Module 1 :: Lecture 2

2.1 Debye Shielding

[Ref. Bellan 1.6]

2.1.1 The Boltzmann Relation

Following the theory cycle of Fig 1.1 of the first lecture:

- Assume nearly collisionless nature of plasma ⇒ idealised theory ⇒ Vlasov theory

- Assume each species $\sigma$ can be modelled as a fluid having density $n_\sigma$, temperature $T_\sigma$ and pressure $P_\sigma = n_\sigma \kappa T_\sigma$ (where $\kappa$ is Boltzmann’s constant) and a mean velocity $u_\sigma^*\,$; i.e. we have made the two-fluid approximation from Vlasov theory. The equation of motion (just force-balance) for each fluid is

$$m_\sigma \frac{du_\sigma^*}{dt} = q_\sigma E - \frac{1}{n_\sigma} \nabla P_\sigma,$$

(2.1)

where $m_\sigma$ is the particle mass, $q_\sigma$ the charge of the particle and $E$ the electric field.

Now we make a series of assumptions that put us on the slow process path of the cycle:

- The inertial term on the LHS of Eq. (2.1) is small and may be dropped

- Inductive fields are small. Thus, the electric field is mostly electrostatic; i.e. $\vec{E} \approx -\nabla \phi$
• Temperature gradients are smeared out by thermal motion so the temperature of each species is spatially uniform (this does not mean the temperature of each species is the same)

• The plasma species are always in thermal equilibrium so that temperature is always well-defined

Using these approximations reduces Eq. (2.1) down to

\[ 0 \approx -n_\sigma q_\sigma \nabla \phi - \kappa T_\sigma \nabla n_\sigma, \]

which tells us simply that the electrostatic forces balance the forces due to the isothermal pressure gradients. We can readily solve Eq. (2.2) to give the Boltzmann relation

\[ n_\sigma(\vec{r}) = n_{\sigma 0} \exp(-q_\sigma \phi(\vec{r})/\kappa T_\sigma), \]

where \( n_{\sigma 0} \) is a constant.

We have to be careful in that Fig. 2.3 is only valid for a slow perturbation, which is relative to the thermal speeds of the species:

• Electrons move fast relative to ions (much smaller mass). Thus, a perturbation that seems 'slow' to an electron may not be slow to the ion.

• Will never see a process be slow for an ion but fast to an electron.

2.1.2 Insertion of a test-particle

In the absence of perturbation, \( \phi = 0 \) everywhere as electron and ion densities were equal throughout the plasma. Inserting a charge \( q_T \) at the origin, without loss of generality, will slightly repel charges of like polarity while attracting the opposite polarity. The resulting, localised separation of charges due to the insertion of the test charge results in an electrostatic field which screens the field of the test-particle; i.e. partially cancels the field due to the perturbation. The effect of this screening is calculated by solving the Poisson equation with both the screening field and field due to the test-particle as the source inhomogeneity:

\[ \nabla^2 \phi = \frac{1}{\epsilon_0} \left[ q_T \delta(\vec{r}) + \sum_{\sigma} n_\sigma(\vec{r}) q_\sigma \right], \]
2.1. DEBYE SHIELDING

where \( q_T \delta(\vec{r}) \) represents the field due to the test particle and \( \sum_{\sigma} n_{\sigma}(\vec{r}) q_{\sigma} \) represents the screening field.

To tie in the Boltzmann relation we assume the insertion of the test charge is slow so that we can replace \( n_{\sigma}(\vec{r}) \) by the Boltzmann relation in Eq. (2.4):

\[
\nabla^2 \phi = \frac{1}{\epsilon_0} \left[ q_T \delta(\vec{r}) + \sum_{\sigma} n_{\sigma 0} \exp(-q_{\sigma} \phi(\vec{r})/\kappa T_{\sigma}) q_{\sigma} \right].
\]

(2.5)

As this is a perturbative analysis, the test charge is assumed to be small, which implies \( |q_{\sigma} \phi| << \kappa T_{\sigma} \). This subsequently allows us to use a Maclaurin series to approximate the exponential in Eq. (2.5) as

\[
\exp(-q_{\sigma} \phi(\vec{r})/\kappa T_{\sigma}) \approx 1 - \frac{q_{\sigma} \phi(\vec{r})}{\kappa T}.
\]

(2.6)

Next the assumption of initial neutrality implies that

\[
\sum_{\sigma} n_{\sigma 0} q_{\sigma} = 0.
\]

(2.7)

Using Eq. (2.6) and Eq. (2.7) in Eq. (2.5) gives

\[
\nabla^2 \phi - \phi \sum_{\sigma} \frac{n_{\sigma 0} q_{\sigma}}{\epsilon_0 \kappa T} = -\frac{q_T}{\epsilon_0} \delta(\vec{r}).
\]

(2.8)

Equation (2.8) is typically written in terms of the effective Debye length, \( \lambda_D \):

\[
\nabla^2 \phi - \frac{1}{\lambda_D^2} \phi = -\frac{q_T}{\epsilon_0} \delta(\vec{r}),
\]

(2.9)

where

\[
\frac{1}{\lambda_D^2} = \sum_{\sigma} \frac{1}{\lambda_{\sigma}^2}
\]

(2.10)

relates the effective Debye length to the species Debye length:

\[
\lambda_{\sigma}^2 = \frac{\epsilon_0 \kappa T_{\sigma}}{n_{\sigma 0} q_{\sigma}^2}.
\]

(2.11)

It is important to note that electrons move at thermal speeds that are too fast for ions to effectively screen. Thus, electron screening only happens via other electrons, while ions are screened by both electrons and ion species.

