Module 1 :: Lecture 9

9.1 Sheath Physics and Langmuir probe theory

[Bellan 2.7]

9.1.1 Introduction

- Use Vlasov theory and the Poisson equation to develop a model for the potential in the steady-state transition region, called the sheath, between a plasma and a conducting wall.

- The extent of the sheath region is on the order of a Debye length.

- The sheath potential normally is solved numerically, but we’ll derive a useful analytic approximation.

- Sheath physics is of particular importance in understanding Langmuir probes (small metal wires), which are used to diagnose low-temperature plasmas.

- In particular, a Langmuir probe is biased at a series of voltages with the resulting currents being subsequently measured to ultimately gain a measurement of the plasma density and electron temperature.

9.1.2 1D Model

We consider a 1D system with a metal wall at $x = 0$ held at a potential of $\phi_p$. The plasma is assumed to be collisionless and unmagnetised, with an ambipolar potential of $\phi_A$ which may be different to the laboratory reference potential (called the ground potential), due to the difference in diffusion rates of electrons and ions out of the plasma. The plasma is assumed to be semi-infinite, extending over the range of $x \in (-\infty, 0)$. 


First consider if $\phi_P = \phi_A$, then neither ions or electrons will accelerate or decelerate when leaving the plasma (i.e. going into the sheath). Thus, both ions and electrons will strike the probe (i.e. wall) at a speed dictated by their thermal velocity. As $m_i \gg m_e$, the electron thermal velocity greatly exceeds that of the ions, the flux on the probe will be mostly due to electrons, in this situation. So, when $\phi_P = \phi_A$, the current collected by the probe will be negative.

Now consider if the probe is biased negative, relative to the plasma. Using, $\overline{\phi}(x) = \phi(x) - \phi_A$ to denote the potential relative to the plasma potential, we have the two limiting behaviours:

$$\lim_{x \to 0} \overline{\phi}(x) = \phi_P - \phi_A$$
$$\lim_{x \gg |\lambda_D|} \overline{\phi}(x) = 0.$$  \hspace{1cm} (9.1)

Next, we assume that in the plasma, i.e. $|x| \gg \lambda_D$, the electron distribution function is Maxwellian, with a temperature $T_e$. However, as we are in steady state, the entire, i.e. in both the plasma and sheath regions, distribution function can only be dependent on constants of motion. As the total energy of any given electron is $mv^2/2 + q_e \overline{\phi}(x)$ is a constant of motion, we deduce that the only distribution function that can be a Maxwellian in the plasma is

$$f_e(x,v) = \frac{n_0}{\sqrt{\pi 2\kappa T_e/m_e}} \exp \left( -\frac{mv^2/2 + q_e \overline{\phi}(x)}{\kappa T_e} \right).$$  \hspace{1cm} (9.2)

By taking the 0th moment, we get the electron density:

$$n_e(x) = \int_{-\infty}^{\infty} f_e(x,v) \, dv = n_0 \exp \left( -\frac{q_e \overline{\phi}(x)}{\kappa T_e} \right).$$  \hspace{1cm} (9.3)

That is, when the probe is biased negative relative to the plasma, only electrons with sufficient energy will ever reach the probe.

Ions also will rarify in the sheath region of a negatively biased probe but for **different** reasons. In particular, consider cold ions in the plasma having velocity denoted $u_0$. In the sheath region, flux conservations dictates that $n_0 u_0 = n_i(x) u_i(x)$ or that

$$n_i(x) = \frac{n_0 u_0}{u_i(x)}.$$  \hspace{1cm} (9.4)
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As a negatively biased probe means that the \( u_i(x) \) is increasing as we approach \( x = 0 \), we see that the ion density will decrease as we penetrate into the sheath.

Next, as quasi-neutrality dictates that the electric field must be zero in the plasma, an energy minimisation argument can be applied to say that \( \phi(x) \) and \( \partial \phi / \partial x \) must both be decreasing as we approach the probe. This indicates that \( \phi \) must have a negative second derivative. On the other hand, the Poisson’s equation for our potential is

\[
\frac{d^2 \phi}{dx^2} = -\frac{e}{\epsilon_0} (n_i(x) - n_e(x)),
\]

which then forces \( n_i(x) > n_e(x) \) everywhere, when combined with the intuition that the second derivative of \( \phi \) must be negative. This inequality can now be used to estimate the inflow velocity of ions into the sheath.

