\beta_i$ values can be obtained by the initial conditions, by computing the RHS of Eq. \((39)\) at $t = t_0$ with known values of $q_i$.

Eq. \((39)\) can be inverted to give $q$ in terms of $\alpha$, $\beta$ and $t$. Differentiating the RHS of Eq. \((38)\) and substituting gives $p$ in terms of $\alpha$, $\beta$ and $t$. That is

$$q = q(\alpha, \beta, t)$$  \hspace{1cm} (40)

$$p = p(\alpha, \beta, t)$$  \hspace{1cm} (41)

Hamilton’s principal function is thus the generator of a canonical transformation to constant coordinates and momenta.

*When solving the Hamilton-Jacobi equations, we are at the same time obtaining a solution to the mechanical problem.*
Hamilton-Jacobi: (cont.)

We also note that
\[
\frac{dS}{dt} = p \cdot \dot{q} - H = L \tag{42}
\]
and so
\[
S = \int L \, dt + \text{constant} \tag{43}
\]

If the Hamiltonian does not depend explicitly on time, Hamilton’s principal function can be written:
\[
S(q, \alpha, t) = W(q, \alpha) - at \tag{44}
\]
where \(W(q, \alpha)\) is Hamilton’s characteristic function. The physical significance can be understood by writing
\[
\frac{dW}{dt} = \frac{\partial W}{\partial q_i} \dot{q}_i \tag{45}
\]
Finally, substituting Eq. (44) into (38) gives $p_i = \partial W/\partial q_i$, and so

$$\frac{dW}{dt} = \mathbf{p} \cdot \dot{\mathbf{q}} \quad \rightarrow \quad W = \int \mathbf{p} \cdot \dot{\mathbf{q}} dt$$

which is an abbreviated action integral.

8. Lecture 9: Summary

- Identity transformation
- Example application of generating functions to solve problems.
- Identity connection transformation
- Infinitesimal canonical transformations and time evolution
- Hamilton-Jacobi theory

next lecture: Example use of Hamilton-Jacobi theory.