1. Lecture 5: Summary

- Approximate action principle for slowly changing oscillatory systems
- Example: Anharmonic oscillator

New: Harmonic oscillator
A particle of mass $m$ oscillates along the $x$-axis under the influence of a spring, with potential $V$

$$V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m \omega_0^2 x^2$$

- Allow spring constant $\omega_0$ to vary slowly in time. That is, $\omega_0 = \omega_0(t)$
- Assume oscillation amplitude small compared to length $l_0$,

Lagrangian is now

$$L = \frac{1}{2} (\dot{x}^2 - \omega_0^2 x^2)$$  \hspace{1cm} (1)
E-L equations yield linear equations for \( x \). Hence, use trial function with fundamental only

\[
\tilde{x} = A \sin \phi(t) .
\]  

(2)

The averaged Lagrangian is

\[
\bar{L} = \int_{0}^{2\pi} \frac{d\phi}{2\pi} L = \frac{mA^2}{4} (\omega^2 - \omega_0^2) .
\]  

(3)

The Euler–Lagrange equation for \( A \) are

\[
\frac{\partial \bar{L}}{\partial A} = \frac{mA}{2} (\omega^2 - \omega_0^2) = 0 .
\]  

(4)

Providing \( A \neq 0 \), then \( \omega = \pm \omega_0(t) \) (which also implies that \( \bar{L} = 0 \)). That is, instantaneous frequency tracks any change in the natural frequency \( \omega_0 \).

The adiabatically conserved action \( J = \partial \bar{L}/\partial \omega = \frac{m\omega A^2}{2} \rightarrow \) Faster oscillation has small amplitude, and vice-versa. \( \square \)
Lecture 5 : Summary (continued)

• Introduced Hamilton (1805-1865)

• Dynamical systems:
  - continuous-time system: \( \dot{z} = f(z, t), \quad t \in \mathbb{R} \)
  - discrete-time system: \( z_{t+1} = f(z, t), \quad t \in \mathbb{Z} \)

• Lagrange’s equations implicitly contain second-order derivatives, \( \ddot{q} \), & so do not form a dynamical system.

• Can form \( 2n \) first-order equations from \( n \) 2nd order equations. E.g.
  \( \ddot{x} = \alpha \dot{x} + \beta x \) becomes

\[
\dot{x} = y, \quad \dot{y} = \alpha y + \beta x
\]  \( (5) \)
Apply reduction to Lagrangian $L(\dot{q}, q, t)$

$$\dot{q} = u, \quad \ddot{u} =?$$  \hspace{1cm} (6)

What is $\ddot{u}$? Expand Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial^2 L}{\partial t \partial \dot{q}} + \left( \frac{\partial q}{\partial t} \cdot \frac{\partial}{\partial q} \right) \frac{\partial L}{\partial q} + \left( \frac{\partial \dot{q}}{\partial t} \cdot \frac{\partial}{\partial \dot{q}} \right) \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \ddot{q}}$$  \hspace{1cm} (7)

For each $q_i$, this becomes

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial^2 L}{\partial t \partial \dot{q}_i} + \sum_{j=1}^{n} \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} + \sum_{j=1}^{n} \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}$$  \hspace{1cm} (8)

which gives

$$\sum_{j=1}^{n} H_{ij} \ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i \partial t} - \sum_{j=1}^{n} \frac{\partial^2 L}{\partial q_i \partial q_j} \dot{q}_j .$$  \hspace{1cm} (9)

with the Hessian matrix $H_{ij} \equiv (\partial^2 L)/(\partial \dot{q}_i \partial \dot{q}_j)$, and so

$$\ddot{u} = H^{-1} \cdot \left[ \frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial \dot{q} \partial t} - \frac{\partial^2 L}{\partial \dot{q} \partial q} \cdot \dot{q} \right]$$  \hspace{1cm} (10)
• Hamilton recognised more natural to replace $u$ with auxillary variables $p = \{p_i|i = 1, \ldots, n\}$, defined by

$$p_i \equiv \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_i}, \quad (11)$$

where $p_i$ is called the generalized momentum canonically conjugate to $q_i$.