Ion energy conservations gives

\[
\frac{1}{2} m_i u_i^2(x) + e\phi(x) = \frac{1}{2} m_i u_0^2,
\]

which can be rewritten to gain an expression for \( u(x) \):

\[
u_i(x) = \sqrt{u_0^2 - 2e\phi(x)/m_i}.
\]

Recalling that we can relate this to the ion density through flux conservation, leads us to

\[
n_i(x) = \frac{n_0}{\sqrt{1 - 2e\phi(x)/m_i u_0^2}}.
\]

The concavity condition \( n_i(x) > n_e(x) \) can now be used to get

\[
\frac{n_0}{\sqrt{1 - 2e\phi(x)/m_i u_0^2}} > \exp \left( -\frac{q_e \phi(x)}{\kappa T_e} \right).
\]

Using the fact that \( \phi \) is negative and a Taylor expansion, we can conclude from the above that

\[
u_0 > \sqrt{\kappa T_e/m_i}.
\]

Thus, to be constant with concavity condition with a negative bias on the probe, it is required that ions enter the sheath with a velocity that is slightly
larger than the ion-acoustic velocity $c_s = \sqrt{\kappa T_e/m_i}$.

The ion-current impinging upon the probe can now be calculated by multiplying the ion flux by area:

\[
I_i = n_0 u_0 q_i A = n_0 c_s e A. \tag{9.11}
\]

The electron current density incident on the probe is given by

\[
J_e = \int_0^\infty q_e v f_e(v,0) \, dv = \frac{n_0 q_e e^{-q_e \Phi(x)/\kappa T_e}}{\sqrt{\pi^2 2 \kappa T_e/m_e}} \int_0^\infty ve^{-mv^2/2\kappa T_e} \, dv = n_0 q_e \sqrt{\frac{\kappa T_e}{2\pi m_e}} e^{-\theta(x)/\kappa T_e}. \tag{9.12}
\]

Thus, the electron current going into the probe is

\[
I_e = -n_0 q_e A \sqrt{\frac{\kappa T_e}{2\pi m_e}} e^{-\theta(x)/\kappa T_e}. \tag{9.13}
\]

The total current going into the probe is given by

\[
I = I_i + I_e = n_0 c_s e A - n_0 q_e A \sqrt{\frac{\kappa T_e}{2\pi m_e}} e^{-\theta(x)/\kappa T_e}. \tag{9.14}
\]

From this, we can see that these currents can cancel each other out if

\[
\sqrt{\frac{2\kappa T_e}{m_i}} = \sqrt{\frac{\kappa T_e}{2\pi m_e}} e^{-\theta(x)/\kappa T_e}; \tag{9.15}
\]

i.e.

\[
e^{-\theta(x)/\kappa T_e} = \ln \sqrt{\frac{m_i}{4\pi m_e}}. \tag{9.16}
\]

Relating this back to the probe potential, we can write:

\[
\phi_P = \phi_A - \frac{\kappa T_e}{e} \ln \sqrt{\frac{m_i}{4\pi m_e}}. \tag{9.17}
\]

The potential differential which corresponds to no net current going through the probe is called the floating potential, as an insulated object put into the plasma will always charge up until it reaches a point where no there is no net current between the object and the plasma.
9.1.3 Use as a plasma diagnostic

The above model can be employed to simply diagnose the plasma density and electron temperature:

1. Put in a probe with a large negative bias so the only current flowing in is due to ions.

2. The collected current is the ion satiation current, \( I_S = n_0 c_s e A \).

3. The ion saturation current is subtracted from all subsequent measurements to give the electron current, \( I_e = I_{measured} - I_S \).

4. The slope of the logarithmic plot of \( I_e \) vs \( \phi \), the potential placed on the probe, should be \( 1/\kappa T_e \). Hence, providing a way to calculate \( T_e \) from the measurements.

5. Once \( T_e \) is known, \( c_s \) can be calculated, which subsequently can lead to a deduction of the plasma density, given knowledge of the probe area, via \( I_S = n_0 c_s e A \).

6. While this is simple to implement, it is not very precise in practice, with uncertainties of two or more being common.