
Due Date: 10am, 29/09/05. Value: 14.2%
(hand in at lecture or put in assignment box in Physics Department.)

1. (a) Write down the Lagrangian in cylindrical polar coordinates \((r, \varphi, z)\), for a particle of charge \(e\) and mass \(m\) moving in an arbitrary scalar potential \(\Phi\) and a vector potential \(\mathbf{A} = e_z A_z\), where \(e_z\) is the unit vector in the \(z\) direction.

(b) Hence, or otherwise, show that the corresponding Hamiltonian is

\[
H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + \frac{(p_z - eA_z)^2}{2m} + e\Phi
\]

(c) Given that the potentials are independent of \(\varphi\) and \(z\), write down two constants of the motion. Show that the particle motion in \(r\) can be described using a one-dimensional Hamiltonian with an effective potential \(V_{eff}(r)\). Given that the potentials are time-independent, write down another integral of the motion. Write down the one-dimensional Hamiltonian equations of motion, and discuss the \(r\) motion of the particle qualitatively assuming \(V_{eff}(r) \to +\infty\) as \(r \to 0\), and considering the two cases \(V_{eff}(r) \to \pm \infty\) as \(r \to \infty\). Sketch a phase-space diagram for the motions in the two cases.

(d) The scalar and vector potentials for the electric and magnetic fields produced by a long straight filament carrying current \(I\) in the \(z\) direction and charged uniformly with a line density \(\lambda\) are \(\Phi(r) = \Phi_0 \ln a/r\) and \(\mathbf{A}(r) = e_z \Psi_0 \ln a/r\), respectively. Here, \(r\) is the distance from the filament, \(a\) is an arbitrary reference distance, and the constants \(\Phi_0\) and \(\Psi_0\) are defined in terms of \(I\) and \(\lambda\) by \(\Phi_0 = \frac{\lambda}{2\pi a_0}\), \(\Psi_0 = \frac{\mu_0 I}{2\pi}\). Show that a charged particle in an evacuated vessel containing such a filament is confined by the magnetic field (i.e. motion is bounded in \(r\)), even when the charges on the particle and the filament are of the same sign. (Consider the initial conditions \(r = a, \varphi = z = 0\), and \(\dot{r} = v_0 > 0, \dot{\varphi} = \omega_0 > 0, \dot{z} = 0\).)

2. Consider the problem of Question 3 in Assignment 2. Using your results:

(a) Sketch a phase space portrait of motion in \(q, \dot{q}\) space showing three topologically distinct orbits: electrons trapped by the wave, and untrapped electrons traveling slower/faster than the wave.

(b) What is the frequency of small oscillations about \(\theta = 0\)?