• New dynamical system is

$$\dot{q} = u(q, p, t), \quad \dot{p} = \frac{\partial L(q, \dot{q}, t)}{\partial q} \bigg|_{\dot{q}=u(q,p,t)} \quad (12)$$

**Problem** : Don’t know the $q$ dependance of $u$ in Eq. (12).

**Solution** : Expand $\partial L(q, u, t)/\partial q$. That is,

$$\frac{\partial L(q, u, t)}{\partial q} = \frac{\partial L(q, u, t)}{\partial q} \bigg|_{u,t} + \frac{\partial u}{\partial q} \cdot \frac{\partial L(q, u, t)}{\partial q} \bigg|_{q,t} \quad (13)$$
• Equation of motion for $\dot{p}$ becomes

$$
\dot{p} = \frac{\partial L(q, u, t)}{\partial q} - \sum_{i=1}^{n} \frac{\partial L}{\partial u_i} \frac{\partial u_i}{\partial q} \equiv -\frac{\partial H}{\partial q},
$$

(14)

where we have defined the Hamiltonian

$$
H(q, p, t) \equiv p \cdot u - L(q, u, t).
$$

(15)

• We also note that

$$
\frac{\partial H}{\partial p} = u(q, p, t) + \sum_{i=1}^{n} \left[p_i - \frac{\partial}{\partial u_i} L(q, u, t)\right] \frac{\partial u_i}{\partial p}
$$

$$
= \dot{q},
$$

(16)

where we have used $u = \dot{q}$ and $p = \partial L(q, \dot{q}, t)/\partial \dot{q}$

• **Finally** we have Hamilton’s equations of motion

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}
$$

(17)
2. Time-Dependent and Autonomous Hamiltonian systems

When there is an external time-varying perturbation (e.g. \( V = V(q, t) \)) then \( H = H(p, q, t) \). A system with no external time-dependent forcing, so \( H = H(p, q) \), is called autonomous.

Consider the time rate of change of the Hamiltonian, \( \dot{H} \equiv \frac{dH}{dt} \), following the phase-space trajectory. Using the Hamiltonian equations of motion,

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} + \dot{q} \cdot \frac{\partial H}{\partial q} + \dot{p} \cdot \frac{\partial H}{\partial p}
\]

\[
= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial q} - \frac{\partial H}{\partial q} \cdot \frac{\partial H}{\partial p}
\]

\[
\equiv \frac{\partial H}{\partial t} \quad \text{(18)}
\]

In an autonomous system, \( \partial H/\partial t = 0 \), so \( \dot{H} = 0 \). That is, \( H \) is a constant of the motion in an autonomous system (as found from Noether’s theorem).
3. Gauge-transformed Lagrangian

Consider the harmonic oscillator Lagrangian for a particle of unit mass

\[ L = \frac{1}{2}(\dot{x}^2 - \omega_0^2 x^2) \, , \]  

where we take \( \omega_0 \) to be constant. Since \( \partial L / \partial \dot{x} = \dot{x} \) and \( \partial L / \partial x = \omega_0^2 x \), we immediately verify that Lagrange’s equation gives the harmonic oscillator equation

\[ \ddot{x} = -\omega_0^2 x \, . \]  

(20)

Now add a gauge term, \( \dot{M} \), to the Lagrangian, taking \( M = \frac{1}{2} \omega_0 x^2 \), giving the new Lagrangian

\[ L' \equiv L + \dot{M} = \frac{1}{2}(\dot{x}^2 + 2\omega_0 x \dot{x} - \omega_0^2 x^2) \, . \]  

(21)

