1. Lecture 4: Summary

• **Point** and **Gauge** Transformations

  - A *point* transformation is a change of generalized coordinates \( q \mapsto Q \) via a diffeomorphism

    \[
    q_i = g_i(Q, t) , \quad i = 1, \ldots, n .
    \]  

    \( (1) \)

  - A Lagrangian *gauge* transformation is a shift of \( L \), defined by

    \[
    L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\partial M}{\partial t} + \dot{q} \cdot \frac{\partial M}{\partial q} ,
    \]  

    \( (2) \)

    where \( M(q, t) \) is any \( C^3 \) function, such that dynamics of the motion remain unchanged. That is, \( S = S' \).
Continuous Symmetries:

- $L$ is said to have a continuous symmetry if it is invariant under the transformation $q \mapsto q_s \equiv h_s(q)$, where $h_{s=0}$ is the identity transformation.

- If $\delta S \equiv \int \delta L \, dt = 0$ on the path $q(t)$, then $\delta S$ vanishes for all members of the family $q_s(t)$. If $q_{s=0}(t) \equiv q(t)$ is a physical path, then so are all the paths $q_s(t)$ generated by the symmetry operation $h_s$.

- Given one solution of the motion we can use a symmetry to generate a continuous family of solutions.
Noether’s theorem: Any time-independent continuous symmetry of the Lagrangian $L_s \equiv L(q_s, \dot{q}_s, t)$ generates an integral of the motion.

– Consider a family $q_s(t)$ obeying the Euler-Lagrange equation,

$$\frac{\partial L_s}{\partial q_s} = \frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{q}_s} \right), \quad (3)$$

– As $L_s$ is constant on this family of paths, then $\frac{\partial L_s}{\partial s} = 0$, &

$$\frac{\partial L_s}{\partial s} = \frac{\partial L_s}{\partial q_s} \cdot \frac{\partial q_s}{\partial s} + \frac{\partial L_s}{\partial \dot{q}_s} \cdot \frac{\partial \dot{q}_s}{\partial s} \quad (4)$$

$$= \frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{q}_s} \right) \cdot \frac{\partial q_s}{\partial s} + \frac{\partial L_s}{\partial \dot{q}_s} \cdot \frac{d}{dt} \left( \frac{\partial q_s}{\partial s} \right) \quad (5)$$

$$\equiv \frac{dI}{dt} \quad (6)$$

where

$$I(q_s, \dot{q}_s, t) \equiv \frac{\partial L}{\partial \dot{q}_s} \cdot \frac{\partial q_s}{\partial s} \quad (7)$$

is the integral of motion $I$ generated by the symmetry $\mathbf{h}$. □
2. Slowly changing oscillatory systems: an approximate action principle

What happens if a rapidly oscillating system slowly evolves? (Eg. gyro-orbi averaged Lagrangian in a tokamak.)

Use a trial function of form

\[ q(t) = \tilde{q}(\phi(t), A_1(t), A_2(t), \ldots) \]  \hspace{1cm} (8)

- \( \tilde{q} \) is a 2\( \pi \)-periodic function of \( \phi \), the phase of rapid oscillations,
- \( A_k \) are a set of slowly varying harmonic amplitudes.

Frequency \( \omega(t) \equiv \dot{\phi}(t) \) is \( \gg \frac{d \ln A_k}{dt} \). Thus,

\[ \dot{q} = \omega(t) \frac{\partial \tilde{q}}{\partial \phi} + \dot{A}_1 \frac{\partial \tilde{q}}{\partial A_1} + \cdots \approx \omega(t) \frac{\partial \tilde{q}}{\partial \phi} . \]  \hspace{1cm} (9)

Approximate the action integral by replacing \( L \) by the averaged (smoothed) Lagrangian, \( \bar{L} \):

\[ S \approx \int_{t_1}^{t_2} dt \, \bar{L}(\omega, A_1(t), A_2(t), \ldots), \quad \bar{L} \equiv \int_0^{2\pi} \frac{d\phi}{2\pi} L \]  \hspace{1cm} (10)
3. Adiabatic invariance

Treat $\phi$ and $A_k$ as new generalized coordinates. The averaging in eq. (10) removed all direct dependence on $\phi$: $\bar{L}$ depends only on its time derivative $\omega$. Thus $\phi$ is now ignorable and the $\phi$ Euler–Lagrange equation,

$$\frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{\phi}} = 0,$$
(11)

is a conservation equation for the adiabatic invariant

$$J \equiv \frac{\partial \bar{L}}{\partial \omega}.$$
(12)

The $A_k$ Euler-Lagrange equations are

$$\frac{\partial \bar{L}}{\partial A_k} = 0, \quad k = 1, 2, \ldots,$$
(13)

give the dispersion relation for $\omega$ and determine the harmonic content.
4. Example: Anharmonic oscillator

A particle of mass $m$ oscillates along the $x$-axis under the influence of a nonideal spring, with potential $V$

$$V(x) = \frac{m\omega_0^2}{2} \left( x^2 + \sigma \frac{x^4}{l_0^2} \right),$$

- $\omega_0$ is the angular frequency of oscillations
- osc. amplitude small compared to characteristic length $l_0$,
- $\sigma = \pm 1$. For “soft” spring $\sigma = -1$, “hard” spring $\sigma = +1$.

