Question 1 [Total marks 16]

(a) [2] Show that the geodesic joining two points in the Euclidean plane is a straight line. [Hint: in Sec. 1.4.1 take \( \tau = x \) so that the length element becomes \( dl = \sqrt{1 + y'^2} \, dx \), where \( y'(x) \equiv dy/dx \).]

(b) [3] If \( f \) does not depend explicitly on \( x \) in the functional

\[
I[y] = \int_{x_1}^{x_2} f(y, y') \, dx ,
\]

show that functions \( y(x) \) for which \( I[y] \) is stationary with respect to infinitesimal variations \( \delta y \) satisfy the identity

\[
y' \frac{\partial f}{\partial y'} - f = \text{const} .
\]

What is the corresponding result in dynamics?

Two point masses \( m_1 \) and \( m_2 \) are joined by a rigid massless rod of length \( l \), and they can slide freely, under the influence of gravity, on a frictionless surface inclined at an angle \( \psi \) to the horizontal.

(c) [2] How many degrees of freedom has this system? Explain the reasoning behind your answer.

(d) [3] Choose generalised coordinates (draw a diagram) and write down a Lagrangian for the system.

(e) [2] Write down as many independent integrals of the motion as you can.

(f) [4] Initially the rod is spinning with an angular speed \( \omega \), and the centre of mass is at rest. Describe mathematically, in words and with a diagram, the subsequent motion.

Question 2 [Total marks 17]

One-dimensional Newtonian dynamics is invariant under the transformation \( x = x' + ut \), where \( x \) is the coordinate measured in the laboratory frame and...
$x'$ is the coordinate measured in a frame moving with uniform velocity $u$.

(a) [2] Draw a diagram to visualise the Galilean transformation described above and work out the transformation equations for velocity, $\dot{x}$, and for momentum, $p \equiv m\dot{x}$.

(b) [1] Verify that the Lagrangians in the laboratory and moving frames, $L = \frac{1}{2}m\dot{x}^2$ and $L' = \frac{1}{2}m\dot{x}'^2$ respectively, both give valid equations of motion for a free-particle (i.e. one not acted on by any forces).

(c) [4] Find the Lagrangian (denoted $\tilde{L}$, say) in the moving frame by the point transformation procedure in Section 2.6.1 of the notes. Show the Lagrangian equations of motion derived from $\tilde{L}$ are the same as those derived from $L'$ but the Hamiltonian is different. Resolve this paradox.

(d) [2] Show that the generating function

$$F_2(x, p') = (x - ut)(p' + mu)$$

generates the Galilean transformation equations for position and momentum and that

$$F_2(x, J) = kxJ,$$

where $k$ is a constant, generates a stretching transformation to a new coordinate $\theta = kx$, and its canonically conjugate momentum $J$.

(e) [5] A particle of charge $e$ and mass $m$ moves in an electrostatic plasma wave with electric potential given by

$$\varphi(x, t) = \varphi_0 \cos(kx - \omega t),$$

where the wavevector $k$, amplitude $\varphi_0$ and frequency $\omega$ are constants. By making a Galilean canonical transformation to a frame moving with the phase velocity, $\omega/k$, combined with a stretching transformation, show that the Hamiltonian can be made the same, with appropriate identifications, as that of the physical pendulum.

(f) [3] Sketch typical particle orbits in phase space over two wavelengths in the $x$-direction and show that the wave separates phase space into three regions: particles that move slower than the wave, particles that move faster than the wave, and particles that move on average at the same speed as the wave. Estimate (to within a numerical factor) the average width, $\Delta v = \Delta p/m$, in velocity of the latter region.
Question 3 [Total marks 16]

A particle of charge $e$ and mass $m$ is in a constant magnetic field $B = m\omega_c/e$ directed parallel to the $z$-axis and a constant electric field $E$ directed parallel to the $x$-axis.

(a) [1] Verify that the electrostatic potential

$$\Phi = -Ex$$

and the vector potential

$$A = \left(\frac{m\omega_c}{e}\right) x e_y$$

give the fields as specified and that the Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{(p_y - m\omega_c x)^2}{2m} + \frac{p_z^2}{2m} - eEx.$$

(b) [2] Write down the Hamiltonian equations of motion and give three integrals of the motion.

(c) [4] Show that $x$ obeys the equation for a harmonic oscillator subject to a forcing term,

$$\ddot{x} + \omega_c^2 x = \frac{p_y\omega_c + eE}{m},$$

and that this is satisfied by

$$x = r_L \sin \omega_c t + \frac{(p_y\omega_c + eE)}{m\omega_c^2},$$

where $r_L$ is an arbitrary constant (the Larmor radius).

(d) [3] By calculating the time averages $\langle \dot{x} \rangle$ and $\langle \dot{y} \rangle$ show that the particle drifts perpendicular to $\mathbf{B}$ at a speed $E/B$ in the $\mathbf{E} \times \mathbf{B}$ direction.

(e) [4] Construct the general solution for $y$ and $z$ and sketch the projection of the motion in the $x,y$ plane.

(f) [2] Now suppose the magnetic field is slowly increased until it is twice its original value. Show that the Larmor radius decreases by a factor $1/\sqrt{2}$. 
