The Effects of Dust on MHD Processes in Astrophysical Plasmas

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We consider a molecular cloud consisting of neutral atomic and molecular species, the ionized atomic and molecular species, the electrons, and negatively charged dust grains. For simplicity a single positively ionized ion species and a single neutral species are assumed, so that a 4-fluid model of the plasma can be used, which employs the fluid momentum equations for plasma ions (singly charged), neutral molecules, charged dust grains and electrons:

\[
\rho_i \left( \frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i \right) = -\nabla p_i + n_i e (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \rho_i \nu_{in}(\mathbf{v}_i - \mathbf{v}_n) - \rho_i \nu_{id} (\mathbf{v}_i - \mathbf{v}_d),
\]

\[
\rho_n \left( \frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n \right) = -\nabla p_n - \rho_n \nu_{ni}(\mathbf{v}_n - \mathbf{v}_i) - \rho_n \nu_{nd} (\mathbf{v}_n - \mathbf{v}_d),
\]

\[
\rho_d \left( \frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_d \cdot \nabla \mathbf{v}_d \right) = -Z_d n_d e (\mathbf{E} + \mathbf{v}_d \times \mathbf{B}) - \rho_d \nu_{dn}(\mathbf{v}_d - \mathbf{v}_n) - \rho_d \nu_{dd} (\mathbf{v}_d - \mathbf{v}_i),
\]

\[
0 = -n_e e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \rho_e \nu_{en}(\mathbf{v}_e - \mathbf{v}_n) - \rho_e \nu_{ed}(\mathbf{v}_e - \mathbf{v}_d) - \rho_e \nu_{ei} (\mathbf{v}_e - \mathbf{v}_i),
\]

where \( \mathbf{E} \) is the wave electric field, \( m_s \) is the species mass, \( \rho_s \) is the species mass density and \( \mathbf{v}_s \) is the species velocity in the wave. \( p_i \) and \( p_n \) are the ion thermal and neutral thermal pressures, and \( \nu_{st} \) is the collision frequency of a particle of species \( s \) with the particles of species \( t \). We have neglected electron inertia in (4), momentum transfer to ions from electrons in (1) and to dust grains from electrons in (3), and the dust thermal pressure gradient in (3).

The fractional ionization \( n_i/n_n \) is assumed fixed, where \( n_n \) is the number density of neutrals. The parameter \( \delta = n_e/n_i < 1 \) measures the charge imbalance of the electrons and ions in the plasma, with the remainder of the negative charge residing on the dust particles, so that the total system is charge neutral. A typical value of \( \delta \) for molecular clouds is \( \delta = 1 - 10^{-4} \). The charge on each dust grain is assumed constant, and for simplicity we also assume that \( \delta \) is constant, even though \( n_n \) and thus \( n_i \) are variable. The neutral mass density obeys the continuity equation

\[
\frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}_n) = 0.
\]

To complete the system of equations, Maxwell’s equations ignoring the displacement current are used, with the conduction current density given by

\[
\mathbf{j} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e - n_d Z_d \mathbf{v}_d).
\]

where equilibrium charge neutrality is expressed by (1).
Equations (4) and (6) lead to the following generalized Ohm’s law:

$$E + \left( \frac{v_i}{\delta} - \frac{(1 - \delta)}{\delta} v_d \right) \times B = \frac{j \times B}{n_e e} - \frac{m_e}{e} \nu_{en} (v_e - v_n) - \frac{m_e}{e} \nu_{ad} (v_e - v_d) - \frac{m_e}{e} \nu_{ei} (v_e - v_i).$$

(7)

The expression for $E$ obtained from (7) can now be substituted into the ion equation (1). We neglect the contribution of the electron collisional momentum transfer terms in (7), compared to the ion momentum transfer terms (i.e., we are neglecting resistivity), so that we may write

$$E + v_i \times B = - \frac{(1 - \delta)}{\delta} (v_i - v_d) \times B + \frac{j \times B}{n_e e}. \quad (8)$$