Consider the trial function

$$x = l_0 \left[ A \cos \omega t + B \cos 3\omega t + C \sin 3\omega t \right],$$

where $\omega$ is the nonlinearly shifted frequency.
Anharmonic oscillator - continued

Trial function is

\[ x = l_0 [A \cos \omega t + B \cos 3\omega t + C \sin 3\omega t] \]

Derivative computes as

\[ \dot{x} \approx l_0 \omega [-A \sin \omega t - 3B \cos 3\omega t + 3C \cos 3\omega t] \quad (14) \]

where \( \dot{A}, \dot{B}, \dot{C} \) are neglected. The averaged Lagrangian is

\[ \bar{L} = \frac{1}{2\pi} \int_0^{2\pi} d\phi L = f(A^2, B^2, C^2, l_0^2, \omega_0^2) \quad (15) \]

Solving

\[ \frac{\partial \bar{L}}{\partial A} = \frac{\partial \bar{L}}{\partial B} = \frac{\partial \bar{L}}{\partial C} = 0 \quad (16) \]

gives \( C = 0 \), and so harmonics are in phase (not surprising as system is periodic). The adiabatic invariant is

\[ J = \frac{\partial \bar{L}}{\partial \omega} = (A^2 + 9B^2)l_0^2 \pi \omega \quad (17) \]
Anharmonic oscillator - continued

Solving $\frac{\partial L}{\partial A} = \frac{\partial L}{\partial B} = 0$ for $\omega$ and $B$, and expand as a series in $A$:

**Hard spring: $\sigma = 1$**

$$\frac{\omega}{\omega_0} = \pm \left( 1 + \frac{3A^2}{4} + \frac{15A^4}{64} + O(A^6) \right)$$

$$B = \frac{A^3}{16} - \frac{21A^5}{64} + O(A^6)$$

$\rightarrow$ System response near natural frequency of spring $\omega_0$.

**Soft spring: $\sigma = -1$**

$$\frac{\omega}{\omega_0} = \pm \left( 0.24 - 0.10A + 0.62A^2 + O(A^3) \right)$$

$$B = -0.56 - 0.26A + 0.31A^2$$

$\rightarrow$ System response very nonlinear
5. Hamilton

- William Rowan Hamilton was born 4 Aug 1805 in Dublin, Ireland. He died 2 Sept 1865 in Dublin, Ireland.

- At age 15 he started studying the works of Newton and Laplace. In 1822 Hamilton found an error in Laplace’s *Méchanique céleste*, which brought him to the attention of the Astronomer Royal of Ireland. He was appointed Professor of Astronomy at Trinity College while he was still an undergraduate aged 21.

- He applied the methods of mechanics to optics, developing both: In 1826(7?) Hamilton presented a memoir, *Theory of Systems of Rays*, to the Royal Irish Academy (a 3rd Supplement was published in 1832). In it he introduced the *characteristic function* for optics.

- While walking in 1843 he discovered/invented *quaternions*, the first noncommutative algebra to be studied, and carved the formula $i^2 = j^2 = k^2 = ijk = -1$ in the stone of Brougham Bridge. Hamilton felt this discovery would revolutionise mathematical physics and he spent the rest of his life working on it.
Sir William Rowan Hamilton

“...the first mathematician of his age”, but an indifferent poet. He was driven to drink by his frustrated love for Catherine Disney.
6. Dynamical systems

Mathematically, a \textit{continuous-time dynamical system} is defined to be a system of first order differential equations

\[ \dot{z} = f(z, t) \, , \quad t \in \mathbb{R} \, , \quad (22) \]

where \( f \) is known as the \textit{vector field} and \( \mathbb{R} \) is the set of real numbers. The space in which \( z \) is defined is called \textit{phase space}.

A \textit{discrete-time dynamical system} is a recursion relation, or \textit{iterated map}

\[ z_{t+1} = f(z, t) \, , \quad t \in \mathbb{Z} \, , \quad (23) \]

where \( f \) is known as the map (or mapping) and \( \mathbb{Z} \) is the set of all integers \( \{ \ldots , -2, -1, 0, 1, 2, \ldots \} \).

Typically (dissipative case), such systems exhibit long-time phenomena like attracting and repelling fixed points and limit cycles, or more complex structures such as strange attractors.
7. Lagrangian Mechanics as a dynamical system

- Lagrange’s equations do not form a dynamical system, because they implicitly contain second-order derivatives, $\ddot{q}$.

- A standard way to obtain a system of first-order equations from a second-order system, is to double the size of the space of time-dependent variables by treating the generalized velocities $u$ as independent of the generalized coordinates, so that the dynamical system is:

\[
\begin{align*}
\dot{q} &= u, \\
\dot{u} &= \ddot{q}(q, u, t).
\end{align*}
\]

The phase space is then of dimension $2n$.