Calculating \( \partial L' / \partial \dot{x} = \dot{x} + \omega_0 x \), and \( \partial L' / \partial x = \omega_0^2 x + \omega_0 \dot{x} \) and substituting into the Lagrangian equation of motion, we see that the gauge contributions cancel. Thus we do indeed recover the harmonic oscillator equation, eq. (20), and the new Lagrangian is a perfectly valid one despite the fact that it is no longer in the natural form, \( T - V \).
4. Gauge-transformed Hamiltonian

Using eq. (21) the canonical momentum is

\[ p = \frac{\partial L'}{\partial \dot{x}} = m (\dot{x} + \omega_0 x) \]  

(22)

The gauge transformation changes the canonical momentum, even though the generalized coordinate \( x \) remains the same. This is an example of a canonical transformation. Inverting eq. (22) we find

\[ \dot{x} = \frac{(p - m\omega_0 x)}{m} \]

Hence

\[
H = \frac{(p - m\omega_0 x)^2}{m} - \left( \frac{(p - m\omega_0 x)^2}{2m} + \omega_0 x (p - m\omega_0 x) - \frac{1}{2} m\omega_0^2 x^2 \right) \\
= \frac{(p - m\omega_0 x)^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 \\
= T + V 
\]

(23)

Thus, even though \( L \) was not of the natural form \( T - V \) in this case, and despite the fact that the functional form of the Hamiltonian has changed, it is still the total energy.
5. Eg 1: Hamiltonian of particle in scalar potential

\[ r = x e_x + y e_y + z e_z \Leftrightarrow q = \{x, y, z\}, \]

\[ L = T - V = \frac{1}{2}m|\dot{r}|^2 - V(r, t). \quad (24) \]

Then, from eq. (??)

\[ p \Leftrightarrow p \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r}. \quad (25) \]

Here the canonical momentum is the same as the ordinary kinematic momentum. Equation (25) is solved trivially to give \( \dot{q} \Leftrightarrow \dot{r} = u(p) \) where \( u(p) = p/m. \) Thus, from eq. (??) we have

\[ H = \frac{|p|^2}{m} - \left(\frac{|p|^2}{2m} - V(r, t)\right) \]

\[ = \frac{|p|^2}{2m} + V(r, t) \]

\[ = T + V. \quad (26) \]
6. Eg 2: Dynamics of harmonic oscillator

An example is the harmonic oscillator Hamiltonian corresponding to the potential,

\[ V = \frac{1}{2}m\omega_0^2x^2, \quad (27) \]

\[ H = \frac{p^2}{2m} + \frac{m\omega_0x^2}{2}. \quad (28) \]

From eq. (??) the Hamiltonian equations of motion are

\[ \dot{x} = \frac{p}{m} \]
\[ \dot{p} = -m\omega_0x. \quad (29) \]

The flow lines of the phase space flow field \((\dot{x}, \dot{p})\) (the \textit{phase portrait}), are elliptical.
7. Eg 3: Dynamics of a physical pendulum

In Lecture 5 we found the Lagrangian

\[ L(\theta, \dot{\theta}) = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos \theta) ; \quad (30) \]

which gives \( p_{\theta} \equiv \partial L / \partial \dot{\theta} = ml^2 \dot{\theta} \). Thus \( \dot{\theta} = p_{\theta} / ml^2 \) so the Hamiltonian, \( H = p_{\theta} \dot{\theta} - L \), becomes

\[ H(\theta, p_{\theta}) = \frac{p_{\theta}^2}{2ml^2} + mgl(1 - \cos \theta) , \quad (31) \]

which again is of the form \( T + V \).

Hamilton’s equations of motion are

\[ \dot{\theta} = \frac{\partial H}{\partial p_{\theta}} = \frac{p_{\theta}}{ml^2} , \]

\[ \dot{p}_{\theta} = -\frac{\partial H}{\partial \theta} = -mgl \sin \theta . \quad (32) \]
8. Pendulum phase portrait

Orbits in phase space can be found by plotting the contours of $H$. There are two topologically distinct classes of orbit:

1. **Rotating** orbits for which the pendulum has enough energy, $H > 2mgl$, to swing entirely over the top.