We now use the strong coupling approximation (Suzuki and Sakai 1996), whereby the ion inertia term (the left hand side) and the ion thermal pressure term are neglected in (1), leaving a balance between the remaining terms. At this point it is useful to normalize the magnetic field by a reference field $B_0$, and define the Alfvén speed based on the field $B_0$ and the ion density: $v_A = B_0/(\mu_0 p_i)$. Eq. (1) may then be written, using (8) and Faraday’s law neglecting the displacement current,

$$\frac{v_A^2}{\delta} (\nabla \times B) \times B = \Omega_m (v_i - v_d) \times B + \nu_{in} (v_i - v_n) + \nu_{id} (v_i - v_d),$$

(9)

where $\Omega_m = \Omega_i (1 - \delta)/\delta$ and $\Omega_i$ is the ion cyclotron frequency, $\Omega_i = B_0 e/m_i$. The presence of dust introduces the first and third terms on the rhs of (9). In the absence of dust, with $\Omega_m = 0$ and $\nu_{id} = 0$, (9) may be solved to give the relative drift velocity of ions and neutrals,

$$v_D = v_i - v_n = \frac{v_A^2}{\delta} (\nabla \times B) \times B/\nu_{in}. \quad (10)$$

This is the strong coupling expression for the relative drift velocity used by Suzuki and Sakai (1996).

The dust equation of motion (3) becomes

$$\frac{\partial v_d}{\partial t} + v_d \cdot \nabla v_d = \frac{\Omega_d v_A^2}{\Omega_i \delta} (\nabla \times B) \times B + \frac{\Omega_d}{\delta} (v_i - v_d) \times B - \nu_{dn} (v_d - v_n) + \nu_{di} (v_i - v_d),$$

(11)

where $\Omega_d = Z_d B_0 e/m_d$ is the dust grain cyclotron frequency. For the dust grains typical of molecular clouds, $\Omega_d \approx 10^{-9} \Omega_i$. 

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Inspection of (11) shows that the acceleration of the dust due to the Lorentz force is proportional to \( \Omega_d/\Omega_i \) and is thus very small compared to that of the ions. To simplify the analysis we consider two limiting cases for the dust fluid velocity \( \mathbf{v}_d \): (a) strongly coupled dust grains and neutral gas, such that \( \mathbf{v}_d = \mathbf{v}_n \), and (b) stationary dust grains, \( \mathbf{v}_d = 0 \). The first case holds approximately if the neutral gas density is high enough that \( \nu_{dn} > \omega \), where \( \omega \) is the characteristic frequency of the waves considered, which we assume here to be higher than the dust cyclotron frequency. For a cloud with \( n_n = 10^4 \text{cm}^{-3} \), \( \nu_{dn} \approx 0.4\Omega_d \) (Pilipp et al. 1987), so that strong dust coupling occurs for higher neutral densities. The second case occurs for high frequencies and lower neutral densities, such that \( \omega > (\Omega_d, \nu_{dn}) \), so that the high dust inertia and small dust collisional coupling implies that the dust is stationary on the time scale of interest. We proceed to discuss the two cases in detail.

II. STRONG DUST COUPLING

For strong dust coupling, the ion equation of motion (9) gives the following expression for the ion-neutral relative drift velocity:

\[
\mathbf{v}_D = \mathbf{v}_i - \mathbf{v}_n = F \frac{v_i^2}{\nu_{id}} \left[ (\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{R}{\rho} \left( (\nabla \times \mathbf{B}) \times \mathbf{B} \right) \right],
\]

where \( \nu_i = \nu_{in} + \nu_{id} \), \( R = \Omega_m \rho_n/\nu_i \rho_0 \), \( B^2 = B_x^2 + B_y^2 + B_z^2 \) and \( F = 1/(1 + R^2 B^2/\rho^2) \). For a small dust number density, \( \nu_{id} << \nu_{in} \), so we approximate \( \nu_i = \nu_{in} \). The inertia of the wave motion is now provided by the neutrals and the dust. Summing the momentum conservation equations (1,2,3) of the ions, neutrals and dust, the equation of motion for the neutral velocity is obtained:

\[
(\rho_n + \rho_d) \left( \frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n \right) = -\nabla p_n + \frac{1}{\rho_0} (\nabla \times \mathbf{B}) \times \mathbf{B}.
\]