- Often used in numerical problems, because the standard numerical integrators require the problem to be posed in terms of systems of first-order differential equations.
Lagrangian dynamical system contd.

For Lagrangian mechanics, expanding out \((d/dt)\partial L/\partial \dot{q}\) using the chain rule and moving all but the highest-order time derivatives to the right-hand side yields

\[
\sum_{j=1}^{n} \frac{\partial^2 L}{\partial \ddot{q}_i \partial \dot{q}_j} \ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \sum_{j=1}^{n} \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j .
\] (26)

The matrix \(H\) acting on \(\ddot{q}\), whose elements are given by

\[
H_{i,j} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} ,
\] (27)

is called the **Hessian matrix** (a kind of generalized mass tensor). For our method we require it to be **nonsingular**, so that its inverse, \(H^{-1}\), exists and we can find \(\ddot{q}\). Then our dynamical system becomes

\[
\dot{q} = u ,
\]

\[
\dot{u} = H^{-1} \cdot \left[ \frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial \dot{q} \partial t} - \frac{\partial^2 L}{\partial \dot{q} \partial q \cdot \dot{q}} \right] .
\] (28)
8. Momentum instead of velocity

- Freedom of choice: many possible choices of auxiliary variable to find $2n$ first-order differential equations - $u$ is only one.

- Hamilton realised that it is very natural to use as the new auxiliary variables the set $p = \{p_i | i = 1, \ldots, n\}$ defined by

$$p_i \equiv \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_i}, \quad (29)$$

where $p_i$ is called the generalized momentum canonically conjugate to $q_i$.

- We shall always assume that eq. (29) can be solved to give $\dot{q}$ as a function of $q$ and $p$

$$\dot{q} = u(q, p, t). \quad (30)$$
9. Equation of motion for $p$

Why choose $p$?

$\partial L/\partial \dot{q}$ occurs *explicitly* in Lagrange’s equations, so we immediately get an equation of motion for $p$

$$\dot{p} = \left. \frac{\partial L(q, \dot{q}, t)}{\partial q} \right|_{\dot{q}=u(q,p,t)}.$$  \hspace{1cm} (31)

Equations (30) and (31) do indeed form a dynamical system, but not very satisfactory. Why?

$u$ is defined only implicitly as a function of the phase-space variables $q$ and $p$, yet the right-hand side of eq. (31) involves a partial derivative in which the $q$-dependence of $u$ is ignored! That is, $\dot{q} = u(q,p,t)$. 
10. Transformation of partial derivatives

We can fix the latter problem by holding $p$ fixed in partial derivatives with respect to $q$, & subtracting a correction term to cancel the contribution coming from the $q$-dependence of $u$. Applying the chain rule and then using eqs. (29) and (30) we get

$$\dot{p} = \frac{\partial L(q, u, t)}{\partial q} - \sum_{i=1}^{n} \frac{\partial L}{\partial u_i} \frac{\partial u_i}{\partial q}$$

$$= \frac{\partial L(q, u, t)}{\partial q} - \sum_{i=1}^{n} p_i \frac{\partial u_i}{\partial q}$$

$$= \frac{\partial}{\partial q} [L(q, u, t) - p \cdot u]$$

$$\equiv - \frac{\partial H}{\partial q}, \quad (32)$$

where we have defined the Hamiltonian

$$H(q, p, t) \equiv p \cdot u - L(q, u, t). \quad (33)$$
11. Hamilton’s equations of motion

Given the importance of $\partial H/\partial q$ it is natural to investigate whether $\partial H/\partial p$ plays a significant role as well. Differentiating eq. (33) we get

$$\frac{\partial H}{\partial p} = u(q, p, t) + \sum_{i=1}^{n} \left[ p_i - \frac{\partial}{\partial u_i} L(q, u, t) \right] \frac{\partial u_i}{\partial p}$$

$$= \dot{q},$$

where the vanishing of the expression in the square bracket and the identification of $u$ with $\dot{q}$ follows from eqs. (29) and (30).

Summarizing eqs. (32) and (34), we now have Hamilton’s equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}.$$  \hspace{1cm} (35)
12. **Summary: Transition from \( L \) to \( H \)**

Invert

\[
\dot{p} \equiv \frac{\partial}{\partial \dot{q}} L(q, \dot{q}, t),
\]

(36)

to make \( p \) an independent variable instead of \( \dot{q} \):

\[
\dot{q} = u(q, p, t).
\]

(37)

Then substitute eq. (38) in

\[
H(q, p, t) \equiv p \cdot \dot{q} - L(q, \dot{q}, t),
\]

(38)

giving the dynamics as a dynamical system, *Hamilton’s equations of motion*:

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = - \frac{\partial H}{\partial q}.
\]

(39)
13. Lecture 5: Summary

- Approximate action principle for slowly changing oscillatory systems
- Example: Anharmonic oscillator
- Hamilton
- Lagrangian Mechanics as a dynamical system
- Canonical momentum
- Transformation of partial derivatives
- Hamilton’s equations of motion

Preparation for next lecture

Revise lecture notes: Sec. 3.1-3.3