One can solve Eq. (2.9) to find the potential is given by

\[
\phi(\vec{r}) = \frac{q_T}{4\pi \epsilon_0 r} e^{-|\vec{r}|/\lambda_D},
\]

(2.12)

whose salient properties can be understood by considering two spatial regimes:
• $r \ll \lambda_D$: The potential looks identical to the potential of the test particle in a vacuum.

• $r \gg \lambda_D$: The test charge is completely screened out by the screening cloud.

As the test charge is completely screened for $r \gg \lambda_D$ it must be that the charge of the screening cloud is equal in magnitude but opposite in sign relative to the original test charge (by Gauss’ law). Subsequently, this analysis only makes sense if there are a large number of particles in the screening cloud; i.e. $\Lambda := 4\pi n_0 \lambda_D^3 / 3 \gg 1$. Later this will be seen to be a requirement for the plasma to be nearly collision less, thus helping to link the circle of assumptions started at the beginning of the section.

Some final points:

• The plasma dimensions must be large compared to $\lambda_D$, else no particle would be outside the screening cloud.

• Any particle in the plasma could be set as "the" test particle and thus Eq. (2.12) is taken to be the time-averaged potential of any selected particle in the plasma.

2.2 Quasi-neutrality

[Ref. Bellan 1.7]

The analysis leading up to $\lambda_D$ assumed the plasma was initially neutral (c.f. Eq. (2.7)). We next seek to close the circle of assumptions by showing that if $\lambda_D$ is small, then the assumption of the plasma being initially neutral is an excellent one to make. Specifically, we show that any realistic electrostatic field can be easily reproduced by small deviations from perfect neutrality.

To show this we perform a worst-case scenario analysis, corresponding to the highly unlikely event of electron thermal fluctuations causing the complete evacuation of electrons from a spatial sphere within the plasma. We wish to find the radius of the largest possible sphere where this could occur; we label the radius of the sphere by $r_M$. The point of this analysis is to show that $r_M$ is relatively small, which implies the largest length scale

\footnote{By selecting a particle as the test particle, we effectively shift to a frame corresponding to that particle’s thermal motion, rendering it stationary in the analysis but still experiencing the effects of the thermal motion of nearby particles.}
2.2. QUASI-NEUTRALITY

where the most gratuitous violation of perfect neutrality can occur is very
small; i.e. macroscopically the plasma is neutral (or quasi-neutral.

We start with a plasma of at temperature $T$ for both ions and electrons
(i.e. the ion thermal velocity is much slower than that of the electrons and
are essentially stationary). Since the sphere of radius $r_M$ is assumed to be
the largest sphere where complete electron depletion can occur, escaping
electrons must come to rest on the surface of this sphere (if not, it wouldn’t
be the largest possible sphere!). Ultimately, we need to relate the thermal
kinetic energy of the electrons to the potential energy stored in this ridiculous
configuration. This can be done by using the work-energy theorem to relate
energy to the work done by the electrons as they move to the surface of the
sphere. However, instead of calculating the work directly, the work-energy
theorem can be employed again to relate the work done by electrons to the
electrostatic potential of the ions left behind in the sphere (this potential
must be the same magnitude but opposite sign of the combined electron
potential as the initial electrostatic potential was zero before the electron
exodus).

The energy density of the electric field due to the ions is $\epsilon_0 E^2 / 2$, with the
electric field pointing in the radially outward direction due to the assumed
spherical symmetry of the ion distribution. The total ion charge within a
sphere is $Q = 4\pi n e r^3 / 3$ and so after the electron exodus, the electric field
at radius $r$ is $\vec{E}(r) = Q / 4\pi \epsilon_0 r^2 = n e r / 3 \epsilon_0$. Now, we can foliate the sphere
of radius $r_M$ by spherical shells and integrate the energy density to find the
total electrostatic energy, which is equal to the work done by the electrons:

$$W = \int_0^{r_M} \frac{\epsilon_0 E^2(r)}{2} 4\pi r^2 dr = \pi r_M^5 \frac{2n_e^2 e^2}{45\epsilon_0}. \quad (2.13)$$

Now we equate this to the total thermal kinetic energy of the electrons:

$$\pi r_M^5 \frac{2n_e^2 e^2}{45\epsilon_0} = \frac{3}{2} n_e kT \times \frac{4}{3} \pi r_M^3,$$

which can be solved to give

$$r_M^2 = 45 \frac{\epsilon_0 kT}{n_e e^2}, \quad (2.14)$$

so that $r_M \approx 7\lambda_D$.

From this we have the following conclusions:

- The largest volume that could become spontaneously fully depleted of
electrons is on the order of a few Debye lengths, but this is a highly
highly unlikely phenomena.
• The plasma is quasi-neutral over length-scales much larger than the Debye length.

• A very similar analysis can be carried out for a biased electrode inserted into the plasma, with the electric field from the electrode replacing that of the test charge. Similarly, the field from the biased electrode is also screened out and the associated screening cloud is on the order of a Debye length. The associated region of non-neutrality is called the sheath.