2. **Librating** orbits for $H < 2mgl$. The equilibrium point $\theta = 0$ or $2\pi$, labelled O in the figure, is a fixed point. The orbits in its neighbourhood, like those of the harmonic oscillator, remain in the neighbourhood for all time (i.e. the fixed point is stable or elliptic). (For $|\theta| \ll 1$ we may expand the cosine so $V \approx mgl\theta^2/2$, i.e. a harmonic oscillator potential.)

The dividing line $H = 2mgl$ between the two topological classes of orbit is called the *separatrix*, and on the separatrix lies another fixed point, labelled X in the figure. Almost all orbits in the neighbourhood of an X point eventually are repelled far away from it, and thus it is referred to as an *unstable* or *hyperbolic* fixed point.
9. Eg 4: Hamiltonian of e.m. potentials

Now consider the case of a charged particle in an electromagnetic field with magnetic vector potential \( \mathbf{A} \) and electrostatic potential \( \Phi \). In Lecture 5 we found \( L = T - e\Phi + e\mathbf{r} \cdot \mathbf{A} \) and thus, from eq. (11),

\[
p \equiv \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\mathbf{A}(\mathbf{r}, t) . \tag{33}
\]

Thus \( \dot{\mathbf{r}} = p/m - e\mathbf{A}/m \) and, from eq. (15) we have

\[
H = \frac{(p - e\mathbf{A}) \cdot p}{m} - \left( \frac{|p - e\mathbf{A}|^2}{2m} + \frac{e(p - e\mathbf{A}) \cdot \mathbf{A}}{m} - e\Phi(\mathbf{r}, t) \right)
= \frac{|p - e\mathbf{A}(\mathbf{r}, t)|^2}{2m} + e\Phi(\mathbf{r}, t)
= T + V . \tag{34}
\]

Thus we find again that, although the Lagrangian cannot be put into the natural form \( T - V(\mathbf{r}) \), the Hamiltonian is still the total energy, kinetic plus electrostatic potential energy.
10. Eg 5: Hamiltonian of generalized \( N \)-body system

Assuming any constraints are independent of \( t \), the Lagrangian is

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{N} m_k \dot{q}_i \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} \dot{q}_j - V(q)
\]

\[
\equiv \frac{\dot{q} \cdot \mu \cdot \dot{q}}{2} - V(q) ; \quad \mu_{i,j}(q) \equiv \sum_{k=1}^{N} m_k \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} . \tag{35}
\]

Hence \( p \equiv \frac{\partial L}{\partial \dot{q}} = \mu(q) \cdot \dot{q} \Rightarrow \dot{q} = \mu^{-1} \cdot p \). \tag{36}

Then, from eq. (15) we have

\[
H = p \cdot \mu^{-1} \cdot p - \left( \frac{p \cdot \mu^{-1} \cdot \mu \cdot \mu^{-1} \cdot p}{2} - V(q) \right)
\]

\[
= \frac{p \cdot \mu^{-1} \cdot p}{2} + V(q) = T + V . \tag{37}
\]
11. **Eg 6: Particle in a central potential**

As a simple, two-dimensional example of a problem in non-Cartesion coordinates we return to the problem of motion in a central potential, expressed in plane polar coordinates:

\[
L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r) .
\]

Comparing with eq. (35) we see that the mass matrix is diagonal

\[
\mu = \begin{bmatrix}
m & 0 \\
0 & mr^2
\end{bmatrix} ,
\]

and thus can be inverted simply taking the reciprocal of the diagonal elements. Hence, from eq. (37)

\[
H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) .
\]
12. Lecture 6: Summary

- Revised derivation of Hamilton’s equations of motion
- In a system with no time dependant forcing, $H(p, q)$ is called autonomous
- Gauge transformtions.

13. Preparation for next lecture

Revised lecture notes: All of Chapter 3.