We normalize the density by \( \rho_0 \), defining \( \rho = (\rho_n + \rho_d)/\rho_0 \), and the pressure by \( p_0 \). The velocity is normalized by the Alfvén velocity based on \( \rho_0 \), \( V_A = B_0/(\mu_0 \rho_0)^{1/2} \), and space and time are normalized by \( I_0 \) and \( \tau_A = I_0/V_A \). The result is

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\beta \nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B}.
\]

The magnetic induction equation gives, using (7) with the collisional electron momentum transfer terms neglected, and neglecting the Hall term,

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \frac{1}{\delta} (\mathbf{v}_i - (1 - \delta) \mathbf{v}_d) \times \mathbf{B} \right].
\]
In the strong dust coupling limit, (15) becomes

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_n \times \mathbf{B}) + \nabla \times \left( \frac{1}{\delta} \mathbf{v}_n \times \mathbf{B} \right).
\] (16)

Substituting (12), and using the normalizations defined above, we obtain

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + A_P \nabla \times \left[ \frac{F}{\rho} \left( ((\nabla \times \mathbf{B}) \times \mathbf{B}) \times \mathbf{B} + (R/\rho) B^2(\nabla \times \mathbf{B}) \times \mathbf{B} \right) \right],
\] (17)

where \( B = |\mathbf{B}| \) and \( A_P = \rho_n/\nu_i \tau_A \rho_i \).

### III. STATIONARY DUST

In the other limit of stationary dust (\( \mathbf{v}_d = 0 \)) the strong ion coupling equation (9) gives

\[
\mathbf{v}_i = \frac{F}{\nu_i} \left[ \frac{v_A^2}{\delta} \left( (\nabla \times \mathbf{B}) \times \mathbf{B} - (R/\rho)((\nabla \times \mathbf{B}) \times \mathbf{B}) \right) \right. \\
\left. - \Omega_m \mathbf{v}_n \times \mathbf{B} + \nu_i \mathbf{v}_n + (\Omega_m^2/\nu_i)(\mathbf{B} \cdot \mathbf{v}_n) \mathbf{B} \right],
\] (18)

which gives, on substitution into (2), the following normalized equation of motion for the neutrals (with \( \rho = \rho_n/\rho_0 \)):

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\beta \nabla \rho + F \left[ (\nabla \times \mathbf{B}) \times \mathbf{B} - (R/\rho)((\nabla \times \mathbf{B}) \times \mathbf{B}) \times \mathbf{B} \right] \\
- (RF/A_D) \mathbf{v} \times \mathbf{B} + (F \rho/A_P) \left[ G \mathbf{v} + (R^2/\rho^2)(\mathbf{B} \cdot \mathbf{v}) \mathbf{B} \right],
\] (19)

where \( G = 1 - (1 + \nu_{nd}/\nu_{ni})/F \).

The magnetic induction equation becomes, using (7),

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \frac{F}{\delta} \left( \mathbf{v} \times \mathbf{B} - (R/\rho)((\mathbf{B} \cdot \mathbf{v}) \mathbf{B} - B^2 \mathbf{v}) \right) \right] \\
+ \frac{A_D}{\delta^2} \nabla \times \left[ \frac{F}{\rho} \left( ((\nabla \times \mathbf{B}) \times \mathbf{B}) \times \mathbf{B} + (R/\rho) B^2(\nabla \times \mathbf{B}) \times \mathbf{B} \right) \right],
\] (20)

The greater complexity of equations (19) and (20) compared to (14) and (17) is found to lead to greater dispersive effects on the waves. However it is interesting to note that the linear dispersion relation, derived from equations (19) and (20), or from the general dispersion relation of Cramer and Vladimirov (1997) in the limit \( \omega \gg \nu_{dn} \), is the same weakly dispersive relation as in the strongly coupled dust case. Thus strong linear dispersion depends on the removal of the strong ion coupling assumption.
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