Module 1 :: Lecture 4/5

4.1 Quantification of Plasma Parameters

[Bellan 1.12]

\[ e = 1.6 \times 10^{-19} C \]
\[ m_e = 9.1 \times 10^{-31} kg \]
\[ \frac{m_p}{m_e} = 1836 \]
\[ \epsilon_0 = 8.85 \times 10^{-1} F/m \]
\[ \kappa = 1.6 \times 10^{-1} J/V, \]

(4.1)

As the temperature is given in electron volts. The Debye length is

\[ \lambda_D = \sqrt{\frac{\epsilon_0 \kappa T}{ne^2}} \]
\[ = 7.4 \times 10^3 \sqrt{T(eV)} \frac{m}{n}. \]

(4.2)

We assume that the typical velocity is related to the temperature by

\[ \frac{1}{2} mv^2 = \frac{3}{2} \kappa T. \]

(4.3)

For electron-electron scattering \( \mu = m_e/2 \) so the small angle scattering cross-section is

\[ \sigma^* = \frac{1}{2\pi} \left( \frac{e^2}{\epsilon_0 \kappa T} \right)^2 \ln \left( \frac{\lambda_D}{b_{\pi/2}} \right) \]
\[ = \frac{1}{2\pi} \left( \frac{e^2}{3\epsilon_0 \kappa T} \right)^2 \ln \Lambda. \]

(4.4)
where

\[
\Lambda = \frac{\lambda_D}{b_{\pi/2}} = \sqrt{\frac{\epsilon_0 \kappa T}{n e^2}} \frac{4 \pi \epsilon_0 m v^2}{e^2} = 6 \pi n \lambda_D^3 \tag{4.5}
\]

is typically very large.

The collision frequency is \(\nu = \sigma^* n V\) so

\[
\nu = n \left( \frac{e^2}{3 \epsilon_0 \kappa T} \right)^2 \sqrt{3 \kappa T m_e} \ln \Lambda
\]

\[
= \frac{e^{5/2}}{2 \times 3^{3/2} \pi \epsilon_0^2 m_e^{1/2} T^{3/2}} \frac{n \ln \Lambda}{T^{3/2}}
\]

\[
= 4 \times 10^{-12} \frac{n \ln \Lambda}{T^{3/2}}, \tag{4.6}
\]

where \(\ln \Lambda\) is between 8 and 25 for most plasmas.

### 4.2 Phase space

[See Howard 2.2]

### 4.3 Distribution function and the Boltzmann Equation

[See Howard 2.3]

### 4.4 Moments of the distribution function

[See Howard 2.4]

### 4.5 Collisions in Vlasov Theory

[Bellan 2.4.1, End of Howard 2.3]
Vlasov theory is not the forum for modelling the detailed behaviour of collisions. Indeed, we only need to develop models for the macroscopic consequences of collisions, as they relate to the Vlasov distribution function.

- We conceptualise Vlasov theory as working on a “rough approximation” of time. That is, we think of the theory as modelling a movie of phase space that has a low frame rate.

- In this concept, fast particle collisions look like vertical jumps in phase space, i.e. a path discontinuity in the path of the particle.

- This jump can be modelled as a paired creation-annihilation event.

- Source and sink terms in the Vlasov equation can be used to model these annihilation events and hence provide a means by which collisions can be brought into Vlasov and subsequent theories.

Consider the Vlasov equation (i.e. Boltzmann but not having the acceleration replaced by a Lorentz term):

$$\frac{\partial f_\alpha}{\partial t} + \nabla_x \cdot (\vec{v} f_\alpha) + \nabla_\vec{v} \cdot (\vec{a} f_\alpha) = \sum_\alpha C_{\sigma \alpha}(f_\sigma), \quad (4.7)$$

where $C_{\sigma \alpha}(f_\sigma)$ is the rate of change of $f_\sigma$ due to collisions of species $\sigma$ with species $\alpha$.

- The dependence the collision term has on $f_\sigma$ now renders this equation as being non-linear.

- As the sum of the collision term is over both species, the Vlasov equations now form a coupled system of PDE.

- Fokker-Planck theory gives the details of how to construct $C_{\sigma \alpha}(f_\sigma)$ and surrounds a more detailed analysis of collisions statistics. In particular, velocity averaging has to be handled with more care than just assuming the thermal velocity is the typical velocity of a particle. This theory is somewhat advanced and is beyond the scope of this course.

- Regardless of the specific form of $C_{\sigma \alpha}(f_\sigma)$, we can immediately impose constraints on this terms using conservations of mass, energy and momentum:
– Conservation of particles (i.e. mass) - Collisions can’t change the number of particles in a particular location:

\[ \int C_{\sigma\alpha}(f_{\sigma}) \, d\vec{v} = 0. \] (4.8)

– Conservation of momentum - Collisions between particles of the same species can not change the total momentum of that species:

\[ \int m_{\sigma} \vec{v} C_{\sigma\sigma}(f_{\sigma}) \, d\vec{v} = 0. \] (4.9)

Moreover, collisions between different species must conserve the total momentum of both species together:

\[ \int m_{\sigma} \vec{v} C_{\sigma\alpha}(f_{\sigma}) \, d\vec{v} + \int m_{\alpha} \vec{v} C_{\alpha\sigma}(f_{\alpha}) \, d\vec{v} = 0. \] (4.10)

– Conservation of energy - Collisions between same species particles can not change the total energy of that species:

\[ \int m_{\sigma} v^2 C_{\sigma\sigma}(f_{\sigma}) \, d\vec{v} = 0, \] (4.11)

while collisions between particles of different species must conserve the total energy held between both species:

\[ \int m_{\sigma} v^2 C_{\sigma\alpha}(f_{\sigma}) \, d\vec{v} + \int m_{\alpha} v^2 C_{\alpha\sigma}(f_{\alpha}) \, d\vec{v} = 0. \] (4.12)